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TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 75th birthday)

Tynysbek Sharipovich Kal'menov was born in the village of Koksaek of the Tolebi district of the Turkestan region (earlier it was the Lenger district of the South-Kazakhstan region of the Kazakh SSR). Although "according to the passport" his birthday was recorded on May 5, his real date of birth is April 6, 1946.

Tynysbek Kal'menov is a graduate of the Novosibirsk State University (1969), and a representative of the school of A.V. Bitsadze, an outstanding scientist, corresponding member of the Academy of Sciences of the USSR. In 1972, he completed his postgraduate studies at the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR. In 1983, he defended his doctoral thesis at the M.V. Lomonosov Moscow State University. Since1989, he is a corresponding member of the Academy of Sciences of the Kazakh SSR, and since 2003, he is an academician of the National Academy of Sciences of the Republic of Kazakhstan.

Tynysbek Kal'menov worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (1972-1985). From 1986 to 1991, he was the dean of the Faculty of Mathematics of Al-Farabi Kazakh State University. From 1991 to 1997, he was the rector of the Kazakh Chemical-Technological University (Shymkent).

From 2004 to 2019, Tynysbek Kal'menov was the General Director of the Institute of Mathematics and Mathematical Modeling. He made it one of the leading scientific centers in the country and the best research institute in Kazakhstan. It suffices to say, that in terms of the number of scientific publications (2015-2018) in international rating journals indexed in the Web of Science, the Institute of Mathematics and Mathematical Modeling was ranked fourth among all Kazakhstani organizations, behind only three large universities: the Nazarbaev University, Al-Farabi National University and L.N. Gumilyov Eurasian National University.

Since 2019, Tynysbek Kal'menov has been working as the head of the Department of Differential Equations of the Institute of Mathematics and Mathematical Modeling. He is a member of the National Scientific Council "Scientific Research in the Field of Natural Sciences", which is the main Kazakhstan council that determines the development of science in the country.

T.Sh. Kal'menov was repeatedly elected to maslikhats of various levels, was a member of the Presidium of the Committee for Supervision and Attestation in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan. He is a Laureate of Lenin Komsomol Prize of the Kazakh SSR (1978), an Honored Worker of Science and Technology of Kazakhstan (1996), awarded with the order "Kurmet" (2008 Pi.) and jubilee medals.

In 2013, he was awarded the State Prize of the Republic of Kazakhstan in the field of science and technology for the series of works "To the theory of initial- boundary value problems for differential equations".

The main areas of scientific interests of academician Tynysbek Kal'menov are differential equations, mathematical physics and operator theory. He has obtained fundamental scientific results, many of which led to the creation of new scientific directions in mathematics.

Tynysbek Kal'menov, using a new maximum principle for an equation of mixed type (Kal'menov's maximum principle), was the first to prove that the Tricomi problem has an eigenfunction, thus he solved the famous problem of the Italian mathematician Francesco Tricomi, set in 1923 This marked the beginning of a new promising direction, that is, the spectral theory of equations of mixed type.

He established necessary and sufficient conditions for the well-posed solvability of the classical Darboux and Goursat problems for strongly degenerate hyperbolic equations.

Tynysbek Kal'menov solved the problem of completeness of the system of root functions of the nonlocal Bitsadze-Samarskii problem for a wide class of multidimensional elliptic equations. This result is final and has been widely recognized by the entire mathematical community.

He developed a new effective method for studying ill-posed problems using spectral expansion of differential operators with deviating argument. On the basis of this method, he found necessary and sufficient conditions for the solvability of the mixed Cauchy problem for the Laplace equation.

Tynysbek Kal'menov was the first to construct boundary conditions of the classical Newton potential. That is a fundamental result at the level of a classical one. Prior to the research of Kal'menov T.Sh., it was believed that the Newton potential gives only a particular solution of an inhomogeneous equation and does not satisfy any boundary conditions. Thanks for these results, for the first time, it was possible to construct the spectral theory of the classical Newton potential.

He developed a new effective method for constructing Green's function for a wide class of boundary value problems. Using this method, Green's function of the Dirichlet problem was first constructed explicitly for a multidimensional polyharmonic equation.

From 1989 to 1993, Tynysbek Kal'menov was the chairman of the Inter- Republican (Kazakhstan, Uzbekistan, Kyrgyzstan, Turkmenistan, Tajikistan) Dissertation Council. He is a member of the International Mathematical Society and he repeatedly has been a member of organizing committee of many international conferences. He carries out a lot of organizational work in training of highly qualified personnel for the Republic of Kazakhstan and preparing international conferences. Under his direct guidance, the First Congress of Mathematicians of Kazakhstan was held. He presented his reports in Germany, Poland, Great Britain, Sweden, France, Spain, Japan, Turkey, China, Iran, India, Malaysia, Australia, Portugal and countries of CIS.

In terms of the number of articles in scientific journals with the impact- factor Web of Science, in the research direction of "Mathematics", the Institute of Mathematics and Mathematical Modeling is on one row with leading mathematical institutes of the Russian Federation, and is ahead of all mathematical institutes in other CIS countries in this indicator.

Tynysbek Kal'menov is one of the few scientists who managed to leave an imprint of their individuality almost in all branches of mathematics in which he has been engaged.

Tynysbek Kal'menov has trained 11 doctors and more than 60 candidate of sciences and PhD, has founded a large scientific school on equations of mixed type and differential operators recognized all over the world. Many of his disciples are now independent scientists recognized in the world of mathematics.

He has published over 150 scientific articles, most of which are published in international mathematical journals, including 14 articles published in "Doklady AN SSSR/ Doklady Mathematics". In the last 5 years alone (2016-2020), he has published more than 30 articles in scientific journals indexed in the Web of Science database. To date, academician Tynysbek Kal'menov has a Hirsch index of 18 in the Web of Science and Scopus databases, which is the highest indicator among all Kazakhstan mathematicians.

Outstanding personal qualities of academician Tynysbek Kalmenov, his high professional level, adherence to principles of purity of science, high exactingness towards himself and his colleagues, all these are the foundations of the enormous authority that he has among Kazakhstan scientists and mathematicians of many countries.

Academician Tynysbek Sharipovich Kalmenov meets his 75th birthday in the prime of his life, and the mathematical community, many of his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal heartily congratulate him on his jubilee and wish him good health, happiness and new successes in mathematics and mathematical education, family well-being and long years of fruitful life.

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AN EXTREMAL PROBLEM ON NON-OVERLAPPING DOMAINS CONTAINING ELLIPSE POINTS

Ya. Zabolotnii, I. Denega

Communicated by S.A. Plaksa

Key words: inner radius of the domain, mutually non-overlapping domains, the Green function, quadratic differential, the Goluzin theorem.

AMS Mathematics Subject Classification: 30C75.

Abstract. An extremal problem of geometric function theory of a complex variable for the maximum of products of the inner radii on a system of n mutually non-overlapping multiply connected domains B_k containing the points a_k, $k = \overline{1,n}$, located on an arbitrary ellipse $\frac{x^2}{d^2} + \frac{y^2}{t^2} = 1$ for which $d^2 - t^2 = 1$, is solved.

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1 Preliminaries

The paper is devoted to investigation of extremal problems in the theory of conformal mappings of multiply connected domains [1]–[20]. The theory of quadratic differentials is one of the important elements in the study of these extremal problems. Moreover, the basic structural theorem of Jenkins [11], which gives a complete description of the global structure of trajectories of a positive quadratic differential on a finite Riemann surface, is one of the key results of the theory.

The start point of the theory of extremal problems on non-overlapping domains is the Lavrent'ev paper [16] who in 1934 solved the problem of the maximum of product of conformal radii of two simply connected non-overlapping domains. This problem caused a whole stream of results of many authors who generalized and strengthened it in different directions. From the proof of this theorem, as a corollary, follows the well-known statement of Koebe-Bieberbach in the theory of univalent functions.

Note, that the problems on non-overlapping domains considered in 1930–1960 were problems corresponding to quadratic differentials with fixed poles. In 1968 P.M. Tamrazov [19] first attracted the attention of experts to the study of the extremal problems associated with quadratic differentials with non-fixed poles possessing a definite freedom. Moreover, he solved a significant extremal problem of the geometric function theory of a complex variable with five free simple poles. Such problems are called extremal problems with free poles. The first problems with free poles on non-overlapping domains were formulated and partially solved by G.P. Bakhtina [4] in 1974–1975.

Further, interest to the study of problems on non-overlapping domains with free poles has grown significantly, as V.N. Dubinin [7]–[9] solved a number of problems using the method of separating transformation developed by him, which is based on the study of the behavior of the Dirichlet integral under some symmetrization transformations. An important achievement of the works of A.K. Bakhtin [1]–[3] is a significant weakening of the requirements for geometry of the relative position of free poles of quadratic differentials corresponding to the studied problems, the development of the method of "control" functionals, the introduction of the concept of radial systems of points, which expanded the classes of extremal problems for which a complete solution was obtained.

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be its one point compactification, U be the open unit disk in $\mathbb{C}$, $\mathbb{R}^+ = (0, \infty)$, $\chi(t) = \frac{1}{2}\left(t + t^{-1}\right)$ be the Zhukovsky function.

A function $g_B(z, a)$ which is continuous in $\overline{\mathbb{C}}$, harmonic in $B\backslash\{a\}$ apart from z, vanishes outside B, and in the neighborhood of a has the following asymptotic expansion

$$g_B(z, a) = -\ln|z - a| + \gamma + o(1), \quad z \to a,$$

is called the (classical) Green function of the domain B with pole at $a \in B$. The inner radius $r(B, a)$ of the domain B with respect to a point a is the quantity e^{γ}.

Since the Green function is a conformal invariant, if a function f maps the domain B conformally and univalently onto a domain $f(B)$, then

$$r(B, a)|f'(a)| = r(f(B), f(a))$$

for each $a \in B$.

The inner radius increases monotonically with the growth of the domain: if $B \subset B'$ then

$$r(B, a) \leqslant r(B', a), \quad a \in B.$$

Let $n, m \in \mathbb{N}$. A set of points $A_{n,m} := \left\{a_{k,p} \in \mathbb{C} : k = \overline{1, n},\ p = \overline{1, m}\right\}$ is called (n, m)-radial system if for all $k = \overline{1, n}$ and $p = \overline{1, m}$ the following inequalities hold

$$0 < |a_{k,1}| < \ldots < |a_{k,m}| < \infty,$$

$$\arg a_{k,1} = \arg a_{k,2} = \ldots = \arg a_{k,m} =: \theta_k =: \theta_k(A_{n,m}),$$

$$0 = \theta_1 < \theta_2 < \ldots < \theta_n < \theta_{n+1} := 2\pi.$$

For $m = 1$, $(n, 1)$-radial system of points is called n-radial. In this case, we use the simpler notations: $a_{k,1} =: a_k$, $k = \overline{1, n}$, $A_{n,1} =: A_n$, $a_{n+1} := a_1$, $a_0 := a_n$.

The quantities $\alpha_k := \alpha_k(A_{n,m}) := \frac{1}{\pi}\left[\theta_{k+1} - \theta_k\right]$, $k = \overline{1, n}$, $\alpha_{n+1} := \alpha_1$, $\alpha_0 := \alpha_n := \frac{1}{\pi}\left[2\pi - \theta_k\right]$, are called the angular parameters of the system $A_{n,m}$. It is obvious that $\sum\limits_{k=1}^{n} \alpha_k(A_{n,m}) = 2$.

The following result was established by G.M. Goluzin [10] using the variational method.

Theorem 1.1. *For functions $f_k(z)$ which univalently map the disc $|z| < 1$ onto mutually non-overlapping domains, $k \in \{1, 2, 3\}$, the following exact estimate holds*

$$\left|\prod_{k=1}^{3} f_k'(0)\right| \leqslant \frac{64}{81\sqrt{3}}|(f_1(0) - f_2(0))(f_1(0) - f_3(0))(f_2(0) - f_3(0))|. \quad (1.1)$$

Equality in (1.1) is attained only for functions $w = f_k(z)$ which conformally and univalently map the disc $|z| < 1$ onto the angles $2\pi/3$ with vertex at point $w = 0$ and bisectors of which pass through points $f_k(0)$, $|f_k(0)| = 1$, and for functions obtained from it by means of any but the same fractional-linear transformation.

Corollary 1.1. *For mutually simply connected non-overlapping domains $D_k \subset \{z : |z| < 1\}$, points $z_k \in D_k$ and function $w = f(z)$ which is regular and univalent in the disc $|z| < 1$, the following estimate holds*

$$\prod_{k=1}^{3} r(D_k, z_k) \leqslant \frac{64}{81\sqrt{3}} \frac{|f(z_1) - f(z_2)||f(z_1) - f(z_3)||f(z_2) - f(z_3)|}{|f'(z_1)f'(z_2)f'(z_3)|}. \quad (1.2)$$

For equality in (1.2) it is necessary that the extremal partition of the plane $\overline{\mathbb{C}}_w$ by the domains $f(D_k)$ should coincide with the extremal partition in the Theorem 1.1 under some fractional-linear transformation. To prove it we need to apply inequality (1.1) to superpositions $f_k(\zeta) = f(g_k(\zeta))$, where $g_k(\zeta)$ are functions mapping conformally and univalently the disc $|\zeta| < 1$ onto domains D_k, respectively, in addition $g_k(0) = z_k$, $k \in \{1, 2, 3\}$.

In the paper [13] E.V. Kostyuchenko proved that the maximum value of multiplication of inner radiuses for three simply connected non-overlapping domains in the disk is attained for three equal sectors. However, this statement remains valid for multiply connected domains D_k. This follows from V.N. Dubinin's generalization of inequality (1.1) to the case of arbitrary meromorphic functions [8, p. 65].

2 Main results

Let $L = \left\{z = x + iy : \frac{x^2}{b^2} + \frac{y^2}{c^2} = 1\right\}$ be the ellipse, where

$$b = \frac{1}{2}\left(\sqrt[3]{\sqrt{10}+3} + \sqrt[3]{\sqrt{10}-3}\right) \quad \text{and} \quad c = \frac{1}{2}\left(\sqrt[3]{\sqrt{10}+3} - \sqrt[3]{\sqrt{10}-3}\right).$$

Then the following proposition is true.

Theorem 2.1. *For any set of different points a_k, $a_k \in L$, $k \in \{1, 2, 3\}$, and any mutually non-overlapping domains B_k such that $a_k \in B_k \subset \overline{\mathbb{C}} \setminus [-1, 1]$, $k \in \{1, 2, 3\}$, the following inequality holds*

$$\prod_{k=1}^{3} r\left(B_k, a_k\right) \leqslant \frac{64}{729}\left(223 - 70\sqrt{10}\right) \prod_{k=1}^{3} \left| \frac{\sqrt{a_k^2 - 1}}{a_k - \sqrt{a_k^2 - 1}} \right|, \tag{2.1}$$

and the equality in (2.1) is attained, in particular, for points $a_k^0 = \frac{1}{2}\left(\sqrt[3]{\sqrt{10}-3} \cdot e^{i(\varphi + \frac{2\pi}{3}(k-1))} + \sqrt[3]{\sqrt{10}+3} \cdot e^{-i(\varphi + \frac{2\pi}{3}(k-1))}\right)$, where $k \in \{1, 2, 3\}$, $\varphi \in \left[0, \frac{2\pi}{3}\right)$, and domains B_k^0 bounded by quarters of the hyperbola

$$\frac{x^2}{\cos^2(\varphi + \frac{2\pi}{3}(k-1) + \frac{\pi}{3})} - \frac{y^2}{\sin^2(\varphi + \frac{2\pi}{3}(k-1) + \frac{\pi}{3})} = 1$$

and

$$\frac{x^2}{\cos^2(\varphi + \frac{2\pi}{3}(k-1) - \frac{\pi}{3})} - \frac{y^2}{\sin^2(\varphi + \frac{2\pi}{3}(k-1) - \frac{\pi}{3})} = 1$$

(which can degenerate into a real or imaginary half-axis) and containing, respectively, the points a_k^0.

Proof. Let B_k^* be the image of the domain B_k by the mapping $w = z - \sqrt{z^2 - 1}$. Consider the branch of the root for which $\sqrt{1} = 1$. Taking into account the invariance of the Green function under conformal and univalent mappings, we get

$$g_{B_k}(z, a_k) = g_{B_k^*}(w, a_k^*) = \ln \frac{1}{|w - a_k^*|} + \ln r(B_k^*, a_k^*) + o(1). \tag{2.2}$$

Note that

$$\ln \frac{1}{|w - a_k^*|} = \ln \left| \frac{1}{z - \sqrt{z^2 - 1} - a_k + \sqrt{a_k^2 - 1}} \right|$$

$$= \ln \frac{1}{|z - a_k|} + \ln \left| \frac{1}{1 - \frac{\sqrt{z^2-1} - \sqrt{a_k^2-1}}{z - a_k}} \right|$$

$$= \ln \frac{1}{|z-a_k|} + \ln \left| \frac{1}{1 - \frac{(\sqrt{z^2-1}-\sqrt{a_k^2-1})(\sqrt{z^2-1}+\sqrt{a_k^2-1})}{(z-a_k)(\sqrt{z^2-1}+\sqrt{a_k^2-1})}} \right|$$

$$= \ln \frac{1}{|z-a_k|} + \ln \left| \frac{1}{1 - \frac{z+a_k}{\sqrt{z^2-1}+\sqrt{a_k^2-1}}} \right|$$

$$= \ln \frac{1}{|z-a_k|} + \ln \left| \frac{\sqrt{z^2-1}+\sqrt{a_k^2-1}}{\sqrt{z^2-1}+\sqrt{a_k^2-1}-z-a_k} \right|$$

$$= \ln \frac{1}{|z-a_k|} + \ln \left| \frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}} \right| + \ln \left| \frac{\sqrt{\frac{z^2-1}{a_k^2-1}}+1}{1+\frac{\sqrt{z^2-1}-z}{\sqrt{a_k^2-1}-a_k}} \right|.$$

Substituting this expression in (2.2) and taking into account that

$$\ln \left| \frac{\sqrt{\frac{z^2-1}{a_k^2-1}}+1}{1+\frac{\sqrt{z^2-1}-z}{\sqrt{a_k^2-1}-a_k}} \right| \to 0 \quad \text{at} \quad z \to a_k,$$

the following equality is true

$$g_{B_k}(z, a_k) = \ln \frac{1}{|z-a_k|} + \ln \left| \frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}} \right| r(B_k^*, a_k^*) + o(1).$$

Hence,

$$r(B_k, a_k) = \left| \frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}} \right| r(B_k^*, a_k^*). \tag{2.3}$$

Moreover, from above-posed considerations, follows the equality

$$\prod_{k=1}^{3} r\left(B_k, a_k\right) = \prod_{k=1}^{3} r(B_k^*, a_k^*) \prod_{k=1}^{3} \left| \frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}} \right|. \tag{2.4}$$

The function $w = z - \sqrt{z^2-1}$ maps the points a_k that lie on the ellipse L onto the points a_k^*, which lie on a circle centered at the origin with the radius $\sqrt[3]{\sqrt{10}-3}$, and domains B_k that contain, respectively, the points a_k of the ellipse L onto the domains $B_k^* \subset U$, which contain, respectively, the points a_k^*.

By Theorem 3.1 [13], the following inequality holds

$$\prod_{k=1}^{3} r(B_k^*, a_k^*) \leqslant \frac{64}{729}\left(223 - 70\sqrt{10}\right). \tag{2.5}$$

The sign of equality in this inequality is attained only for the sectors of the unit circle with the central angle $\frac{2\pi}{3}$ and for the points a_k^* lying on the bisectors and on the circle of the radius $\sqrt[3]{\sqrt{10}-3}$.

Combining two previous inequalities (2.4) and (2.5), we obtain

$$\prod_{k=1}^{3} r\left(B_k, a_k\right) \leqslant \frac{64}{729}\left(223 - 70\sqrt{10}\right) \prod_{k=1}^{3} \left| \frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}} \right|,$$

and the equality in this inequality is attained, in particular, for the points a_k^0 and domains B_k^0 specified in the statement of the theorem which are, respectively, inverse images of the points a_k^* and sectors for which equality in inequality (2.5) holds by the mapping $w = z - \sqrt{z^2-1}$. □

Theorem 2.1 is valid also in a more general form.

Theorem 2.2. *For any set of different fixed points a_k, $a_k \in \overline{\mathbb{C}} \setminus [-1,\, 1]$, $k \in \{1,2,3\}$, and any mutually non-overlapping domains B_k such that $a_k \in B_k \subset \overline{\mathbb{C}} \setminus [-1,\, 1]$, $k \in \{1,2,3\}$, the following inequality holds*

$$\prod_{k=1}^{3} r\left(B_k, a_k\right) \leqslant \frac{64}{729}\left(223 - 70\sqrt{10}\right) \prod_{k=1}^{3} \left| \frac{\sqrt{a_k^2-1}}{a_k - \sqrt{a_k^2-1}} \right|, \tag{2.6}$$

and the equality in (2.6) *is attained, in particular, for the points a_k^0 and domains B_k^0 described in Theorem* 2.1.

Proof. Note, that inequality (2.5) holds for an arbitrary configuration of the points $a_k^* \in B_k^* \subset U$, $k \in \{1,2,3\}$, where B_k^* are mutually non-overlapping domains. Next, we repeat the reasoning in the proof of Theorem 2.1. □

Remark 1. Note, that the case in which the points $a_k \in L$ is interesting primarily due to the fact that the equality in inequality (2.6) is achieved for this case and extremal configurations of the domains can be written in an explicit form.

Remark 2. Theorem 2.2 gives a different estimate for the product of the inner radii of three non-overlapping domains than Goluzin's theorem [10]. The estimate obtained in Theorem 2.2 is more accurate for many individual cases.

For example, for configurations of domains B_k^0 and points a_k^0 described in Theorem 2.1 for the case $\varphi = 0$ in inequality (2.6) we get equality $\prod\limits_{k=1}^{3} r\left(B_k, a_k\right) \approx 0,0589$. By Goluzin's theorem the following estimate holds

$$\prod_{k=1}^{3} r\left(B_k, a_k\right) \leqslant \frac{64}{81\sqrt{3}} |a_1^0 - a_2^0||a_1^0 - a_2^0||a_2^0 - a_3^0||a_1^0 - a_3^0| \approx 1,7778.$$

Note, that the estimate (2.6) can be applied only to configurations of domains B_k and points a_k for which $a_k \in B_k \subset \overline{\mathbb{C}} \setminus [-1,\, 1]$, $k \in \{1,2,3\}$.

The following theorem is true for the case $n > 3$ and all domains B_k, $k = \overline{1,n}$, contained in some circle.

Theorem 2.3. *Let $n \in \mathbb{N}$, $n > 3$, $R \in \mathbb{R}^+$, $R > \frac{1}{2}\left(\sqrt[3]{\sqrt{10}+3} + \sqrt[3]{\sqrt{10}-3}\right)$ and $U_R = \{z : |z| < R\}$. Then, for any system of different points $A_n = \{a_k\}_{k=1}^n \in L$ and for any collection of mutually non-overlapping domains $\{B_k\}_{k=1}^n$, $a_k \in B_k \subset U_R \setminus [-1,\, 1]$, $k = \overline{1,n}$, the inequality*

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{2}{n}\right)^n \cdot \left(\sqrt{10}-3\right)^{\frac{n}{3}} \cdot \prod_{k=1}^{n} \left| \frac{\sqrt{a_k^2-1}}{a_k - \sqrt{a_k^2-1}} \right|$$
$$\left(\prod_{k=1}^{n} \chi\left(\left(\sqrt[3]{\sqrt{10}-3} \right)^{\frac{1}{\alpha_k}} \right) \chi\left(\left(\sqrt[3]{\sqrt{10}-3} \right)^{\frac{1}{\alpha_{k-1}}} \right) \right)^{\frac{1}{2}}$$

holds. The sign of equality is attained, when a_k and B_k, $k=\overline{1,n}$, are, respectively, the poles and circular domains of the quadratic differential

$$Q(z)dz^2 = -\frac{\left(\frac{z}{2}+\frac{1}{2z}\right)^{n-2}\left(\frac{1}{4}-\frac{1}{2z^2}+\frac{1}{4z^4}\right)\left(1+\left(\frac{z}{2}+\frac{1}{2z}\right)^n\right)^2}{\left[\left(\frac{z}{2}+\frac{1}{2z}\right)^n-(\sqrt{2}-1)^2\right]^2\left[\left(\frac{z}{2}+\frac{1}{2z}\right)^n-(\sqrt{2}+1)^2\right]^2}dz^2. \tag{2.7}$$

Proof. Similarly to Theorem 2.2, let the function $w=z-\sqrt{z^2-1}$ map the points a_k that lie on the ellipse L onto the points a_k^*, which lie on a circle centered at the origin with radius $\sqrt[3]{\sqrt{10}-3}$, and domains B_k that contain, respectively, the points a_k of the ellipse L onto the domains $B_k^*\subset U$, which contain, respectively, the points a_k^*.

Then, we construct $(n,2)$-radial system of points $A_{n,2}^*=\{a_{k,p}^*\}$ and suppose that $a_{k,p}^*:=R^2(\overline{a}_{k,l}^*)^{-1}$, $p+l=3$, $k=\overline{1,n}$, $p\in\{1,2\}$, and system of domains $\{B_{k,p}^*\}$, $a_{k,p}^*\in B_{k,p}^*$, and the domains $B_{k,p}^*$ are symmetric to the domains $B_{k,l}^*$, $k+l=3$, $k=\overline{1,n}$, $p\in\{1,2\}$.

By using the Theorem 3.1.1 [1], the following estimate holds

$$\prod_{k=1}^{n}\prod_{p=1}^{2} r(B_{k,p}^*,a_{k,p}^*)$$

$$\leqslant (2R)^{2n}\left(\prod_{k=1}^{n}\alpha_k\right)^2\left(\prod_{k=1}^{n}\mu_k(R)\right)^{\frac{1}{2}}\left[\prod_{k=1}^{n}\chi\left(\left|\frac{a_{k,1}^*}{R}\right|^{\frac{1}{\alpha_k}}\right)\chi\left(\left|\frac{a_{k,1}^*}{R}\right|^{\frac{1}{\alpha_{k-1}}}\right)\right],$$

where the values $\{\mu_k(R)\}_{k=1}^n\subset\mathbb{R}^+$ for a given R are the coefficients of displacements of the system $A_{n,2}$ (see, for example, [1, p. 75–76]). In this case, $\mu_k(R)\leqslant m^{-2m}$, $R\in\mathbb{R}^+$.

Taking into account that

$$\prod_{k=1}^{n}\prod_{p=1}^{2}|a_{k,p}^*| = R^{2n},$$

$$r(B_{k,3-p}^*,a_{k,3-p}^*) = r(B_{k,p}^*,a_{k,p}^*)\cdot\frac{R^2}{|a_{k,p}^*|^2},$$

we obtain

$$\prod_{k=1}^{n}\prod_{p=1}^{2} r(B_{k,p}^*,a_{k,p}^*) = \left[\prod_{k=1}^{n} r(B_{k,1}^*,a_{k,1}^*)\right]^2\cdot R^{2n}\cdot\left[\prod_{k=1}^{n}|a_{k,1}^*|\right]^{-2}$$

$$\leqslant (2R)^{2n}\left(\frac{1}{2}\right)^{2n}\left(\prod_{k=1}^{n}\alpha_k\right)^2\prod_{k=1}^{n}\chi\left(\left|\frac{a_{k,1}^*}{R}\right|^{\frac{1}{\alpha_k}}\right)\chi\left(\left|\frac{a_{k,1}^*}{R}\right|^{\frac{1}{\alpha_{k-1}}}\right).$$

From here it follows that

$$\left[\prod_{k=1}^{n} r(B_{k,1}^*,a_{k,1}^*)\right]^2 \leqslant \left(\prod_{k=1}^{n}\alpha_k\right)^2\prod_{k=1}^{n}\chi\left(\left|\frac{a_{k,1}^*}{R}\right|^{\frac{1}{\alpha_k}}\right)\chi\left(\left|\frac{a_{k,1}^*}{R}\right|^{\frac{1}{\alpha_{k-1}}}\right)\cdot\left[\prod_{k=1}^{n}|a_{k,1}^*|\right]^2.$$

Extracting the square root of both sides of the inequality we obtain

$$\prod_{k=1}^{n} r(B_{k,1}^*,a_{k,1}^*) \leqslant \left(\prod_{k=1}^{n}\alpha_k\right)\cdot\left(\prod_{k=1}^{n}\chi\left(\left|\frac{a_{k,1}^*}{R}\right|^{\frac{1}{\alpha_k}}\right)\chi\left(\left|\frac{a_{k,1}^*}{R}\right|^{\frac{1}{\alpha_{k-1}}}\right)\right)^{\frac{1}{2}}\cdot\prod_{k=1}^{n}|a_{k,1}^*|.$$

Since $a_{k,1}^* \in B_{k,1}^* \subset U$, $k=\overline{1,n}$, and $\prod\limits_{k=1}^{n} \alpha_k \leqslant \left(\frac{2}{n}\right)^n$, we consequently deduce

$$\prod_{k=1}^{n} r\left(B_{k,1}^*, a_{k,1}^*\right) \leqslant \left(\frac{2}{n}\right)^n \cdot \left(\prod_{k=1}^{n} \chi\left(\left|a_{k,1}^*\right|^{\frac{1}{\alpha_k}}\right) \chi\left(\left|a_{k,1}^*\right|^{\frac{1}{\alpha_{k-1}}}\right)\right)^{\frac{1}{2}} \cdot \prod_{k=1}^{n} |a_{k,1}^*|,$$

the sign of equality in this inequality is attained, if the points $a_{k,1}^*$ and the domains $B_{k,1}^*$, $a_{k,1}^* \in B_{k,1}^* \subset U$, $k=\overline{1,n}$, are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{w^{n-2}(1+w^n)^2}{\left[w^n - (\sqrt{2}-1)^2\right]^2 \left[w^n - (\sqrt{2}+1)^2\right]^2} dw^2. \tag{2.8}$$

Therefore, using equality (2.3), we get

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{2}{n}\right)^n \cdot \left(\prod_{k=1}^{n} \chi\left(\left|a_k^*\right|^{\frac{1}{\alpha_k}}\right) \chi\left(\left|a_k^*\right|^{\frac{1}{\alpha_{k-1}}}\right)\right)^{\frac{1}{2}} \cdot \prod_{k=1}^{n} |a_k^*| \cdot \prod_{k=1}^{n} \left|\frac{\sqrt{a_k^2-1}}{a_k - \sqrt{a_k^2-1}}\right|.$$

By replacing the variable in quadratic differential (2.8) by the formula

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

we have

$$dw = \frac{1}{2}\left(1 - \frac{1}{z^2}\right) dz,$$

then

$$dw^2 = \frac{1}{4}\left(1 - \frac{2}{z^2} + \frac{1}{z^4}\right) dz^2.$$

Thus, we obtain quadratic differential (2.7).

□

Using the method of the proof of Theorem 2.1 we obtain the following statement about estimate of the maximum of the product of the inner radii on the system of n mutually non-overlapping domains B_k containing the points a_k, $k=\overline{1,n}$, located on an arbitrary ellipse.

Let $M = \left\{z = x+iy : \frac{x^2}{d^2} + \frac{y^2}{t^2} = 1,\ d^2 - t^2 = 1\right\}$ and let $d^* = d - \sqrt{d^2-1}$. Then the following theorem is valid.

Theorem 2.4. *Let $n \in \mathbb{N}$, $n \geqslant 3$. Then, for any system of different points a_k such that $a_k \in M$, $k=\overline{1,n}$, and for any collection of mutually non-overlapping domains $\{B_k\}_{k=1}^n$, $a_k \in B_k \subset \overline{\mathbb{C}}\backslash[-1, 1]$, $k=\overline{1,n}$, the inequality*

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) \tag{2.9}$$

$$\leqslant \left(\frac{4(d-\sqrt{d^2-1})}{n}\right)^n \left(\frac{1-(d-\sqrt{d^2-1})^n}{1+(d-\sqrt{d^2-1})^n}\right)^n \prod_{k=1}^{n} \left|\frac{\sqrt{a_k^2-1}}{a_k - \sqrt{a_k^2-1}}\right|$$

holds. The sign of equality is attained, if a_k and B_k, $k=\overline{1,n}$, are, respectively, the poles and circular domains of the quadratic differential

$$Q(z)dz^2 = -\frac{\left(\frac{z}{2}+\frac{1}{2z}\right)^{n-2}\left(\left(\frac{z}{2}+\frac{1}{2z}\right)^n + 1\right)\left(\frac{1}{4} - \frac{1}{2z^2} + \frac{1}{z^4}\right)}{\left(\left(\frac{z}{2}+\frac{1}{2z}\right)^n - (d^*)^n\right)^2 \left(1 - \left(\frac{z}{2}+\frac{1}{2z}\right)^n (d^*)^n\right)^2} dz^2. \tag{2.10}$$

Proof. Let B_k^* be the image of the domain B_k, $k=\overline{1,n}$, by the mapping $w=z-\sqrt{z^2-1}$. Next, we consider branch of the root for which $\sqrt{1}=1$. Then, the points a_k^* are the images of the points a_k at this mapping and lie on circle with the center at the origin and radius $d^*=d-\sqrt{d^2-1}$, besides $B_k^*\subset U$, $k=\overline{1,n}$. From the results of paper [7], for domains B_k^* and points a_k^*, $k=\overline{1,n}$, the following inequality holds

$$\prod_{k=1}^{n} r\left(B_k^*, a_k^*\right) \leqslant \left(\frac{4d^*}{n}\right)^n \left(\frac{1-(d^*)^n}{1+(d^*)^n}\right)^n, \tag{2.11}$$

the sign of equality in this inequality is attained, if the points a_k^* and the domains B_k^*, $k=\overline{1,n}$, are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{w^{n-2}(w^n+1)}{\left(w^n-(d^*)^n\right)^2\left(1-w^n(d^*)^n\right)^2}dw^2. \tag{2.12}$$

Using equality (2.3) and the condition $d^*=d-\sqrt{d^2-1}$, from inequality (2.11) we obtain (2.9). The poles and circular domains of quadratic differential (2.10) are the inverse images, respectively, the poles and circular domains of the quadratic differential (2.12) by mapping $w=z-\sqrt{z^2-1}$. Thus, equality in inequality (2.9) holds for it. □

Remark 3. Note, that by some linear transformation $w=pz+z_0$ we can transform an arbitrary ellipse $\frac{x-x_0}{d_0^2}+\frac{y-y_0}{t_0^2}=1$ on the complex plane onto an ellipse of the form $\frac{x^2}{d^2}+\frac{y^2}{t^2}=1$ for which $d^2-t^2=1$. Moreover, the inner radii of respective domains in this transformation will be treated as $|p|:1$. Therefore, in order to obtain an estimate of the product of the inner radii of non-overlapping domains containing points of an arbitrary ellipse, it is necessary to transform it onto the ellipse M by an appropriate linear transformation and apply Theorem 2.4.

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