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TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 75th birthday)

Tynysbek Sharipovich Kal'menov was born in the village of Koksaek of the Tolebi district of the Turkestan region (earlier it was the Lenger district of the South-Kazakhstan region of the Kazakh SSR). Although "according to the passport" his birthday was recorded on May 5, his real date of birth is April 6, 1946.

Tynysbek Kal'menov is a graduate of the Novosibirsk State University (1969), and a representative of the school of A.V. Bitsadze, an outstanding scientist, corresponding member of the Academy of Sciences of the USSR. In 1972, he completed his postgraduate studies at the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR. In 1983, he defended his doctoral thesis at the M.V. Lomonosov Moscow State University. Since1989, he is a corresponding member of the Academy of Sciences of the Kazakh SSR, and since 2003, he is an academician of the National Academy of Sciences of the Republic of Kazakhstan.

Tynysbek Kal'menov worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (1972-1985). From 1986 to 1991, he was the dean of the Faculty of Mathematics of Al-Farabi Kazakh State University. From 1991 to 1997, he was the rector of the Kazakh Chemical-Technological University (Shymkent).

From 2004 to 2019, Tynysbek Kal'menov was the General Director of the Institute of Mathematics and Mathematical Modeling. He made it one of the leading scientific centers in the country and the best research institute in Kazakhstan. It suffices to say, that in terms of the number of scientific publications (2015-2018) in international rating journals indexed in the Web of Science, the Institute of Mathematics and Mathematical Modeling was ranked fourth among all Kazakhstani organizations, behind only three large universities: the Nazarbaev University, Al-Farabi National University and L.N. Gumilyov Eurasian National University.

Since 2019, Tynysbek Kal'menov has been working as the head of the Department of Differential Equations of the Institute of Mathematics and Mathematical Modeling. He is a member of the National Scientific Council "Scientific Research in the Field of Natural Sciences", which is the main Kazakhstan council that determines the development of science in the country.

T.Sh. Kal'menov was repeatedly elected to maslikhats of various levels, was a member of the Presidium of the Committee for Supervision and Attestation in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan. He is a Laureate of Lenin Komsomol Prize of the Kazakh SSR (1978), an Honored Worker of Science and Technology of Kazakhstan (1996), awarded with the order "Kurmet" (2008 Pi.) and jubilee medals.

In 2013, he was awarded the State Prize of the Republic of Kazakhstan in the field of science and technology for the series of works "To the theory of initial- boundary value problems for differential equations".

The main areas of scientific interests of academician Tynysbek Kal'menov are differential equations, mathematical physics and operator theory. He has obtained fundamental scientific results, many of which led to the creation of new scientific directions in mathematics.

Tynysbek Kal'menov, using a new maximum principle for an equation of mixed type (Kal'menov's maximum principle), was the first to prove that the Tricomi problem has an eigenfunction, thus he solved the famous problem of the Italian mathematician Francesco Tricomi, set in 1923 This marked the beginning of a new promising direction, that is, the spectral theory of equations of mixed type.

He established necessary and sufficient conditions for the well-posed solvability of the classical Darboux and Goursat problems for strongly degenerate hyperbolic equations.

Tynysbek Kal'menov solved the problem of completeness of the system of root functions of the nonlocal Bitsadze-Samarskii problem for a wide class of multidimensional elliptic equations. This result is final and has been widely recognized by the entire mathematical community.

He developed a new effective method for studying ill-posed problems using spectral expansion of differential operators with deviating argument. On the basis of this method, he found necessary and sufficient conditions for the solvability of the mixed Cauchy problem for the Laplace equation.

Tynysbek Kal'menov was the first to construct boundary conditions of the classical Newton potential. That is a fundamental result at the level of a classical one. Prior to the research of Kal'menov T.Sh., it was believed that the Newton potential gives only a particular solution of an inhomogeneous equation and does not satisfy any boundary conditions. Thanks for these results, for the first time, it was possible to construct the spectral theory of the classical Newton potential.

He developed a new effective method for constructing Green's function for a wide class of boundary value problems. Using this method, Green's function of the Dirichlet problem was first constructed explicitly for a multidimensional polyharmonic equation.

From 1989 to 1993, Tynysbek Kal'menov was the chairman of the Inter- Republican (Kazakhstan, Uzbekistan, Kyrgyzstan, Turkmenistan, Tajikistan) Dissertation Council. He is a member of the International Mathematical Society and he repeatedly has been a member of organizing committee of many international conferences. He carries out a lot of organizational work in training of highly qualified personnel for the Republic of Kazakhstan and preparing international conferences. Under his direct guidance, the First Congress of Mathematicians of Kazakhstan was held. He presented his reports in Germany, Poland, Great Britain, Sweden, France, Spain, Japan, Turkey, China, Iran, India, Malaysia, Australia, Portugal and countries of CIS.

In terms of the number of articles in scientific journals with the impact- factor Web of Science, in the research direction of "Mathematics", the Institute of Mathematics and Mathematical Modeling is on one row with leading mathematical institutes of the Russian Federation, and is ahead of all mathematical institutes in other CIS countries in this indicator.

Tynysbek Kal'menov is one of the few scientists who managed to leave an imprint of their individuality almost in all branches of mathematics in which he has been engaged.

Tynysbek Kal'menov has trained 11 doctors and more than 60 candidate of sciences and PhD, has founded a large scientific school on equations of mixed type and differential operators recognized all over the world. Many of his disciples are now independent scientists recognized in the world of mathematics.

He has published over 150 scientific articles, most of which are published in international mathematical journals, including 14 articles published in "Doklady AN SSSR/ Doklady Mathematics". In the last 5 years alone (2016-2020), he has published more than 30 articles in scientific journals indexed in the Web of Science database. To date, academician Tynysbek Kal'menov has a Hirsch index of 18 in the Web of Science and Scopus databases, which is the highest indicator among all Kazakhstan mathematicians.

Outstanding personal qualities of academician Tynysbek Kalmenov, his high professional level, adherence to principles of purity of science, high exactingness towards himself and his colleagues, all these are the foundations of the enormous authority that he has among Kazakhstan scientists and mathematicians of many countries.

Academician Tynysbek Sharipovich Kalmenov meets his 75th birthday in the prime of his life, and the mathematical community, many of his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal heartily congratulate him on his jubilee and wish him good health, happiness and new successes in mathematics and mathematical education, family well-being and long years of fruitful life.

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IDEAL CONNES-AMENABILITY OF LAU PRODUCT OF BANACH ALGEBRAS

A. Minapoor, A. Bodaghi, O.T. Mewomo

Communicated by E. Kissin

Key words: amenability, derivation, ideal amenability, ideal Connes-amenability, Lau product algebra.

AMS Mathematics Subject Classification: Primary 46H25, 46H20; Secondary 46H35

Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and θ be a non-zero character on $\mathcal{B}$. In the current paper, we study the ideal Connes-amenability of the algebra $\mathcal{A} \times_\theta \mathcal{B}$ so-called the θ-Lau product algebra. We also prove that if $\mathcal{A} \times_\theta \mathcal{B}$ is ideally Connes-amenable, then both $\mathcal{A}$ and $\mathcal{B}$ are ideally Connes-amenable. As a result, we show that $l^1(S) \times_\theta l^1(S)$ is ideally Connes-amenable, where S is a Rees matrix semigroup.

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1 Introduction

Johnson in [8] initiated the concept of amenability for Banach algebras. After this pioneering work of Johnson, several modifications of the original notion of amenability in Banach algebras were introduced, see [13] for details and more information. In [4], Gorgi (this author is the same as Gordji) and Yazdanpanah introduced a notion of amenability on Banach algebras which was called ideal amenability. They connected this notion of amenability to the weak amenability and amenability of Banach algebras, and showed that ideal amenability is different from amenability and weak amenability. Further investigations on this notion of amenability and its approximate version can be found in [3, 11, 12].

When there is a natural weak*-topology on the algebra, it is suggested to restrict the attention to those derivations which enjoy certain weak*-continuity. This is successfully done by Johnson, Kadison and Ringrose for von Neumann algebras [9]. Due to some important contribution of Connes, Helemskii coined the term Connes-amenability for this concept [5]. Later V. Runde extended this notion to the setting of dual Banach algebras [20] (see also [19] and [21]).

Suppose that $\mathcal{A}, \mathcal{B}$ are Banach algebras and $\theta \in \triangle(\mathcal{B})$, where $\triangle(\mathcal{B})$ is the set of all non-zero characters on $\mathcal{B}$. The θ-Lau product of $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \times_\theta \mathcal{B}$ and is defined as the space $\mathcal{A} \times \mathcal{B}$ with the multiplication

$$(a, b) \times_\theta (c, d) = (ac + \theta(b)c + \theta(d)a, bd), \tag{1.1}$$

for all $a, c \in \mathcal{A}$ and $b, d \in \mathcal{B}$. The algebra $\mathcal{A} \times_\theta \mathcal{B}$ with the norm $\|(a, b)\| = \|a\| + \|b\|$ and the above multiplication is a Banach algebra that is called Lau product algebra. In fact, this product was introduced by Lau in [10] for a certain class of Banach algebras and by Monfared [22] for the general case. This product not only induces some new examples of Banach algebras which are interesting in their own but also they are known as a fertile source of (counter) examples in functional analysis and abstract harmonic analysis. A very familiar example, which is of special interest, is the case in which $\mathcal{B} = \mathbb{C}$ and θ is the identity character i on $\mathcal{B}$. In this case, we get the unitization $\mathcal{A}^\sharp = \mathcal{A} \times_i \mathbb{C}$ of $\mathcal{A}$. Monfared [22] showed that $\mathcal{A} \times_\theta \mathcal{B}$ is amenable if and only if $\mathcal{A}, \mathcal{B}$ are amenable and moreover he proved that if $\mathcal{A}, \mathcal{B}$ are weakly amenable, then $\mathcal{A} \times_\theta \mathcal{B}$ is weakly amenable but if $\mathcal{A} \times_\theta \mathcal{B}$ is weakly amenable, then $\mathcal{B}$ is weakly amenable and $\mathcal{A}$ is cyclic amenable. The Connes-amenability of $\mathcal{A} \times_\theta \mathcal{B}$ was investigated by Razi and Pourabbas in [18]. They showed that if $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras, then $\mathcal{A} \times_\theta \mathcal{B}$ is a dual Banach algebra and vice versa. Furthermore, they proved that Connes-amenability of $\mathcal{A} \times_\theta \mathcal{B}$ implies Connes-amenability of both Banach algebras $\mathcal{A}$ and $\mathcal{B}$.

Ideal Connes-amenability for dual Banach algebras was studied for the first time in [16]. Among other things, the authors in [16] showed that von Neumann algebras are always ideally Connes-amenable. In addition, for a locally compact group G, the Fourier–Stieltjes algebra of G is ideally Connes-amenable, but not ideally amenable. Recently, the first author investigated the ideal Connes-amenability of discrete Beurling algebras and l^1-Munn algebras in [14] and [15], respectively.

In this paper, we are mainly concerned with the investigation of ideal Connes-amenability of $\mathcal{A} \times_\theta \mathcal{B}$ and show that if $\mathcal{A} \times_\theta \mathcal{B}$ is ideally Connes-amenable, then $\mathcal{A}, \mathcal{B}$ are ideally Connes-amenable. Finally, we present some examples regarding to the ideal Connes-amenability of Lau product algebras.

2 Ideal Connes-amenability

We start this section by recalling some standard notions which are available in [1, 13]. Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \longrightarrow X$ is called a *derivation* if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathcal{A}).$$

The space of all derivations of $\mathcal{A}$ into X is denoted by $\mathcal{Z}^1(\mathcal{A}, X)$. For each $x \in X$, the map $a \mapsto \delta_x(a) := a \cdot x - x \cdot a$ is a derivation, and these maps form the space $\mathcal{B}^1(\mathcal{A}, X)$ of *inner* derivations. The quotient space $\mathcal{H}^1(\mathcal{A}, X) = \mathcal{Z}^1(\mathcal{A}, X)/\mathcal{B}^1(\mathcal{A}, X)$ is the first *cohomology group* of $\mathcal{A}$ with coefficients in X.

Let X be a $\mathcal{A}$-bimodule. Then, the dual space X^* of X is also a Banach $\mathcal{A}$-bimodule by the following module actions:

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle, \qquad \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle, \qquad (a \in \mathcal{A},\ x \in X,\ f \in X^*).$$

In this case, X^* is said to be the *dual Banach $\mathcal{A}$-bimodule*. With the above notations, a Banach algebra A is called *amenable* if $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule X. Moreover, $\mathcal{A}$ is called *weakly amenable* if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. For $n \in \mathbb{N}$, $\mathcal{A}$ is called *n-weakly amenable* if $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$, where $\mathcal{A}^{(n)}$ is n-th dual of $\mathcal{A}$. Furthermore, $\mathcal{A}$ is said to be *n-ideally amenable* if $H^1(\mathcal{A}, \mathcal{I}^{(n)}) = \{0\}$ for every closed two sided ideal $\mathcal{I}$ in $\mathcal{A}$ [4].

A Banach algebra $\mathcal{A}$ is said to be dual if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = (\mathcal{A}_*)^*$. One can see that a Banach algebra which is also a dual space is a dual Banach algebra if and only if the multiplication map is separately w^*-continuous [20]. Examples of dual Banach algebras include all Von Neumann algebras, the algebra $B(E) = (E\hat{\otimes}E^*)^*$ of all bounded operators on a reflexive Banach space E, the measure algebra $M(G) = C_0(G)^*$, the Fourier-Stieljes algebra $B(G) = C^*(G)^*$, and the second dual B^{**} of an Arens regular Banach algebra B.

Let $\mathcal{A}$ be a Banach algebra. A dual Banach $\mathcal{A}$-bimodule X is called *normal* if for each $x \in X$, the maps $a \mapsto a \cdot x$ and $b \mapsto x \cdot b$ from $\mathcal{A}$ into X are w^*-continuous, and $\mathcal{A}$ is said to be *Connes-amenable* if for every normal dual Banach $\mathcal{A}$-bimodule X, every w^*-continuous derivation $D : \mathcal{A} \longrightarrow X$ is inner [20]. We denote $\mathcal{Z}^1_{w^*}(\mathcal{A}, X^*)$ and $\mathcal{B}^1_{w^*}(\mathcal{A}, X^*)$ for the w^*-continuous derivations and inner w^*-continuous derivations from $\mathcal{A}$ into X^*, respectively and $\mathcal{H}^1_{w^*}(\mathcal{A}, X^*) = \mathcal{Z}^1_{w^*}(\mathcal{A}, X^*)/\mathcal{B}^1(\mathcal{A}, X^*)$.

Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{I}$ be a *weak**-closed two-sided ideal of $\mathcal{A}$ (such ideals are also dual Banach algebra by [16, Lemma 2.1]). A dual Banach algebra $\mathcal{A}$ is $\mathcal{I}$-Connes-amenable if $H^1_{w^*}(\mathcal{A},\mathcal{I}) = \{0\}$ and $\mathcal{A}$ is *ideally Connes-amenable* if it is $\mathcal{I}$-Connes-amenable for every *weak**-closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$ [16].

It is shown in [16, Proposition 2.3] that every ideally Connes-amenable dual Banach algebra is unital. Hence, we note that Proposition 2.6 of [16] [the ideal Connes-amenability of a dual Banach algebra $\mathcal{A}$ and $\mathcal{A}^\#$ (the unitization of $\mathcal{A}$) are equivalent] is a trivial result.

Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras and $\mathcal{I}$, $\mathcal{J}$ be w^*-closed two-sided ideals in $\mathcal{A}$ and $\mathcal{B}$ respectively. Given $\theta \in \triangle(\mathcal{B})$. If $\mathcal{J} \subset \ker\theta$, then $\mathcal{I} \times_\theta \mathcal{J}$ is a w^*-closed two-sided ideal of $\mathcal{A} \times_\theta \mathcal{B}$, where $\ker\theta$ is the kernel of θ. In particular, $\mathcal{A} \times_\theta \mathcal{J}$ is a w^*-closed two-sided ideal of $\mathcal{A} \times_\theta \mathcal{B}$; see [22, Proposition 2.6].

The upcoming lemma is a tool to achieve our aim in this paper, shows that the ideal Connes-amenability of $\mathcal{A} \times_\theta \mathcal{B}$ implies the ideal Connes-amenability of both dual Banach algebras $\mathcal{A}$ and $\mathcal{B}$.

Lemma 2.1. *Let $\mathcal{I}$ and $\mathcal{J}$ be w^*-closed two-sided ideals in dual Banach algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Given $\theta \in \triangle(\mathcal{B})$ and $\mathcal{J} \subset \ker\theta$. A mapping $D : \mathcal{A} \times_\theta \mathcal{B} \longrightarrow \mathcal{I} \times_\theta \mathcal{J}$ is a w^*-continuous derivation if and only if $D(a,b) = (D_\mathcal{A}(a) + T_\mathcal{A}(b), D_\mathcal{B}(b) + T_\mathcal{B}(a))$ for all $a \in \mathcal{A}, b \in \mathcal{B}$, where*

1. *$D_\mathcal{A} : \mathcal{A} \longrightarrow \mathcal{I}$ and $D_\mathcal{B} : \mathcal{B} \longrightarrow \mathcal{J}$ are w^*-continuous derivations;*
2. *$T_\mathcal{A} : \mathcal{B} \longrightarrow \mathcal{I}$ is a bounded linear operator such that $aT_\mathcal{A}(b) = T_\mathcal{A}(b)c = 0$ and*
$$T_\mathcal{A}(bd) = \theta(b)T_\mathcal{A}(d) + \theta(d)T_\mathcal{A}(b), \tag{2.1}$$
for all $a, c \in \mathcal{A}, b, d \in \mathcal{B}$;
3. *$T_\mathcal{B} : \mathcal{A} \longrightarrow \mathcal{J}$ is a bounded linear operator such that $T_\mathcal{B}(ac) = 0$, and $\theta(b)T_\mathcal{B}(c) = bT_\mathcal{B}(c)$ for all $a, c \in \mathcal{A}$ and $b \in \mathcal{B}$.*

Moreover, $D = \delta_{(i,j)}$ for some $i \in \mathcal{I}$, $j \in \mathcal{J}$ if and only if $D_\mathcal{B} = \delta_j$, $D_\mathcal{A} = \delta_i$, $T_\mathcal{B} = 0$ and $T_\mathcal{A} = 0$.

Proof. A straightforward verification shows that $D : \mathcal{A} \times_\theta \mathcal{B} \longrightarrow \mathcal{I} \times_\theta \mathcal{J}$ is a bounded linear operator if and only if there exist bounded linear mappings $D_\mathcal{A} : \mathcal{A} \longrightarrow \mathcal{I}$, $D_\mathcal{B} : \mathcal{B} \longrightarrow \mathcal{J}$, $T_\mathcal{A} : \mathcal{B} \longrightarrow \mathcal{I}$, $T_\mathcal{B} : \mathcal{A} \longrightarrow \mathcal{J}$ such that
$$D(a,b) = (D_\mathcal{A}(a) + T_\mathcal{A}(b), D_\mathcal{B}(b) + T_\mathcal{B}(a)), \tag{2.2}$$
for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. In addition, D is a derivation if and only if
$$D((a,b) \times_\theta (c,d)) = D(a,b) \times_\theta (c,d) + (a,b) \times_\theta D(c,d) \tag{2.3}$$
for all $a, c \in \mathcal{A}$ and $b, d \in \mathcal{B}$ (here and the rest of the proof). By the definition of D and (1.1), relation (2.3) holds if and only if
$$\begin{aligned} T_\mathcal{A}(bd) + D_\mathcal{A}(ac) + \theta(b)D_\mathcal{A}(c) &= T_\mathcal{A}(b)c + D_\mathcal{A}(a)c + \theta(d)T_\mathcal{A}(b) + aT_\mathcal{A}(d) \\ &\quad + aD_\mathcal{A}(c) + \theta(b)T_\mathcal{A}(d) + \theta(b)D_\mathcal{A}(c), \end{aligned} \tag{2.4}$$
and
$$D_\mathcal{B}(bd) + T_\mathcal{B}(ac) + \theta(b)T_\mathcal{B}(c) + \theta(d)T_\mathcal{B}(a) = bD_\mathcal{B}(d) + bT_\mathcal{B}(c) + D_\mathcal{B}(b)d + T_\mathcal{B}(a)d. \tag{2.5}$$
Putting $b = d = 0$ in (2.4), we have
$$D_\mathcal{A}(ac) = aD_\mathcal{A}(c) + D_\mathcal{A}(a)c.$$

The above equality shows that $D_{\mathcal{A}}$ is a derivation on $\mathcal{A}$. Once more, by putting $b = d = 0$ in (2.5), we find $T_{\mathcal{B}}(ac) = 0$ which is a part of assertion (3). Setting $a = c = 0$ in (2.4) and (2.5), we get

$$T_{\mathcal{A}}(bd) = \theta(b)T_{\mathcal{A}}(d) + \theta(d)T_{\mathcal{A}}(b)$$

and

$$D_{\mathcal{B}}(bd) = bD_{\mathcal{B}}(d) + D_{\mathcal{B}}(b)d.$$

It follows from the relations above that (2.1) is valid and, moreover, $D_{\mathcal{B}}$ is a derivation. Replacing (a, d) by $(0, 0)$ in (2.4), we obtain $T_{\mathcal{A}}(b)c = 0$. On the other hand, by letting $a = d = 0$ in (2.5), we obtain

$$\theta(b)T_{\mathcal{B}}(c) = bT_{\mathcal{B}}(c).$$

Again, by putting $b = c = 0$ in (2.4), we find $aT_{\mathcal{A}}(d) = 0$. Consequently, we conclude that the above statements hold if and only if conditions (1), (2) and (3) are satisfied.

We now let $D = \delta_{(i,j)}$ for some $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Then, by (1.1)

$$D(a, d) = (a, b) \times_\theta (i, j) - (i, j) \times_\theta (a, b) = (ai - ia, bj - jb). \tag{2.6}$$

Setting $b = 0$ in (2.6) and in (2.2), we arrive

$$(D_{\mathcal{A}}(a), T_{\mathcal{B}}(a)) = (ai - ia, 0).$$

Thus, $D_{\mathcal{A}}(a) = \delta_i(a)$, and $T_{\mathcal{B}}(a) = 0$ for all $a \in \mathcal{A}$. Similarly, one can obtain $T_{\mathcal{A}}(b) = 0$, and hence $D_{\mathcal{B}}(b) = \delta_j(b)$.

Conversely, assume that $D_{\mathcal{A}} = \delta_i$, $D_{\mathcal{B}} = \delta_j$, $T_{\mathcal{B}} = 0$ and $T_{\mathcal{A}} = 0$. Then, by (1.1) and (2.2)

$$D(a, b) = (ai - ia, bj - jb) = (a, b) \times_\theta (i, j) - (i, j) \times_\theta (a, b)$$

for all $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$. This means that $D = \delta_{(i,j)}$, which completes the proof. □

A direct application of Lemma 2.1 gives the following result.

Theorem 2.1. *If $\mathcal{A} \times_\theta \mathcal{B}$ is ideally Connes-amenable, then so are $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{I}$ be a w^*-continuous derivation, where $\mathcal{I}$ is an arbitrary w^*-closed two-sided ideal of $\mathcal{A}$. Define $\overline{D} : \mathcal{A} \times_\theta \mathcal{B} \longrightarrow \mathcal{I} \times_\theta \mathcal{J}$ via $\overline{D}(a, b) := (D(a), 0)$, where $\mathcal{J}$ is a w^*-closed two-sided ideal in $\mathcal{B}$, such that $\mathcal{J} \subset \ker\theta$, for example $\mathcal{J} = \{0\}$. Lemma 2.1 implies that $\overline{D}$ is a w^*-continuous derivation and so it is inner. Thus, D is inner and hence $\mathcal{A}$ is ideally Connes-amenable.

For ideal Connes-amenability of $\mathcal{B}$, let $D : \mathcal{B} \longrightarrow \mathcal{J}$ be a w^*-continuous derivation, where $\mathcal{J}$ is an arbitrary w^*-closed two-sided ideal of $\mathcal{B}$. Define $\overline{D} : \mathcal{A} \times_\theta \mathcal{B} \longrightarrow \mathcal{A} \times_\theta \mathcal{J}$ through $\overline{D}(a, b) := (0, D(b))$. Applying again Lemma 2.1, we find out that $\overline{D}$ is a w^*-continuous derivation and therefore it is inner. This shows that $\mathcal{B}$ is ideally Connes-amenable. □

It is known from [16, Proposition 2.3] that every ideally Connes-amenable dual Banach algebra $\mathcal{A}$ is unital and automatically $\mathcal{A}^2$ is dense in $\mathcal{A}$. Here we recall from Lemma 2.1 of [16] that if $\mathcal{I}$ is a weak*-closed two-sided ideal of a dual Banach algebra $\mathcal{A} = (\mathcal{A}_*)^*$, then it is a dual Banach algebra, where $\mathcal{A}_*$ is the predual of $\mathcal{A}$. Therefore, $\mathcal{I}$ has a predual $\mathcal{I}_*$ and so we can write $\mathcal{I} = (\mathcal{I}_*)^*$. We use this fact and present the next proposition that shows under some mild conditions, for a dual Banach algebra $\mathcal{A}$, the set $\mathcal{A}^2$ is dense in $\mathcal{A}$.

Proposition 2.1. *Let $\mathcal{I} = (\mathcal{I}_*)^*$ be a w^*-closed two-sided ideals in $\mathcal{A}$ such that $\overline{\mathcal{I}_*^2} \neq \mathcal{I}_*$. If every w^*-continuous operator $D : \mathcal{A} \longrightarrow \mathcal{I}$ satisfies*

$$D(ac) = D(a)c = cD(a) = 0 \tag{2.7}$$

for all $a, c \in \mathcal{A}$, then D is inner. In particular, $D = 0$ if and only if $\mathcal{A}^2$ is dense in $\mathcal{A}$.

Proof. Clearly, our assumption implies that D is inner. If $\mathcal{A}^2$ is dense in $\mathcal{A}$, then for $a \in \mathcal{A}$, there is a net $\{a_\alpha b_\alpha\}$ such that $a_\alpha b_\beta \to a$. Thus, $D(a) = \lim_\alpha \lim_\beta D(a_\alpha b_\alpha) = 0$.

Conversely, suppose contrary to our claim, that $\mathcal{A}^2$ is not dense in $\mathcal{A}$. Take non-zero $f \in \mathcal{A}^*$ such that $f|_{\mathcal{A}^2} = 0$. Given $0 \neq \lambda \in \mathcal{I}$ such that $\lambda|_{\mathcal{I}_*^2} = 0$. Then, $D : \mathcal{A} \longrightarrow \mathcal{I} = (\mathcal{I}_*)^*$ defined by $D(a) = f(a)\lambda$ is a w^*-continuous derivation that satisfies (2.7). For each $x \in \mathcal{I}$, we have $f(a)\lambda(x) = D(ax) = 0$ for all $a \in \mathcal{A}$. This implies that $f = 0$ and leads us to a contradiction. □

Let $\mathcal{A}$ be a unital dual Banach algebra with identity $e_\mathcal{A}$. Suppose the bounded linear mappings $D_\mathcal{A} : \mathcal{A} \longrightarrow \mathcal{I}$, $T_\mathcal{B} : \mathcal{B} \longrightarrow \mathcal{J}$, $T_\mathcal{A} : \mathcal{B} \longrightarrow \mathcal{I}$, $T_\mathcal{B} : \mathcal{A} \longrightarrow \mathcal{J}$ are as in Lemma 2.1. Since $\mathcal{A}$ is unital, by the definition of $T_\mathcal{B}$, we conclude that $T_\mathcal{B} = 0$. Thus, every derivation $D : \mathcal{A} \times_\theta \mathcal{B} \longrightarrow \mathcal{I} \times_\theta \mathcal{J}$ is as the form

$$D(a, b) = (D_\mathcal{A}(a) + T_\mathcal{A}(b), D_\mathcal{B}(b)),$$

where $T_\mathcal{A}(b) = -e_\mathcal{A}\theta(D_\mathcal{B}(b))$. Let $\mathcal{J} \subset \ker\theta$. Then, $D_\mathcal{A} : \mathcal{A} \longrightarrow \mathcal{I}$ and $D_\mathcal{B} : \mathcal{B} \longrightarrow \mathcal{J}$ are w^*-continuous derivations. If $D_\mathcal{B} = \delta_j$ for some $j \in \mathcal{J}$, then

$$T_\mathcal{A}(b) = -e_\mathcal{A}\theta(D_\mathcal{B}(b)) = 0.$$

Hence, $T_\mathcal{A} = 0$. It now follows from Lemma 2.1 that $D = \delta_{(i,j)}$ if and only if $D_\mathcal{A} = \delta_i$ and $D_\mathcal{B} = \delta_j$ for some $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Summing up, we get

Theorem 2.2. *Let $\mathcal{A}$ be a unital dual Banach algebra, $\mathcal{B}$ be a dual Banach algebra and $\theta \in \triangle(\mathcal{B})$. Suppose that $\mathcal{I}$ is an arbitrary w^*-closed two-sided ideal of $\mathcal{A}$ and $\mathcal{J} \subset \ker\theta$ is a w^*-closed two sided ideal of $\mathcal{B}$. Then, $H^1_{w^*}(\mathcal{A} \times_\theta \mathcal{B}, \mathcal{I} \times_\theta \mathcal{J}) = H^1_{w^*}(\mathcal{A}, \mathcal{I}) \bigoplus H^1_{w^*}(\mathcal{B}, \mathcal{J})$.*

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-weakly amenable, commutative dual Banach algebras. Then, $\mathcal{A} \times_\theta \mathcal{B}$ is ideally Connes-amenable.*

Proof. We firstly note that by hypothese, $\mathcal{A}$ is 2-ideally amenable. In addition, it follows from [16, Theorem 2.11] that $\mathcal{A}$ is ideally Connes-amenable, and so Proposition 2.3 from [16] implies that $\mathcal{A}$ is unital. Since $\mathcal{A}$ and $\mathcal{B}$ are 2-weakly amenable, $\mathcal{A} \times_\theta \mathcal{B}$ is 2-weakly amenable by [17, Corollary 4.11]. On the other hand, $\mathcal{A} \times_\theta \mathcal{B}$ is commutative and hence it is 2-ideally amenable by [16, Theorem 2.11]. Thus, we conclude that $\mathcal{A} \times_\theta \mathcal{B}$ is ideally Connes-amenable. □

Corollary 2.1. *Let $\mathcal{A}$ be a commutative dual Banach algebra and $H^1_{w^*}(\mathcal{A}, \mathcal{A}) = \{0\}$. Then, $\mathcal{A} \times_\theta \mathcal{A}$ is ideally Connes-amenable.*

Proof. The result follows from Theorem 2.3 and [16, Proposition 2.9]. □

Example. Let $(\mathbb{Z}, +)$ be the group of integers. Define $\omega_\alpha : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{R}^+$ by $\omega_\alpha(k, m) = (1+ \mid k \mid)^\alpha(1+ \mid k + m \mid)^\alpha$, where $0 < \alpha < 1/2$. By ([16, Example 2.18], $l^1(\mathbb{Z} \times \mathbb{Z}, \omega_\alpha)$ is 2-ideally amenable. For $0 < \alpha, \beta < 1/2$, set $\mathcal{A} := l^1(\mathbb{Z} \times \mathbb{Z}, \omega_\alpha)$ and $\mathcal{B} := l^1(\mathbb{Z} \times \mathbb{Z}, \omega_\beta)$. It now follows from Theorem 2.3 that $\mathcal{A} \times_\theta \mathcal{B}$ is ideally Connes-amenable.

Note that for two Banach algebras $\mathcal{A}$ and $\mathcal{B}$, we have

$$\mathcal{A}^{\star\star} \times_\theta \mathcal{B}^{\star\star} \cong (\mathcal{A} \times_\theta \mathcal{B})^{\star\star}.$$

As a result of [16, Corollary 2.16], ideal Connes-amenability of $\mathcal{A}^{\star\star} \times_\theta \mathcal{B}^{\star\star}$, implies that of $\mathcal{A} \times_\theta \mathcal{B}$.

Let S be a non-empty set. The set of all mappings from S to the complex numbers set $\mathbb{C}$ is denoted by $\mathbb{C}^S$. Consider

$$l^1(S) = \left\{ f \in \mathbb{C}^S : \sum_{s \in S} |f(s)| < \infty \right\},$$

with the norm $\|\cdot\|_1$ given by $\|f\|_1 = \sum_{s\in S} |f(s)|$ for $f \in l^1(S)$. Now suppose that S is a semigroup. For $f, g \in l^1(S)$, we set

$$(f \star g)(t) = \left\{\sum f(r)g(s) : r, s \in S,\, rs = t\right\} \quad (t \in S)$$

so that $f \star g \in l^1(S)$. It is standard that $(l^1(S), \star)$ is a Banach algebra, called the *semigroup algebra on* S. Clearly, $l^1(S)$ is commutative if and only if S is abelian.

Let G be a group, and let $m, n \in \mathbb{N}$, the zero adjoined to G is 0. That is $x0 = 0x = 0$ for $x \in G$. A *Rees semigroup* has the form $S = \mathcal{M}(G, P, m, n)$, where $P = (a_{ij}) \in \mathbb{M}_{n,m}(G)$, the collection of $n \times m$ matrices with components in G. For $x \in G$, $i \in \mathbb{N}_m$, and $j \in \mathbb{N}_n$, let $(x)_{ij}$ be the element of $\mathbb{M}_{m,n}(G^0)$ with x in the $(i, j)^{\text{th.}}$ place and 0 elsewhere. As a set, S consists of the collection of all these matrices $(x)_{ij}$. Multiplication in S is given by the formula

$$(x)_{ij}(y)_{k\ell} = (xa_{jk}y)_{i\ell} \quad (x, y \in G,\ i, k \in \mathbb{N}_m,\ j, \ell \in \mathbb{N}_n).$$

It is shown in [7, Lemma 3.2.2] that S is a semigroup. Similarly, we have the semigroup $\mathcal{M}^0(G, P, m, n)$, where the elements of this semigroup are those of $\mathcal{M}(G, P, m, n)$, together with the element 0, identified with the matrix that has 0 in each place and the components of P are now allowed to belong to G^0. The matrix P is called the *sandwich matrix* in each case. The semigroup $\mathcal{M}^0(G, P, m, n)$ is a *Rees matrix semigroup with a zero over* G. We write $\mathcal{M}^0(G, P, n)$ for $\mathcal{M}^0(G, P, n, n)$ in the case where $m = n$. For details on Rees semigroups, see [7, Section 3.2] and [2, Chapter 3].

We also recall that a Brandt semigroup S over a group G with index set J is the semigroup consisting of elementary $J \times J$ matrices over $G \cup \{0\}$ and a zero matrix $\mathbf{0}$. We write $S = \{(g)_{ij} : g \in G, i, j \in J\} \cup \{\mathbf{0}\}$, with multiplication given by

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ \mathbf{0} & \text{if } j \neq k. \end{cases}$$

Theorem 2.4. *Let $S = \mathcal{M}^0(G, P, n)$ be a Rees matrix semigroup with a zero over the locally compact group G and Sandwich matrix P. If $l^1(S)$ is commutative, then $l^1(S) \times_\theta l^1(S)$ is ideally Connes-amenable.*

Proof. By [6, Theorem 3.1], $l^1(S)$ is 2-weakly amenable. Since $l^1(S)$ is commutative, it is 2-ideally amenable. Thus, Theorem 2.11 from [16] implies that $l^1(S)$ is ideally Connes-amenable and hence it is unital. Now, Theorem 2.3 shows that $l^1(S) \times_\theta l^1(S)$ is ideally Connes-amenable. □

Corollary 2.2. *Let S be a Brandt semigroup over a group G with index set J. If $l^1(S)$ is commutative, then $l^1(S) \times_\theta l^1(S)$ is ideally Connes-amenable.*

Proof. This follows by Theorem 2.4 and this fact that $S \cong \mathcal{M}^0(G, P, n)$, for some group G; see [7]. □

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