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TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 75th birthday)

Tynysbek Sharipovich Kal'menov was born in the village of Koksaek of the Tolebi district of the Turkestan region (earlier it was the Lenger district of the South-Kazakhstan region of the Kazakh SSR). Although "according to the passport" his birthday was recorded on May 5, his real date of birth is April 6, 1946.

Tynysbek Kal'menov is a graduate of the Novosibirsk State University (1969), and a representative of the school of A.V. Bitsadze, an outstanding scientist, corresponding member of the Academy of Sciences of the USSR. In 1972, he completed his postgraduate studies at the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR. In 1983, he defended his doctoral thesis at the M.V. Lomonosov Moscow State University. Since1989, he is a corresponding member of the Academy of Sciences of the Kazakh SSR, and since 2003, he is an academician of the National Academy of Sciences of the Republic of Kazakhstan.

Tynysbek Kal'menov worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (1972-1985). From 1986 to 1991, he was the dean of the Faculty of Mathematics of Al-Farabi Kazakh State University. From 1991 to 1997, he was the rector of the Kazakh Chemical-Technological University (Shymkent).

From 2004 to 2019, Tynysbek Kal'menov was the General Director of the Institute of Mathematics and Mathematical Modeling. He made it one of the leading scientific centers in the country and the best research institute in Kazakhstan. It suffices to say, that in terms of the number of scientific publications (2015-2018) in international rating journals indexed in the Web of Science, the Institute of Mathematics and Mathematical Modeling was ranked fourth among all Kazakhstani organizations, behind only three large universities: the Nazarbaev University, Al-Farabi National University and L.N. Gumilyov Eurasian National University.

Since 2019, Tynysbek Kal'menov has been working as the head of the Department of Differential Equations of the Institute of Mathematics and Mathematical Modeling. He is a member of the National Scientific Council "Scientific Research in the Field of Natural Sciences", which is the main Kazakhstan council that determines the development of science in the country.

T.Sh. Kal'menov was repeatedly elected to maslikhats of various levels, was a member of the Presidium of the Committee for Supervision and Attestation in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan. He is a Laureate of Lenin Komsomol Prize of the Kazakh SSR (1978), an Honored Worker of Science and Technology of Kazakhstan (1996), awarded with the order "Kurmet" (2008 Pi.) and jubilee medals.

In 2013, he was awarded the State Prize of the Republic of Kazakhstan in the field of science and technology for the series of works "To the theory of initial- boundary value problems for differential equations".

The main areas of scientific interests of academician Tynysbek Kal'menov are differential equations, mathematical physics and operator theory. He has obtained fundamental scientific results, many of which led to the creation of new scientific directions in mathematics.

Tynysbek Kal'menov, using a new maximum principle for an equation of mixed type (Kal'menov's maximum principle), was the first to prove that the Tricomi problem has an eigenfunction, thus he solved the famous problem of the Italian mathematician Francesco Tricomi, set in 1923 This marked the beginning of a new promising direction, that is, the spectral theory of equations of mixed type.

He established necessary and sufficient conditions for the well-posed solvability of the classical Darboux and Goursat problems for strongly degenerate hyperbolic equations.

Tynysbek Kal'menov solved the problem of completeness of the system of root functions of the nonlocal Bitsadze-Samarskii problem for a wide class of multidimensional elliptic equations. This result is final and has been widely recognized by the entire mathematical community.

He developed a new effective method for studying ill-posed problems using spectral expansion of differential operators with deviating argument. On the basis of this method, he found necessary and sufficient conditions for the solvability of the mixed Cauchy problem for the Laplace equation.

Tynysbek Kal'menov was the first to construct boundary conditions of the classical Newton potential. That is a fundamental result at the level of a classical one. Prior to the research of Kal'menov T.Sh., it was believed that the Newton potential gives only a particular solution of an inhomogeneous equation and does not satisfy any boundary conditions. Thanks for these results, for the first time, it was possible to construct the spectral theory of the classical Newton potential.

He developed a new effective method for constructing Green's function for a wide class of boundary value problems. Using this method, Green's function of the Dirichlet problem was first constructed explicitly for a multidimensional polyharmonic equation.

From 1989 to 1993, Tynysbek Kal'menov was the chairman of the Inter- Republican (Kazakhstan, Uzbekistan, Kyrgyzstan, Turkmenistan, Tajikistan) Dissertation Council. He is a member of the International Mathematical Society and he repeatedly has been a member of organizing committee of many international conferences. He carries out a lot of organizational work in training of highly qualified personnel for the Republic of Kazakhstan and preparing international conferences. Under his direct guidance, the First Congress of Mathematicians of Kazakhstan was held. He presented his reports in Germany, Poland, Great Britain, Sweden, France, Spain, Japan, Turkey, China, Iran, India, Malaysia, Australia, Portugal and countries of CIS.

In terms of the number of articles in scientific journals with the impact- factor Web of Science, in the research direction of "Mathematics", the Institute of Mathematics and Mathematical Modeling is on one row with leading mathematical institutes of the Russian Federation, and is ahead of all mathematical institutes in other CIS countries in this indicator.

Tynysbek Kal'menov is one of the few scientists who managed to leave an imprint of their individuality almost in all branches of mathematics in which he has been engaged.

Tynysbek Kal'menov has trained 11 doctors and more than 60 candidate of sciences and PhD, has founded a large scientific school on equations of mixed type and differential operators recognized all over the world. Many of his disciples are now independent scientists recognized in the world of mathematics.

He has published over 150 scientific articles, most of which are published in international mathematical journals, including 14 articles published in "Doklady AN SSSR/ Doklady Mathematics". In the last 5 years alone (2016-2020), he has published more than 30 articles in scientific journals indexed in the Web of Science database. To date, academician Tynysbek Kal'menov has a Hirsch index of 18 in the Web of Science and Scopus databases, which is the highest indicator among all Kazakhstan mathematicians.

Outstanding personal qualities of academician Tynysbek Kalmenov, his high professional level, adherence to principles of purity of science, high exactingness towards himself and his colleagues, all these are the foundations of the enormous authority that he has among Kazakhstan scientists and mathematicians of many countries.

Academician Tynysbek Sharipovich Kalmenov meets his 75th birthday in the prime of his life, and the mathematical community, many of his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal heartily congratulate him on his jubilee and wish him good health, happiness and new successes in mathematics and mathematical education, family well-being and long years of fruitful life.

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POISSON – JENSEN FORMULAS AND BALAYAGE OF MEASURES

B.N. Khabibullin

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Key words: subharmonic function, potential, Riesz measure, Green's function, harmonic measure, Poisson – Jensen formula, balayage of measures, Jensen measure.

AMS Mathematics Subject Classification: 31B05, 31A05, 31B15, 31A15, 26A51.

Abstract. Our main results are certain developments of the classical Poisson – Jensen formula for subharmonic functions. The basis of the classical Poisson – Jensen formula is the natural duality between harmonic measures and Green's functions. Our generalizations use some duality between the balayage of measures and their potentials.

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1 Introduction

1.1 On the classical Poisson-Jensen formula

Let D be a *bounded domain* in the d-dimensional Euclidean space $\mathbb{R}^d$ with the *closure* $\operatorname{clos} D$ in $\mathbb{R}^d$ and the *boundary* ∂D in $\mathbb{R}^d$. Then, for any $x \in D$ there are the *extended harmonic measure* $\omega_D(x,\cdot)$ *for* D *at* $x \in D$, which is a Borel probability measure on $\mathbb{R}^d$ with support on ∂D, and the *generalized Green's function* $g_D(\cdot,x)$ *for* D *with pole at* x extended by zero values on the complement $\mathbb{R}^d \setminus \operatorname{clos} D$ and by the upper semicontinuous regularization on ∂D from D [22], [2], [50], [23], [14], [46] (see also (2.7) and (2.9) in Subsection 2.3 below).

Let $u \not\equiv -\infty$ be a *subharmonic function on* $\operatorname{clos} D$, i.e., on an open set containing $\operatorname{clos} D$, with its *Riesz measure* $\varDelta_u$ on this open set (see for details Subsections 1.2 and formula (1.6)).

Classical Poisson – Jensen formula ([22, Theorem 5.27], [50, Section 4.5])**.**

$$u(x) = \int_{\partial D} u \,\mathrm{d}\omega_D(x,\cdot) - \int_{\operatorname{clos} D} g_D(\cdot,x)\,\mathrm{d}\varDelta_u \quad \text{for each } x \in D. \tag{1.1}$$

For $s \in \mathbb{R}$, we set

$$k_s(t) := \begin{cases} \ln t & \text{if } s = 0,\\ -\frac{s}{|s|}t^{-s} & \text{if } s \in \mathbb{R}\setminus 0,\end{cases} \qquad t \in \mathbb{R}^+ \setminus 0, \tag{1.2k}$$

$$K_{d-2}(y,x) := \begin{cases} k_{d-2}\bigl(|y-x|\bigr) & \text{if } y \neq x,\\ -\infty & \text{if } y = x \text{ and } d \geq 2,\\ 0 & \text{if } y = x \text{ and } d = 1,\end{cases} \quad (y,x) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{1.2K}$$

The following functions

$$p\colon y \underset{y\,\in\,\mathbb{R}^d}{\longmapsto} g_D(y,x) + K_{d-2}(y,x), \qquad q\colon y \underset{y\,\in\,\mathbb{R}^d}{\longmapsto} K_{d-2}(y,x) \tag{1.3}$$

are subharmonic with the Riesz probability measures $\varDelta_p = \omega_D(x,\cdot)$ and $\varDelta_q = \delta_x$, where δ_x is the Dirac measure at $x \in D$: $\delta_x\bigl(\{x\}\bigr) = 1$. The following symmetric equivalent form of classical Poisson – Jensen formula (1.1) immediately follows from the suitable definitions of harmonic measures and Green's functions and is briefly discussed in Subsection 2.3.

Symmetrization of the classical Poisson – Jensen formula . *If we choose p, q as in* (1.3) *and put $S = \operatorname{clos} D$, then* (1.1) *can be rewritten in the symmetric form*

$$\int_S u \,\mathrm{d}\varDelta_q + \int_S p \,\mathrm{d}\varDelta_u = \int_S u \,\mathrm{d}\varDelta_p + \int_S q \,\mathrm{d}\varDelta_u. \tag{1.4}$$

Equality (1.4) reflects the fact that the *Laplace operator* $\triangle$ is *self-adjoint* for some formal bilinear integral form $(u, \triangle w) := \int u \triangle w = \int \triangle u\, w = (\triangle u, w)$, where $w := q - p$. The following result is a special case of our Main Theorem from Subsection 2.2, but already significantly develops the classical Poisson – Jensen formula (1.3)–(1.4).

Theorem 1.1. *Let $S \subset \mathbb{R}^d$ be a non-empty compact set, and $p \not\equiv -\infty$, and $q \not\equiv -\infty$ be a pair of subharmonic functions on S with the Riesz measures $\varDelta_p$ and $\varDelta_q$, respectively. If p and q are harmonic outside S and $p = q$ outside S, then the symmetric Poisson – Jensen formula* (1.4) *holds for each subharmonic function $u \not\equiv -\infty$ on S with the Riesz measure $\varDelta_u$.*

In the case in which p, q, and u are smooth, Theorem 1.1 is an easy corollary of Green's identity, but its proof for arbitrary subharmonic functions p, q and u requires careful justification. The Main Lemma and Main Theorem of our work establish exact relations between Poisson – Jensen formulas and the concept of balayage of measures and subharmonic functions.

Our Main Lemma is formulated in Subsection 2.1 and gives a symmetric Poisson – Jensen formula for measures and their potentials. The Main Lemma is proved in Section 3. The proof of the Main Lemma use Theorem 3.1 from Subsection 3.1 on representations for pairs of subharmonic functions. This result of Liouville type is also of independent interest.

The Main Theorem is formulated in Subsection 2.2 and gives a full symmetric Poisson – Jensen formula for subharmonic integrands. The proof of the Main Theorem in Section 4 essentially uses the Main Lemma. Theorem 1.1 is deduced from the Main Theorem in Subsection 2.2. The next Subsection 2.3 contains a discussion of classical symmetric Poisson – Jensen formula (1.3)–(1.4) as a corollary of Theorem 1.1.

Our Duality Theorems 1–3 in Section 5 give a complete description of potentials of measures obtained as a certain process of balayage of measures with compact supports. In order to prove Duality Theorems 1–3, we use both the Main Lemma and the Main Theorem. A significant number of references to the author's work in the bibliography are connected, in particular, with the fact that we indicate a list of works in which previous versions of duality theorems were applied to various problems of function theory.

We proceed to precise and detailed definitions and formulations.

1.2 Basic notation, definitions, and conventions

The reader can skip this Subsection 1.2 and return to it only if necessary.

Sets, topology, order

We denote by $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{R}$, and $\mathbb{R}^+ := \{x \in \mathbb{R} \colon x \geq 0\}$ the sets of *natural,* of *real,* and of *positive* numbers, each endowed with its natural order ($\leq$, $\sup / \inf$), algebraic, geometric and topological structure. We denote singleton sets by a symbol without curly brackets. So, $\mathbb{N}_0 :=$

$\{0\}\cup\mathbb{N} =: 0\cup\mathbb{N}$, and $\mathbb{R}^+\setminus 0 := \mathbb{R}^+\setminus\{0\}$ is the set of *strictly positive* numbers, etc. The *extended real line* $\overline{\mathbb{R}} := -\infty\sqcup\mathbb{R}\sqcup+\infty$ is the order completion of $\mathbb{R}$ by the *disjoint union* $\sqcup$ with $+\infty := \sup\mathbb{R}$ and $-\infty := \inf\mathbb{R}$ equipped with the order topology with two ends $\pm\infty$, $\overline{\mathbb{R}}^+ := \mathbb{R}^+\sqcup+\infty$; $\inf\varnothing := +\infty$, $\sup\varnothing := -\infty$ for the *empty set* $\varnothing$ etc. The same symbol 0 is also used, depending on the context, to denote zero vector, zero function, zero measure, etc.

We denote by $\mathbb{R}^d$ the *Euclidean space of* $d\in\mathbb{N}$ *dimensions* with the *Euclidean norm* $|x| := \sqrt{x_1^2+\cdots+x_d^2}$ of $x=(x_1,\dots,x_d)\in\mathbb{R}^d$, and we denote by $\mathbb{R}^d_\infty := \mathbb{R}^d\sqcup\infty$ the *Alexandroff* (Aleksandrov) *one-point compactification* of $\mathbb{R}^d$ obtained by adding one extra point ∞. For a subset $S\subset\mathbb{R}^d_\infty$ or a subset $S\subset\mathbb{R}^d$ we let $\complement S := \mathbb{R}^d_\infty\setminus S$, $\operatorname{clos} S$, $\operatorname{int} S := \complement(\operatorname{clos}\complement S)$, and $\partial S := \operatorname{clos} S\setminus\operatorname{int} S$ denote its *complement, closure, interior,* and *boundary* always in $\mathbb{R}^d_\infty$, and S is equipped with the topology induced from $\mathbb{R}^d_\infty$. If S' is a relatively compact subset in S, i.e., $\operatorname{clos} S'\subset S$, then we write $S'\Subset S$. We denote by $B(x,t) := \{y\in\mathbb{R}^d\colon |y-x|<t\}$, $\overline{B}(x,t) := \{y\in\mathbb{R}^d\colon |y-x|\leq t\}$, $\partial\overline{B}(x,t) := \overline{B}(x,t)\setminus B(x,t)$ an *open ball,* a *closed ball,* a *sphere of radius* $t\in\mathbb{R}^+$ *centered at* $x\in\mathbb{R}^d$, respectively.

Let T be a topological space, and S be a subset in T. We denote by $\operatorname{Conn}_T S$ or $\operatorname{Conn}_T(S)$ the set of all connected components of $S\subset T$ in T.

Throughout this paper $O\neq\varnothing$ will denote an *open subset in* $\mathbb{R}^d$, and $D\neq\varnothing$ is a *domain in* $\mathbb{R}^d$, i.e., an open connected subset in $\mathbb{R}^d$.

Measures and charges

The convex cone over $\mathbb{R}^+$ of all Borel, or Radon, positive measures $\mu\geq 0$ on the σ-algebra $\operatorname{Bor}(S)$ of all *Borel subsets* of S is denoted by $\operatorname{Meas}^+(S)$; $\operatorname{Meas}^+_{\text{cmp}}(S)\subset\operatorname{Meas}^+(S)$ is the subcone of $\mu\in\operatorname{Meas}^+(S)$ with compact *support* $\operatorname{supp}\mu$ in S, $\operatorname{Meas}(S) := \operatorname{Meas}^+(S)-\operatorname{Meas}^+(S)$ is the vector lattice over $\mathbb{R}$ of *charges,* or signed measures, on S, $\operatorname{Meas}^{+1}(S)$ is the convex set of *probability* measures on S, $\operatorname{Meas}^{1+}_{\text{cmp}}(S) := \operatorname{Meas}^{1+}(S)\cap\operatorname{Meas}_{\text{cmp}}(S)$, and $\operatorname{Meas}_{\text{cmp}}(S) := \operatorname{Meas}^+_{\text{cmp}}(S)-\operatorname{Meas}^+_{\text{cmp}}(S)$. For a charge $\mu\in\operatorname{Meas}(S)$, we let $\mu^+ := \sup\{0,\mu\}$, $\mu^- := (-\mu)^+$ and $\mu := \mu^++\mu^-$ respectively denote its *upper, lower,* and *total variations,* and $\mu(x,t) := \mu\bigl(\overline{B}(x,t)\bigr)$.

For an *extended numerical function* $f\colon S\to\overline{\mathbb{R}}$ we allow values $\pm\infty$ for Lebesgue integrals [22, Chapter 3, Definiftion 3.3.2] (see also [9])

$$\int_S f\,\mathrm{d}\mu\in\overline{\mathbb{R}},\quad \mu\in\operatorname{Meas}^+(S), \tag{1.5}$$

and we say that f is *μ-integrable on S* if the integral in (1.5) is finite.

Subharmonic functions

We denote by $\operatorname{sbh}(O)$ the convex cone over $\mathbb{R}^+$ of all *subharmonic* (locally convex if $d=1$) functions on O, including functions that are identically equal to $-\infty$ on some components $C\in\operatorname{Conn}_{\mathbb{R}^d_\infty}(O)$. Thus, $\operatorname{har}(O) := \operatorname{sbh}(O)\cap\bigl(-\operatorname{sbh}(O)\bigr)$ is the vector space over $\mathbb{R}$ of all *harmonic* (locally affine if $d=1$) functions on O. Each function

$$u\in\operatorname{sbh}_*(O) := \bigl\{u\in\operatorname{sbh}(O)\colon u\not\equiv-\infty \text{ on each } C\in\operatorname{Conn}_{\mathbb{R}^d_\infty}(O)\bigr\}$$

is associated with its *Riesz measure*

$$\varDelta_u := c_d\triangle u\in\operatorname{Meas}^+(O),\quad c_d := \frac{\Gamma(d/2)}{2\pi^{d/2}\max\{1,d-2\}}, \tag{1.6}$$

where $\triangle$ is the *Laplace operator* acting in the sense of the theory of distributions (or generalized functions), and Γ is the gamma function. If $u \equiv -\infty$ on $C \in \operatorname{Conn}_{\mathbb{R}^d_\infty}(O)$, then we set $\varDelta_{-\infty}(S) := +\infty$ for each $S \subset C$. Given $S \subset \mathbb{R}^d$, we set

$$\begin{aligned}\operatorname{Sbh}(S) &:= \bigcup\Bigl\{\operatorname{sbh}(O') \colon S \subset O' \overset{\text{open}}{=} \operatorname{int} O' \subset \mathbb{R}^d\Bigr\},\\ \operatorname{Sbh}_*(S) &:= \bigcup\Bigl\{\operatorname{sbh}_*(O') \colon S \subset O' \overset{\text{open}}{=} \operatorname{int} O' \subset \mathbb{R}^d\Bigr\},\\ \operatorname{Har}_*(S) &:= \bigcup\Bigl\{\operatorname{har}(O') \colon S \subset O' \overset{\text{open}}{=} \operatorname{int} O' \subset \mathbb{R}^d\Bigr\}.\end{aligned}$$

Consider a binary relation $\cong \subset \operatorname{Sbh}(S) \times \operatorname{Sbh}(S)$ on $\operatorname{Sbh}(S)$ defined by the rule: $U \cong V$ if *there is an open set* $O' \supset S$ *in* $\mathbb{R}^d$ *such that* $U \in \operatorname{sbh}(O')$, $V \in \operatorname{sbh}(O')$, *and* $U(x) = V(x)$ *for each* $x \in O'$. This relation $\cong$ is an *equivalence relation* on $\operatorname{Sbh}(S)$, on $\operatorname{Sbh}_*(S)$, and on $\operatorname{Har}(S)$. The quotient sets of $\operatorname{Sbh}(S)$, of $\operatorname{Sbh}_*(S)$, and of $\operatorname{Har}(S)$ by $\cong$ are denoted below by $\operatorname{sbh}(S) := \operatorname{Sbh}(S)/\cong$, $\operatorname{sbh}_*(S) := \operatorname{Sbh}_*(S)/\cong$, and $\operatorname{har}(S) := \operatorname{Har}(S)/\cong$, respectively. The equivalence class $[u]$ of u is denoted without square brackets as simply u, and we do not distinguish between the equivalence class $[u]$ and the function u when possible. So, for $u, v \in \operatorname{sbh}(S)$, we write *"$u = v$ on S"* if $[u] = [v]$ in $\operatorname{sbh}(S) := \operatorname{Sbh}(S)/\cong$, or, equivalently, $u \cong v$ on $\operatorname{Sbh}(S)$, and we write $u \not\equiv -\infty$ if $u \in \operatorname{sbh}_*(S)$. The concept of the Riesz measure $\varDelta_u$ of $u \in \operatorname{sbh}(S)$ is correctly and uniquely defined by the restriction $\varDelta_u\bigm|_S$ of the Riesz measure $\varDelta_u$ to S. For $u \in \operatorname{sbh}(S)$ and $v \in \operatorname{sbh}(S)$, the concepts *"$u \leq v$ on S"*, and *"$u = v$ outside S"*, *"$u \leq v$ outside S"*, *"u is harmonic outside S"*, etc. defined naturally: $u(x) \leq v(x)$ *for each* $x \in S$, and *there exits an open set* $O' \supset S$ such that $u(x) = v(x)$ *for each* $x \in O' \setminus S$, $u(x) \leq v(x)$ *for each* $x \in O' \setminus S$, *the restriction* $u\bigm|_{O'\setminus S}$ *is harmonic on* $O' \setminus S$, respectively. So, Theorem 1.1 in Introduction is formulated precisely in this interpretation.

Balayage

Let $S \in \operatorname{Bor}(\mathbb{R}^d)$ and H be a set of *upper semicontinuous* functions $f \colon S \to \overline{\mathbb{R}} \setminus +\infty$. A measure $\omega \in \operatorname{Meas}^+_{\text{cmp}}(S)$ is called the *balayage* of a measure $\varDelta \in \operatorname{Meas}^+_{\text{cmp}}(S)$ for S with respect to H [49], [8], [41, Definition 5.2], or, briefly, ω *is a H-balayage of* $\varDelta$, and we write $\varDelta \preceq_H \omega$ or $\omega \succeq_H \varDelta$ if

$$\int_S h \,\mathrm{d}\varDelta \leq \int_S h \,\mathrm{d}\omega \quad \text{for each } h \in H \text{ in accordance with (1.5).} \tag{1.7}$$

If $\varDelta \preceq_H \omega$ and $\omega \preceq_H \varDelta$, then we write $\varDelta \simeq_H \omega$. The following properties are obvious.

1. The binary relation $\preceq_H$ (respectively $\simeq_H$) on $\operatorname{Meas}^+_{\text{cmp}}(S)$ is a *preorder,* i.e., a reflexive and transitive relation, (respectively, an *equivalence*) on $\operatorname{Meas}^+_{\text{cmp}}(S)$.

2. If H contains a strictly positive (respectively, negative) constant, then $\varDelta(S) \leq \omega(S)$ ($\varDelta(S) \geq \omega(S)$, respectively).

3. If $H = -H$, then the order $\preceq_H$ is the equivalence $\simeq_H$. So, if $H = \operatorname{har}(S)$, then ω is a $\operatorname{har}(S)$-balayage of $\varDelta$ if and only if $\varDelta \simeq_{\operatorname{har}(S)} \omega$, i.e.,

$$\int_S h \,\mathrm{d}\varDelta = \int_S h \,\mathrm{d}\omega \quad \text{for each } h \in \operatorname{har}(S) \quad \text{and} \quad \varDelta(S) = \omega(S). \tag{1.8}$$

4. If $\varDelta \preceq_{\operatorname{sbh}(S)} \omega$, then $\varDelta \preceq_{\operatorname{har}(S)} \omega$. The converse is not true [43, Example XIB2], [48, Example].

5. If $\omega \in \operatorname{Meas}^+_{\text{cmp}}(O)$ is a $\bigl(\operatorname{sbh}(O) \cap C^\infty(O)\bigr)$-balayage of $\varDelta \in \operatorname{Meas}^+_{\text{cmp}}(O)$, where $C^\infty(O)$ is the class of all infinitely differentiable functions on O, then $\varDelta \preceq_{\operatorname{sbh}(O)} \omega$, since for each function $u \in \operatorname{sbh}(O)$ there exists a sequence of functions $u_j \underset{j \in \mathbb{N}}{\in} \operatorname{sbh}(O) \cap C^\infty(O)$ decreasing to it [14, Chapter 4, Section 10, Approximation Theorem].

Remark 1. Balayage of charges and measures with a non-compact support also occur frequently and are used in Analysis. So, a bounded domain $D \subset \mathbb{R}^d$ is called a *quadrature domain* (for harmonic functions) if there is a charge $\varDelta \in \operatorname{Meas}_{\mathrm{cmp}}(D)$ such that the restriction of the Lebesgue measure λ to D is a balayage of $\varDelta$ with respect to the class of all harmonic λ-integrable functions on D. In connection with the quadrature domains, see very informative overview [19, Chapter 3] and the bibliography therein.

Potentials

For a *charge* $\mu \in \operatorname{Meas}_{\mathrm{cmp}}(O)$ its *potential*

$$\operatorname{pt}_\mu \colon \mathbb{R}^d_\infty \to \overline{\mathbb{R}}, \quad \operatorname{pt}_\mu(y) \overset{(1.2)}{:=} \int_O K_{d-2}(x,y)\,\mathrm{d}\mu(x), \tag{1.9}$$

is uniquely determined on [3], [40, 3.1]

$$\operatorname{Dom}\operatorname{pt}_\mu := \left\{ y \in \mathbb{R}^d \colon \inf \left\{ \int_0^1 \frac{\mu^-(y,t)}{t^{d-1}}\,\mathrm{d}t, \int_0^1 \frac{\mu^+(y,t)}{t^{d-1}}\,\mathrm{d}t \right\} < +\infty \right\} \tag{1.10}$$

by values in $\overline{\mathbb{R}}$, and the set $E := (\complement \operatorname{Dom}\operatorname{pt}_\mu) \setminus \infty$ is *polar* with zero *outer capacity*

$$\operatorname{Cap}^*(E) := \inf_{E \subset O' \overset{\text{open}}{=} \operatorname{int} O'} \sup_{\substack{C \overset{\text{closed}}{=} \operatorname{clos} C \overset{\text{compact}}{\Subset} O \\ \nu \in \operatorname{Meas}^{1+}(C)}} k_{d-2}^{-1} \left(\iint K_{d-2}(x,y)\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y) \right).$$

Evidently $\operatorname{pt}_\mu \in \operatorname{har}\bigl(\mathbb{R}^d \setminus \operatorname{supp}|\mu|\bigr)$, and if $\mu \in \operatorname{Meas}^+_{\mathrm{cmp}}(\mathbb{R}^d)$, then $\operatorname{pt}_\mu \in \operatorname{sbh}_*(\mathbb{R}^d)$.

Inward filling of sets with respect to an open set

Let O be an open set in $\mathbb{R}^d$. The union of $S \subset O$ with all components $C \in \operatorname{Conn}_O(O \setminus S)$ such that $C \Subset O$ will be called the *inward filling* of S with respect to O, and we denote this union by $\operatorname{in-fill}_O S$ or $\operatorname{in-fill}_O(S)$, although in [16, Section 1.7], [6], [17, Section 12], [35, Section 1] another notation $\hat{S}$ was used. Denote by O_∞ the *Alexandroff one-point compactification* of O with underlying set $O \sqcup \infty$, where the extra point $\infty \notin O$ can be identified with the boundary ∂O or the complement $\complement O$, considered as a single point. The following elementary properties of the inward filling will often be used without mentioning them.

Proposition 1.1 ([16, Section 6.3], [17], [6], [18]). *Let S be a compact set in an open set $O \subset \mathbb{R}^d$. Then*

(i) $\operatorname{in-fill}_O S$ *is a compact subset in O, and* $\operatorname{in-fill}_O\bigl(\operatorname{in-fill}_O S\bigr) = \operatorname{in-fill}_O S$;

(ii) *the set $O_\infty \setminus \operatorname{in-fill}_O S$ is connected and locally connected subset in O_∞;*

(iii) *the inward filling of S with respect to O coincides with the complement in O_∞ of connected component of $O_\infty \setminus S$ containing the point ∞, i. e., $\operatorname{in-fill}_O S = O_\infty \setminus C_\infty$, where $\infty \in C_\infty \in \operatorname{Conn}_{O_\infty}(O_\infty \setminus S)$;*

(iv) *if $O' \subset \mathbb{R}^d_\infty$ is an open subset and $O \subset O'$, then* $\operatorname{in-fill}_O S \subset \operatorname{in-fill}_{O'} S$;

(v) *if $S' \subset S$ is a compact subset in O, then* $\operatorname{in-fill}_O S' \subset \operatorname{in-fill}_O S$;

(vi) $\mathbb{R}^d \setminus \operatorname{in-fill}_O S$ *has only finitely many components:* $\#\operatorname{Conn}_{\mathbb{R}^d_\infty}(\mathbb{R}^d \setminus \operatorname{in-fill}_O S) < +\infty$.

2 Poisson – Jensen formulas

2.1 Main result for measures and their potentials

Main Lemma. *Let* $\varDelta \in \operatorname{Meas}^+_{\rm cmp}(O)$, $\omega \in \operatorname{Meas}^+_{\rm cmp}(O)$, *and*

$$S_O := \text{in-fill}_O(\operatorname{supp}\varDelta \cup \operatorname{supp}\omega). \tag{2.1}$$

The following seven statements are equivalent.

I. $\varDelta \preceq_{\operatorname{har}(O)} \omega$.

II. $\varDelta \simeq_{\operatorname{har}(S_O)} \omega$.

III. $\operatorname{pt}_{\varDelta} = \operatorname{pt}_{\omega}$ *on* $\mathbb{R}^d \setminus S_O$.

IV. *There are a compact subset* S *in* O, *a function* $q \in \operatorname{sbh}_*(S)$ *with the Riesz measure* $\varDelta_q = \varDelta$, *and a function* $p \in \operatorname{sbh}_*(S)$ *with the Riesz measure* $\varDelta_p = \omega$ *such that* q *and* p *are harmonic outside* S, *and* $q = p$ *outside* S.

V. *The symmetric Poisson – Jensen formula for potentials is valid:*

$$\int u \,\mathrm{d}\varDelta + \int_B \operatorname{pt}_{\omega} \mathrm{d}\varDelta_u = \int u \,\mathrm{d}\omega + \int_B \operatorname{pt}_{\varDelta} \mathrm{d}\varDelta_u \tag{2.2f}$$

for each $B \in \operatorname{Bor}(\mathbb{R}^d)$ such that $S_O \subset B \Subset O$ (2.2B)

and for each $u \in \operatorname{sbh}_*(\operatorname{clos} B)$. (2.2u)

VI. *For each* $q \in \operatorname{sbh}_*(S_O)$ *with* $\varDelta_q = \varDelta$ *there is* $p \in \operatorname{sbh}_*(S_O)$ *with* $\varDelta_p = \omega$ *such that*

$$\int u \,\mathrm{d}\varDelta + \int_{S_O} p \,\mathrm{d}\varDelta_u = \int u \,\mathrm{d}\omega + \int_{S_O} q \,\mathrm{d}\varDelta_u \quad \text{for each } u \in \operatorname{sbh}_*(O). \tag{2.3}$$

VII. *There are a compact subset* $S \supset S_O$ *in* O *and a pair of functions* $q \in \operatorname{sbh}_*(S_O)$ *and* $p \in \operatorname{sbh}_*(S_O)$ *with the Riesz measures* $\varDelta_q = \varDelta$ *and* $\varDelta_p = \omega$, *respectively, such that the equality in* (2.3) *is fulfilled for each special subharmonic function*

$$u_x \colon y \underset{y \in \mathbb{R}^d}{\longmapsto} K_{d-2}(y, x), \quad x \in O \setminus S,$$

instead of all functions $u \in \operatorname{sbh}_*(O)$ *in* (2.3).

The proof of the Main Lemma will be given only after some preparation in Section 3.

2.2 Full version of the Poisson – Jensen formula for subharmonic integrands

The starting point of the Main Lemma is a pair of measures $\varDelta, \omega \in \operatorname{Meas}^+_{\rm cmp}(O)$. Our Main Theorem is a functional counterpart of the Main Lemma. The starting point in it is now a pair of subharmonic functions from (2.4s).

Main Theorem . *Let*

$$\varnothing \neq S \overset{\text{closed}}{=} \operatorname{clos} S \overset{\text{compact}}{\Subset} O \overset{\text{open}}{=} \operatorname{int} O \subset \mathbb{R}^d, \quad S_O := \text{in-fill}_O\, S, \tag{2.4S}$$

$$q \in \operatorname{sbh}_*(O) \cap \operatorname{har}(O \setminus S), \quad p \in \operatorname{sbh}_*(O) \cap \operatorname{har}(O \setminus S), \tag{2.4s}$$

$$S_{\neq} := \bigl\{x \in O \colon q(x) \neq p(x)\bigr\}. \tag{2.4$\neq$}$$

The following four statements are equivalent.

[I] $S_{\neq} \subset S_O$*, i.e.,* $q = p$ *on* $O \setminus S_O$*.*

[II] *There is a function* $h \in \operatorname{har}(O)$ *such that*

$$\begin{cases} q = \operatorname{pt}_{\Delta_q} + h \\ p = \operatorname{pt}_{\Delta_p} + h \end{cases} \text{on } O \text{ and } \operatorname{pt}_{\Delta_q} = \operatorname{pt}_{\Delta_p} \text{on } \mathbb{R}^d \setminus S_O, \tag{2.5}$$

where $\Delta_q \in \operatorname{Meas}^+(S)$ *and* $\Delta_p \in \operatorname{Meas}^+(S)$ *are the Riesz measures of* q *and* p*.*

[III] *The full symmetric Poisson – Jensen formula is valid:*

$$\int_S u \, \mathrm{d}\Delta_q + \int_B p \, \mathrm{d}\Delta_u = \int_S u \, \mathrm{d}\Delta_p + \int_B q \, \mathrm{d}\Delta_u \quad \text{for each } B \in \operatorname{Bor}(\mathbb{R}^d) \tag{2.6f}$$

$$\textit{if } S_O \cap S_{\neq} \subset B \Subset O \text{ and for each } u \in \operatorname{sbh}_*(S_O \cup \operatorname{clos} B). \tag{2.6B}$$

[IV] (2.6) *holds for a sequence of sets* $B_j \underset{j \in \mathbb{N}_0}{\in} \operatorname{Bor}(O)$ *such that* $B_0 := S_O \underset{j \in \mathbb{N}}{\subset} B_j \underset{j \in \mathbb{N}}{\Subset} O$ *and* $\bigcup_{j \in \mathbb{N}} B_j = O$ *instead of all* $B \in \operatorname{Bor}(\mathbb{R}^d)$ *with* $S_O \cap S_{\neq} \subset B \Subset O$ *in* (2.6) *and for each special subharmonic function* $u_x \colon y \underset{y \in \mathbb{R}^d}{\longmapsto} K_{d-2}(y, x)$ *with* $x \in O \setminus B_0 = O \setminus S_O$ *instead of all functions* $u \in \operatorname{sbh}_*(S_O \cup \operatorname{clos} B)$ *in* (2.6B)*.*

We can now prove Theorem 1.1 of the Introduction.

Proof of Theorem 1.1. There is an open set $O \subset \mathbb{R}^d$ such that $u \in \operatorname{sbh}_*(O)$, $q \in \operatorname{sbh}_*(O)$ and $p \in \operatorname{sbh}_*(O)$ are harmonic on $O \setminus S$, and also $q = p$ on $O \setminus S$. Evidently, in notation (2.4), we have $S_O \cap S_{\neq} \subset S \subset S_O \Subset O$ and $u \in \operatorname{sbh}_*(S_O)$. Theorem 1.1 with (1.4) follows from implication [I]⇒[III] of the Main Theorem, since we can choose $B := S$ in (2.6). □

2.3 In detail on the classical Poisson – Jensen formula

If $x \in D \Subset O$, then the *extended harmonic measure* $\omega_D(x, \cdot) \in \operatorname{Meas}^{1+}(\partial D) \subset \operatorname{Meas}^{1+}_{\text{cmp}}(\mathbb{R}^d)$ *(for* D *at* x*)* defined on sets $B \in \operatorname{Bor}(\mathbb{R}^d)$ by

$$\omega_D(x, B) := \sup \left\{ u(x) \colon u \in \operatorname{sbh}(D), \ \limsup_{D \ni y' \to y \in \partial D} u(y') \leq \begin{cases} 1 \text{ for } y \in B \cap \partial D \\ 0 \text{ for } y \notin B \cap \partial D \end{cases} \right\} \tag{2.7}$$

is a $\operatorname{har}(O)$-balayage of δ_x with obvious equalities

$$\text{in-fill}\bigl(\operatorname{supp} \delta_x \cup \operatorname{supp} \omega_D(x, \cdot)\bigr) = \text{in-fill}(x \cup \partial D) = \operatorname{clos} D,$$

the potential (see [46, Chapter 4, Section 1, paragraph 2])

$$\begin{aligned} \operatorname{pt}_{\omega_D(x,\cdot) - \delta_x}(y) &= \operatorname{pt}_{\omega_D(x,\cdot)}(y) - \operatorname{pt}_{\delta_x}(y) \\ &= \int_{\partial D} K_{d-2}(y, x') \, \mathrm{d}_{x'} \omega_D(x, x') - K_{d-2}(y, x) = g_D(y, x), \quad y \in \mathbb{R}^d_\infty, \quad x \in D, \end{aligned} \tag{2.8}$$

is equal to the *generalized Green's function* $g_D(\cdot,x)\colon \mathbb{R}^d_\infty \to \overline{\mathbb{R}}^+$ *(for D with pole at x and $g_D(x,x):=+\infty$)* defined on $\mathbb{R}^d_\infty\setminus x$ by upper semicontinuous regularization

$$g_D(y,x):=\breve{g}^*(y,x):=\limsup_{\mathbb{R}^d\ni y'\to y}\breve{g}(y',x)\in\overline{\mathbb{R}}^+\quad\text{for each } y\in\mathbb{R}^d_\infty\setminus x, \text{ where}$$
$$\breve{g}(y,x):=\sup\left\{u(y)\colon u\in\operatorname{sbh}(\mathbb{R}^d\setminus x),\ \begin{cases}u(y')\leq 0 \text{ for each } y\notin\operatorname{clos}D,\\ \limsup\limits_{x\neq y\to x}\dfrac{u(y)}{-K_{d-2}(x,y)}\leq 1\end{cases}\right\}. \tag{2.9}$$

Equalities (2.8) give (1.3) with $\varDelta_p=\omega_D(x,\cdot)$ and $\varDelta_q=\delta_x$. Thus, Theorem 1.1 implies symmetric Poisson – Jensen formula (1.4) which can be written in detail as

$$\int_{\operatorname{clos}D}u\,\mathrm{d}\delta_x+\int_{\operatorname{clos}D}\Bigl(g_D(\cdot,x)+K_{d-2}(\cdot,x)\Bigr)\,\mathrm{d}\varDelta_u$$
$$=\int_{\operatorname{clos}D}u\,\mathrm{d}\omega_D(x,\cdot)+\int_{\operatorname{clos}D}K_{d-2}(\cdot,x)\,\mathrm{d}\varDelta_u. \tag{2.10}$$

The latter coincides with classical Poisson – Jensen formula (1.1).

2.4 The Poisson – Jensen formula for the Arens – Singer, and Jensen measures and potentials

Our results presented in this Subsection 2.4 are intermediate between classical Poisson – Jensen formula (1.1) and symmetric Poisson – Jensen formula (1.4) of Theorem 1.1. If $x\in O$ and $\delta_x\preceq_{\operatorname{har}(O)}\omega\in\operatorname{Meas}^+_{\mathrm{cmp}}(O)$, then ω is called an *Arens – Singer measure on O at x* [15, Chapter 3], [52], [30], [35, Definition 1], [34], [36], [45], or representing measure. We denote by $AS_x(O)\subset\operatorname{Meas}^{1+}_{\mathrm{cmp}}(O)$ the class of all Arens – Singer measures on O at x. If $\omega\in AS_x(O)$, then the potential [50, Section 3.1], [35, Definition 2], [40, Sections 3.1, 3.2], [11]

$$\operatorname{pt}_{\omega-\delta_x}(y))\overset{(1.9)}{=}\operatorname{pt}_\omega-\operatorname{pt}_{\delta_x}(y)\overset{(1.2)}{=}\operatorname{pt}_\omega(y)-K_{d-2}(y,x),\quad y\in\mathbb{R}^d_\infty\setminus x,$$

satisfies conditions [35, Section 1] (see also Duality Theorem 3 in Section 5 below)

$$\begin{aligned}\operatorname{pt}_{\omega-\delta_x}&\in\operatorname{sbh}(\mathbb{R}^d_\infty\setminus x),\quad \operatorname{pt}_{\omega-\delta_x}(\infty):=0,\\ \operatorname{pt}_{\omega-\delta_x}&\equiv 0\quad\text{on } \mathbb{R}^d_\infty\setminus\text{in-fill}_O(x\cup\operatorname{supp}\omega),\\ \operatorname{pt}_{\omega-\delta_x}(y)&\leq -K_{d-2}(x,y)+O(1)\quad\text{as } x\neq y\to x.\end{aligned} \tag{2.11}$$

If $x\in O$ and $\delta_x\preceq_{\operatorname{sbh}(O)}\omega$, then this measure ω is called a *Jensen measure on O at x* [15, Chapter 3], [43], [44], [12], [13], [51], [10], [20], [21], [24], [42], [7], [38]. The class of these measures is denoted by $J_x(O)$, and properties (2.11) are supplemented by the *positivity property* $\operatorname{pt}_{\omega-\delta_x}\geq 0$ on $\mathbb{R}^d_\infty\setminus x$ for all measures $\omega\in J_x(O)\subset AS_x(O)$. These measures can be considered as generalizations of the extended harmonic measures (2.7).

By the implication I⇒V of the Main Lemma with $\varDelta:=\delta_x$, we obtain the following our version [35, Proposition 1.2, formula (1.3)] of the Poisson-Jensen formula for Arens – Singer measures $\omega\in AS_x(O)$, generalizing classical Poisson-Jensen formula (1.1).

Poisson – Jensen formula for Arens – Singer and Jensen measures *If $\omega\in AS_x(O)$, then*

$$u(x)=\int u\,\mathrm{d}\omega-\int\operatorname{pt}_{\omega-\delta_x}\mathrm{d}\varDelta_u\quad\textit{for each } u\in\operatorname{sbh}_*(O) \textit{ with } u(x)\neq-\infty. \tag{2.12}$$

If $\omega\in J_x(O)$ and $\omega\neq\delta_x$, then the restriction $u(x)\neq-\infty$ in (2.12) can be removed.

For $x \in \mathbb{R}^d$, a function $V \in \operatorname{sbh}_*\bigl(\mathbb{R}^d_\infty \setminus x\bigr)$ is called an *Arens – Singer potential on O with pole at $x \in O$* [15, Chapter 3], [52], [34], [35], [36, Definition 6], [45, Section 4] if there is $S_V \Subset O$ such that

$$V \equiv 0 \quad \text{on } \complement S_V \quad \text{and} \quad \limsup_{x\neq y\to x} \frac{V(y)}{-K_{d-2}(x,y)} \leq 1. \tag{2.13}$$

We denote by $ASP_x(O)$ the class of all Arens – Singer potentials on O with pole at $x \in O$. A *positive* Arens – Singer potential is called a *Jensen potential on O with pole at $x \in O$* [15, Chapter 3], [1], [35], [47], [36, Definition 8], [44, Section IIIC], [37], [42], [7]. We denote by $JP_x(O)$ the class of all Jensen potentials on O with pole at $x \in O$. These potentials can be considered as generalizations of Green's functions (2.9). For $V \in ASP_x(O)$, we choose (cf. (1.3))

$$p\colon y \longmapsto V(y) + K_{d-2}(y,x), \quad q\colon y \longmapsto K_{d-2}(y,x) \quad \text{for } y \in \mathbb{R}^d. \tag{2.14}$$

Then these subharmonic functions on $\mathbb{R}^d$ are harmonic and coincide outside $\operatorname{clos} S_V$ by (2.13), and the implication [I]⇒[III] of the Main Theorem give the equality (cf. (2.10))

$$\int_O u \,\mathrm{d}\delta_x + \int_{S_O} \Bigl(V(y) + K_{d-2}(\cdot,x)\Bigr)\,\mathrm{d}\varDelta_u = \int_O u\,\mathrm{d}\varDelta_V + \int_{S_O} K_{d-2}(\cdot,x)\,\mathrm{d}\varDelta_u,$$

where $S_O = \text{in-fill}\Biggl(x \cup \Bigl(\bigcup\bigl\{y \in O\colon V \notin \operatorname{har}(y)\bigr\}\Bigr)\Biggr)$. Hence we obtain

Poisson – Jensen formula for Arens – Singer and Jensen potentials *If $V \in ASP_x(O)$, then*

$$u(x) = \int u\,\mathrm{d}\varDelta_V - \int V\,\mathrm{d}\varDelta_u \quad \text{for each } u \in \operatorname{sbh}_*(O) \text{ with } u(x) \neq -\infty. \tag{2.15}$$

If $V \in JP_x(O)$ and $V \not\equiv 0$ on $\complement x$, then the restriction $u(x) \neq -\infty$ in (2.15) can be removed.

3 Proof of the Main Lemma

3.1 Representations for pairs of subharmonic functions

Proposition 3.1. *If $\mu \in \operatorname{Meas}_{\mathrm{cmp}}(\mathbb{R}^d)$, then*

$$\operatorname{pt}_\mu \in \operatorname{sbh}(\mathbb{R}^d) \bigcap \operatorname{har}(\mathbb{R}^d \setminus \operatorname{supp}\mu), \tag{3.1h}$$

$$\operatorname{pt}_\mu(x) \overset{(1.2k)}{=} \mu(\mathbb{R}^d)k_{d-2}\bigl(|x|\bigr) + O\bigl(1/|x|^{d-1}\bigr), \quad x \to \infty. \tag{3.1∞}$$

Proof. For $d = 1$, we have

$$\bigl|\operatorname{pt}_\mu(x) - \mu(\mathbb{R})|x|\bigr| \leq \int \bigl||x-y| - |x|\bigr|\,\mathrm{d}|\mu|(y) \leq \int |y|\,\mathrm{d}|\mu|(y) = O(1), \quad |x| \to +\infty.$$

See [50, Theorem 3.1.2] for $d = 2$.

For $d > 2$ and $|x| \geq 2\sup\bigl\{|y|\colon y \in \operatorname{supp}\mu\bigr\}$, we have

$$\begin{aligned}
\bigl|\operatorname{pt}_\mu(x) - \mu(\mathbb{R}^d)k_{d-2}\bigl(|x|\bigr)\bigr| &= \left|\int \left(\frac{1}{|x|^{d-2}} - \frac{1}{|x-y|^{d-2}}\right)\mathrm{d}\mu(y)\right| \\
&\leq \int \left|\frac{1}{|x|^{d-2}} - \frac{1}{|x-y|^{d-2}}\right| \mathrm{d}|\mu|(y) \leq \frac{2^{d-2}}{|x|^{2d-4}} \int \bigl||x-y|^{d-2} - |x|^{d-2}\bigr|\,\mathrm{d}|\mu|(y) \\
&\leq \frac{2^{d-2}}{|x|^{2d-4}} \int |y||x|^{d-3} \sum_{k=0}^{d-3} \Bigl(\frac{3}{2}\Bigr)^k \mathrm{d}|\mu|(y) \leq 2\frac{3^{d-2}}{|x|^{d-1}} \int |y|\,\mathrm{d}|\mu|(y) = O\Bigl(\frac{1}{|x|^{d-1}}\Bigr),
\end{aligned}$$

and we get (3.1∞). □

Theorem 3.1. *Let $O\subset \mathbb{R}^d$ be an open set, and let $p\in \operatorname{sbh}_*(O)$ and $q\in \operatorname{sbh}_*(O)$ be a pair of functions such that p and q are harmonic outside a compact subset in O. If there is a compact set $S\Subset O$ such that $p=q$ on $O\setminus S$, then, for the Riesz measures $\varDelta_p\in \operatorname{Meas}^+_{\rm cmp}(O)$ of p and $\varDelta_q\in \operatorname{Meas}^+_{\rm cmp}(O)$ of q, we have*

$$\varDelta_p(O)=\varDelta_q(O),\quad \operatorname{pt}_{\varDelta_p}=\operatorname{pt}_{\varDelta_q}\quad \text{on } \mathbb{R}^d\setminus S, \tag{3.2}$$

and there is a harmonic function H on O such that

$$\begin{cases} p=\operatorname{pt}_{\varDelta_p}+H\\ q=\operatorname{pt}_{\varDelta_q}+H\end{cases} \qquad \text{on } O,\qquad H\in \operatorname{har}(O). \tag{3.3}$$

Proof. By Weyl's lemma for the Laplace equation, we have

$$\begin{cases} \triangle(p-\operatorname{pt}_{\varDelta_p})\overset{(1.6)}{=}\frac{1}{c_d}(\varDelta_p-\varDelta_p)=0\\ \triangle(q-\operatorname{pt}_{\varDelta_q})\overset{(1.6)}{=}\frac{1}{c_d}(\varDelta_q-\varDelta_q)=0\end{cases} \qquad \Longrightarrow \qquad \begin{cases} h_p:=p-\operatorname{pt}_{\varDelta_p}\in \operatorname{har}(O)\\ h_q:=q-\operatorname{pt}_{\varDelta_q}\in \operatorname{har}(O)\end{cases}$$

and obtain the representations

$$\begin{cases} p=\operatorname{pt}_{\varDelta_p}+h_p\\ q=\operatorname{pt}_{\varDelta_Q}+h_q\end{cases} \quad \text{on } O \text{ with } h_p\in \operatorname{har}(O) \text{ and } h_q\in \operatorname{har}(O). \tag{3.4}$$

Let us first consider separately several cases.

The case $O:=\mathbb{R}^d$ with the notation $P:=p$ and $Q:=q$. Put

$$h\overset{(3.4)}{:=}h_P-h_Q\in \operatorname{har}(\mathbb{R}^d). \tag{3.5}$$

By the conditions of Theorem 3.1 and Proposition 3.1, we have

$$\begin{split} h(x)\overset{(3.5)}{=}h_P(x)-h_Q(x)\overset{(3.4)}{=}-\operatorname{pt}_{\varDelta_P}(x)+\operatorname{pt}_{\varDelta_Q}(x)+\bigl(P(x)-Q(x)\bigr)\\ \overset{(3.1\infty)}{=}bk_{d-2}\bigl(|x|\bigr)+O\bigl(|x|^{1-d}\bigr),\quad |x|\to+\infty,\quad \text{where } b:=\varDelta_Q(\mathbb{R}^d)-\varDelta_P(\mathbb{R}^d). \end{split} \tag{3.6}$$

The case $d>2$. If $d\geq 3$, then, in view of (3.6), this harmonic function h is bounded on $\mathbb{R}^d$. By Liouville's Theorem [5, Chapter 3], h is constant, and $h_P-h_Q=h\overset{(3.6)}{\equiv}0$ on $\mathbb{R}^d$. In particular, $|b|=\bigl|b+|x|^{d-2}h(x)\bigr|\overset{(3.6)}{=}O\bigl(1/|x|\bigr)$ as $x\to\infty$, i.e., $b=0$. Thus, for $H:=h_P=h_Q$, by (3.4), we obtain representations (3.3) together with equality $\operatorname{pt}_{\varDelta_P}=\operatorname{pt}_{\varDelta_Q}$ on $\mathbb{R}^d\setminus S$, as required.

The case $d=2$. Using (3.6) we obtain $\bigl|h(x)-b\log|x|\bigr|\overset{(3.6)}{=}O\bigl(1/|x|\bigr)$ as $x\to\infty$. Hence, this harmonic function h is bounded from below if $b\geq 0$ or bounded from above if $b<0$. Therefore, by Liouville's Theorem, h is constant, $b=0$, i.e., $\varDelta_P(\mathbb{R}^2)\overset{(3.6)}{=}\varDelta_Q(\mathbb{R}^2)$, and $h\overset{(3.6)}{\equiv}0$ on $\mathbb{R}^2$. Thus, we obtain (3.3) together with equality (3.2).

The case $d=1$. Using (3.6) we obtain $\bigl|h(x)-b|x|\bigr| \overset{(3.6)}{=} O(1)$ as $x\to\infty$. Hence, this affine function h on $\mathbb{R}$ is bounded from below if $b\geq 0$ or bounded from above if $b<0$. Therefore, h is constant, $b=0$, i.e., $\varDelta_P(\mathbb{R}) \overset{(3.6)}{=} \varDelta_Q(\mathbb{R})$, and $h \overset{(3.6)}{\equiv} C$ on $\mathbb{R}$ for a constant $C\in\mathbb{R}$. Thus,

$$\begin{cases} P(x)=\operatorname{pt}_{\varDelta_P}(x)+ax+b+C \\ Q(x)=\operatorname{pt}_{\varDelta_Q}(x)+ax+b \end{cases} \quad \text{for } x\in\mathbb{R} \text{ with } h_Q(x) \underset{x\,\in\,\mathbb{R}}{\equiv} ax+b, \tag{3.7}$$

Definition (1.9) of potentials in the case $d=1$ immediately implies

Lemma 3.1. *Let $\varDelta\in\operatorname{Meas}^+_{\rm cmp}(\mathbb{R})$, and $s_l:=\inf\operatorname{supp}\varDelta$, $s_r:=\sup\operatorname{supp}\varDelta$. Then*

$$\operatorname{pt}_{\varDelta}(x)=\begin{cases} \varDelta(\mathbb{R})x-\int y\,\mathrm{d}\varDelta(y) & \text{if } x\geq s_r, \\ -\varDelta(\mathbb{R})x+\int y\,\mathrm{d}\varDelta(y) & \text{if } x\leq s_l. \end{cases}$$

We set

$$\begin{cases} t:=\varDelta_P(\mathbb{R})=\varDelta_Q(\mathbb{R})\in\mathbb{R}^+, \\ S_l:=\inf(S\cup\operatorname{supp}\varDelta_P\cup\operatorname{supp}\varDelta_Q)\in\mathbb{R}, \\ S_r:=\sup(S\cup\operatorname{supp}\varDelta_P\cup\operatorname{supp}\varDelta_Q)\geq S_l. \end{cases}$$

In view of $P(x)\equiv Q(x)$ for $x\in\mathbb{R}\setminus S$, by Lemma 3.1, we have

$$\begin{cases} tx-\int y\,\mathrm{d}\varDelta_P(y)+ax+b+C=tx-\int y\,\mathrm{d}\varDelta_Q(y)+ax+b & \text{if } x\geq S_r, \\ -tx+\int y\,\mathrm{d}\varDelta_P(y)+ax+b+C=-tx+\int y\,\mathrm{d}\varDelta_Q(y)+ax+b & \text{if } x\leq S_l, \end{cases}$$

whence

$$\begin{cases} -\int y\,\mathrm{d}\varDelta_P(y)+C=-\int y\,\mathrm{d}\varDelta_Q(y), \\ \int y\,\mathrm{d}\varDelta_P(y)+C=\int y\,\mathrm{d}\varDelta_Q(y). \end{cases}$$

Adding these equalities, we obtain $C=0$. Thus, we get (3.3) together with (3.2).

The general case of an open set $O\subset\mathbb{R}^d$. Let us start again with the representations (3.4). We set

$$\mathsf{S} \overset{\text{closed}}{:=} S\bigcup\operatorname{supp}\varDelta_q\bigcup\operatorname{supp}\varDelta_p \overset{\text{compact}}{\Subset} O, \tag{3.8S}$$

$$w:=p-q,\quad \varDelta_w \overset{(1.6)}{:=} c_d\,\triangle w=\varDelta_p-\varDelta_q\in\operatorname{Meas}(\mathsf{S})\subset\operatorname{Meas}_{\rm cmp}(O). \tag{3.8w}$$

This difference $w\in\operatorname{sbh}_*(O)-\operatorname{sbh}_*(O)$ of subharmonic functions, i.e., a δ-subharmonic function [3], [4], [40, 3.1], is uniquely defined on O outside a polar set (cf. (1.10))

$$\operatorname{Dom}w:=\left\{x\in O\colon\ \inf\left\{\int_0\frac{\varDelta_w^-(x,t)}{t^{d-1}}\,\mathrm{d}t,\int_0\frac{\varDelta_w^+(x,t)}{t^{d-1}}\,\mathrm{d}t\right\}<+\infty\right\} \overset{(3.8S)}{\subset} \mathsf{S}, \tag{3.9}$$

and $w\equiv 0$ on $O\setminus\mathsf{S}$ since $p=q$ outside $S\subset\mathsf{S}$ in (3.8w), and $p,q\in\operatorname{har}(O\setminus\mathsf{S})$. The Riesz charge $\varDelta_w \overset{(3.8)}{\in} \operatorname{Meas}_{\rm cmp}(O)$ of this δ-subharmonic function w on O is also uniquely determined on O with $\operatorname{supp}|\varDelta_w|\subset\mathsf{S}$ [3, Theorem 2]. The function $w\colon O\setminus\operatorname{Dom}w\to\overline{\mathbb{R}}$ can be extended from O to the whole of $\mathbb{R}^d\setminus\operatorname{Dom}w$ by zero values:

$$w\equiv 0\quad\text{on } \mathbb{R}^d\setminus\mathsf{S} \overset{(3.8S)}{\supset} \mathbb{R}^d\setminus O,\quad \varDelta_w=\varDelta_p-\varDelta_q \overset{(3.8w)}{\in} \operatorname{Meas}(\mathsf{S}). \tag{3.10}$$

This function w on $\mathbb{R}^d\setminus \operatorname{Dom} w$ is still a δ-subharmonic function, but already on $\mathbb{R}^d$, since δ-subharmonic functions are defined locally [3, Theorem 3]. The Riesz charge of this δ-subharmonic function $w\colon \mathbb{R}^d\setminus \operatorname{Dom} d\to \overline{\mathbb{R}}$ on $\mathbb{R}^d$ is the same charge $\varDelta_d \overset{(3.8w)}{\in} \operatorname{Meas}(\mathsf{S})$. There is a canonical representation [3, Definition 5] of w such that [3, Theorem 5]

$$w=P-Q\quad \text{on } \mathbb{R}^d\setminus \operatorname{Dom} w,\quad \text{where } P,Q\in \operatorname{sbh}_*(\mathbb{R}^d)\cap \operatorname{har}(\mathbb{R}^d\setminus \mathsf{S}) \tag{3.11d}$$

are functions with the Riesz measures

$$\begin{cases}\varDelta_P \overset{(1.6)}{:=} c_d \bigtriangleup P=\varDelta_w^+ \overset{(3.11d)}{\in} \operatorname{Meas}^+(\mathsf{S}),\\ \varDelta_Q \overset{(1.6)}{:=} c_d \bigtriangleup Q=\varDelta_w^- \overset{(3.11d)}{\in} \operatorname{Meas}^+(\mathsf{S}),\end{cases} \tag{3.11$\varDelta$}$$

$$P \overset{(3.10),(3.11d)}{\equiv} Q \quad \text{on } \mathbb{R}^d\setminus \mathsf{S}, \tag{3.11$\equiv$}$$

and there is a function $s\in \operatorname{sbh}_*(O)$ with the Riesz measure

$$\varDelta_s=\varDelta_p-\varDelta_w^+ \overset{(3.10),(3.11\varDelta)}{=} \varDelta_q-\varDelta_w^-\in \operatorname{Meas}^+(\mathsf{S}) \tag{3.11s}$$

$$\text{such that } \begin{cases} p=P+s,\\ q=Q+s\end{cases} \quad \text{on } O. \tag{3.11r}$$

By (3.11d) and (3.11≡), all conditions of Theorem 3.1 are fulfilled for the functions P,Q from (3.11) instead of p,q, but in the case $\mathbb{R}^d$ instead of O and S instead of S. Thus, we have (3.2) in the form

$$\varDelta_w^+(O) \overset{(3.11\varDelta)}{=} \varDelta_P(\mathbb{R}^d) \overset{(3.2)}{=} \varDelta_Q(\mathbb{R}^d) \overset{(3.11\varDelta)}{=} \varDelta_w^-(O), \tag{3.12$\varDelta$}$$

$$\operatorname{pt}_{\varDelta_w^+} \overset{(3.11\varDelta)}{=} \operatorname{pt}_{\varDelta_P}=\operatorname{pt}_{\varDelta_Q} \overset{(3.11\varDelta)}{=} \operatorname{pt}_{\varDelta_w^-} \quad \text{on } \mathbb{R}^d\setminus \mathsf{S}, \tag{3.12p}$$

and representations (3.3) in the form

$$\begin{cases} P \overset{(3.3)}{=} \operatorname{pt}_{\varDelta_P}+h \overset{(3.12p)}{=} \operatorname{pt}_{\varDelta_w^+}+h\\ Q \overset{(3.3)}{=} \operatorname{pt}_{\varDelta_Q}+h \overset{(3.12p)}{=} \operatorname{pt}_{\varDelta_w^-}+h\end{cases} \quad \text{on } \mathbb{R}^d, \quad h\in \operatorname{har}(\mathbb{R}^d). \tag{3.13}$$

Hence, by representation (3.11r), we obtain the following representations

$$\begin{gathered}\begin{cases} p \overset{(3.11r),(3.13)}{=} \operatorname{pt}_{\varDelta_w^+}+h+s,\\ q \overset{(3.11r),(3.13)}{=} \operatorname{pt}_{\varDelta_w^-}+h+s\end{cases} \quad \text{on } O,\\ h\in \operatorname{har}(\mathbb{R}^d),\quad \operatorname{pt}_{\varDelta_w^+} \overset{(3.12p)}{=} \operatorname{pt}_{\varDelta_w^-} \text{ on } \mathbb{R}^d\setminus \mathsf{S},\quad \varDelta_w^+(O) \overset{(3.12\varDelta)}{=} \varDelta_w^-(O).\end{gathered} \tag{3.14}$$

Besides, the function $l \overset{(3.11s)}{:=} s-\operatorname{pt}_{\varDelta_s}$ is harmonic on O by Weyl's lemma for the Laplace equation $\bigtriangleup (s-\operatorname{pt}_{\varDelta_s}) \overset{(3.11s)}{=} \varDelta_s-\varDelta_s=0$. Hence

$$\begin{gathered}\begin{cases} p \overset{(3.14)}{=} \operatorname{pt}_{\varDelta_w^+}+\operatorname{pt}_{\varDelta_s}+h+l,\\ q \overset{(3.14)}{=} \operatorname{pt}_{\varDelta_w^-}+\operatorname{pt}_{\varDelta_s}+h+l\end{cases} \quad \text{on } O, \text{ where } h\in \operatorname{har}(\mathbb{R}^d) \text{ and } l\in \operatorname{har}(O),\\ \operatorname{pt}_{\varDelta_w^+}+\operatorname{pt}_{\varDelta_s} \overset{(3.14)}{=} \operatorname{pt}_{\varDelta_w^-}+\operatorname{pt}_{\varDelta_s} \text{ on } \mathbb{R}^d\setminus \mathsf{S},\quad \varDelta_w^+(O) \overset{(3.14)}{=} \varDelta_w^-(O).\end{gathered} \tag{3.15}$$

By construction, we have

$$\begin{cases}\mathrm{pt}_{\varDelta_w^+}+\mathrm{pt}_{\varDelta_s}=\mathrm{pt}_{\varDelta_w^+ +\varDelta_s}\overset{(3.11s)}{=}\mathrm{pt}_{\varDelta_p},\\ \mathrm{pt}_{\varDelta_w^-}+\mathrm{pt}_{\varDelta_s}=\mathrm{pt}_{\varDelta_w^- +\varDelta_s}\overset{(3.11s)}{=}\mathrm{pt}_{\varDelta_q},\end{cases}$$
$$\varDelta_p(O)=(\varDelta_w^+ +\varDelta_s)(O)\overset{(3.11s)}{=}(\varDelta_w^- +\varDelta_s)(O)=\varDelta_p(O).$$

Hence, if we set $H:=h+l\in \mathrm{har}(O)$, then, by (3.15), we obtain exactly (3.3), as well as (3.2), with the only difference being that in (3.2) we have $\mathsf{S}\overset{(3.8S)}{\supset} S$ instead of S. Moreover, it immediately follows from representation (3.3) and the condition $p=q$ on $\mathsf{S}\setminus S\overset{(3.8S)}{\subset} O\setminus S$ that $\mathrm{pt}_{\varDelta_p}=\mathrm{pt}_{\varDelta_q}$ on $\mathbb{R}^d\setminus S=(\mathbb{R}^d\setminus \mathsf{S})\bigcup(\mathsf{S}\setminus S)$. □

3.2 Duality between balayage of measures and their potentials

In this subsection, the equivalence of the first four statements of the Main Lemma according to the scheme

$$\begin{array}{ccc} \mathrm{I} & \longrightarrow & \mathrm{II}\\ & \nwarrow & \downarrow\\ \mathrm{IV} & \longleftrightarrow & \mathrm{III}\end{array}\tag{3.16}$$

will be established. We write A$\overset{proof}{\Longrightarrow}$B if the implication A⇒B is proved or discussed below.

I$\overset{proof}{\Longrightarrow}$II. By Proposition 1.1(i-ii) and [16, Theorem 1.7], if $h\in\mathrm{har}(S_O)$ is harmonic on the inward filling $S_O\overset{(2.1)}{=}\text{in-fill}(\operatorname{supp}\varDelta\cup\operatorname{supp}\omega)=\text{in-fill}\,S_O\Subset O$ of S, then there are functions $h_k\underset{k\in\mathbb{N}}{\in}\mathrm{har}(O)$ such that the sequence $(h_k)_{k\in\mathbb{N}}$ converges to h in the space $C(S_O)$ of all continuous functions on the compact set $S_O\Subset O$ with sup-norm. Hence,

$$\int_{S_O}h\,\mathrm{d}\varDelta=\int_{S_O}\lim_{k\to\infty}h_k\,\mathrm{d}\varDelta=\lim_{k\to\infty}\int_{S_O}h_k\,\mathrm{d}\varDelta=\lim_{k\to\infty}\int_O h_k\,\mathrm{d}\varDelta$$
$$\overset{\mathrm{I},(1.8)}{=}\lim_{k\to\infty}\int_O h_k\,\mathrm{d}\omega=\lim_{k\to\infty}\int_{S_O}h_k\,\mathrm{d}\omega=\int_{S_O}\lim_{k\to\infty}h_k\,\mathrm{d}\omega=\int_{S_O}h\,\mathrm{d}\varDelta.$$

Statement II of the Main Lemma is established.

II$\overset{proof}{\Longrightarrow}$III. If $x\notin S_O$, then the subharmonic function

$$u_x\colon y\underset{y\in\mathbb{R}^d}{\longmapsto}K_{d-2}(y,x)\tag{3.17}$$

is harmonic on S_O. Hence, for $x\notin S_O$,

$$\mathrm{pt}_\varDelta(x)=\int_{\operatorname{supp}\varDelta}K_{d-2}(\cdot,x)\,\mathrm{d}\varDelta=\int_{S_O}K_{d-2}(\cdot,x)\,\mathrm{d}\varDelta\overset{(3.17)}{=}\int_{S_O}u_x\,\mathrm{d}\varDelta$$
$$\overset{\mathrm{II},(1.8)}{=}\int_{S_O}u_x\,\mathrm{d}\omega\overset{(3.17)}{=}\int_{S_O}K_{d-2}(\cdot,x)\,\mathrm{d}\omega=\int_{\operatorname{supp}\omega}K_{d-2}(\cdot,x)\,\mathrm{d}\varDelta=\mathrm{pt}_\omega(x).$$

The statement III of the Main Lemma is established.

III$\overset{proof}{\Longrightarrow}$IV. This implication is obvious if we choose $p:=\mathrm{pt}_\omega$ and $q:=\mathrm{pt}_\varDelta$.

IV$\overset{proof}{\Longrightarrow}$III. This implication is a special case of Theorem 3.1 with the conclusion (3.2).

III$\overset{proof}{\Longrightarrow}$I. We use the following

Lemma 3.2 ([16, Lemma 1.8]). *Let F be a compact subset in $\mathbb{R}^d$, $h \in \operatorname{har}(F)$, and $b \in \mathbb{R}^+ \setminus 0$. Then there are points $y_1, y_2, \dots, y_m$ in $\mathbb{R}^d \setminus F$ and constants $a_1, a_2, \dots, a_m \in \mathbb{R}$ such that*

$$\Bigl|h(x) - \sum_{j=1}^{m} a_j k_{d-2}\bigl(|x-y_j|\bigr)\Bigr| < b \quad \text{for all } x \in F. \tag{3.18}$$

Applying Lemma 3.2 to the compact set $F := S_O \Subset O$ and a function $h \in \operatorname{har}(O)$, we obtain

$$\begin{aligned}\Bigl|\int_O h\,\mathrm{d}(\omega-\Delta)\Bigr| &= \Bigl|\int_{S_O} h\,\mathrm{d}(\omega-\Delta)\Bigr| \overset{\text{III}}{=} \Bigl|\int_{S_O} h\,\mathrm{d}(\omega-\Delta) - \sum_{j=1}^{m} a_j \bigl(\overbrace{\operatorname{pt}_\omega(y_j) - \operatorname{pt}_\Delta(y_j)}^{0}\bigr)\Bigr| \\ &\overset{(1.9)}{=} \Bigl|\int_{S_O} h\,\mathrm{d}(\omega-\Delta) - \sum_{j=1}^{m} a_j\Bigl(\int_{S_O} K_{d-2}(y,y_j)\,\mathrm{d}\omega(y) - \int_{S_O} K_{d-2}(y,y_j)\,\mathrm{d}\Delta(y)\Bigr)\Bigr| \\ &\overset{(1.2\mathrm{K})}{=} \Bigl|\int_{S_O} h(y)\,\mathrm{d}(\omega-\Delta)(y) - \int_{S_O}\sum_{j=1}^{m} a_j k_{d-2}\bigl(|y-y_j|\bigr)\,\mathrm{d}(\omega-\Delta)(y)\Bigr| \\ &\overset{(3.18)}{\leq} \sup_{y\in S_O}\Bigl|h(y) - \sum_{j=1}^{m} a_j k_{d-2}\bigl(|y-y_j|\bigr)\Bigr|\bigl(\omega(O)+\Delta(O)\bigr) \overset{(3.18)}{\leq} b\bigl(\omega(O)+\Delta(O)\bigr)\end{aligned}$$

for each $b \in \mathbb{R}^+ \setminus 0$. Hence $\Delta \preceq_{\operatorname{har}(O)} \omega$. Thus, we obtain Statement I and complete the proof of implications (3.16).

3.3 The symmetric Poisson – Jensen formula for measures and their potentials

In this subsection, we complete the proof of the Main Lemma by establishing the implications

$$\bigl(\text{II} \cap \text{III}\bigr) \longrightarrow \text{V} \longrightarrow \text{VI} \longrightarrow \text{VII} \longrightarrow \text{IV}, \tag{3.19}$$

where $\text{II} \cap \text{III}$ means that Statements II and III are simultaneously satisfied, and the equivalence $(\text{II} \cap \text{III}) \Leftrightarrow \text{IV}$ of the extreme statements $(\text{II} \cap \text{III})$ and IV of (3.19) has already been proved in the previous Subsection 3.2; see (3.16).

$(\text{II} \cap \text{III}) \overset{proof}{\Longrightarrow} \text{V}$. Let $u \overset{(2.2\mathrm{u})}{\in} \operatorname{sbh}_*(\operatorname{clos} B)$, where $S_O \overset{(2.2\mathrm{B})}{\subset} B \Subset O$. We can choose an open set O' such that $B \Subset O' \Subset O$ and $u \in \operatorname{sbh}_*(\operatorname{clos} O')$. Consider first the case

$$-\infty < \int u\,\mathrm{d}\Delta, \quad \text{where } \operatorname{supp}\Delta \overset{(2.1)}{\subset} S_O \Subset O'. \tag{3.20}$$

Let

$$\mu' := \Delta_u \big|_{\operatorname{clos} O'} \tag{3.21}$$

be the restriction of Riesz measure of $u \in \operatorname{sbh}_*(\operatorname{clos} O')$ to $\operatorname{clos} O' \Subset O$. By the Riesz Decomposition Theorem [50, Theorem 3.7.1], [22, Theorem 3.9], [2, Theorem 4.4.1], [23, Theorem 6.18] we obtain the representation

$$u = \operatorname{pt}_{\mu'} + h \quad \text{on } O', \quad \text{where } h \in \operatorname{har}(O') \text{ is continuous and bounded on } S_O. \tag{3.22}$$

Integrating this representation with respect to $\mathrm{d}\omega$ and $\mathrm{d}\Delta$, we obtain

$$\int u\,\mathrm{d}\omega \overset{(3.22)}{=} \int \operatorname{pt}_{\mu'}\,\mathrm{d}\omega + \int h\,\mathrm{d}\omega, \quad \operatorname{supp}\omega \overset{(2.1)}{\subset} S_O, \tag{3.23ω}$$

$$\int u\,\mathrm{d}\Delta \overset{(3.22)}{=} \int \operatorname{pt}_{\mu'}\,\mathrm{d}\Delta + \int h\,\mathrm{d}\Delta, \quad \operatorname{supp}\Delta \overset{(2.1)}{\subset} S_O, \tag{3.23Δ}$$

where the three integrals in (3.23$\varDelta$) are finite, although in equality (3.23ω) the first two integrals can take simultaneously the value of $-\infty$, but the last integral in (3.23ω) is finite. Therefore, the difference (3.23ω)$-$(3.23$\varDelta$) of these two equalities is well defined:

$$\int u \,\mathrm{d}\omega - \int u \,\mathrm{d}\varDelta \overset{(3.23)}{=} \int \mathrm{pt}_{\mu'} \,\mathrm{d}\omega - \int \mathrm{pt}_{\mu'} \,\mathrm{d}\varDelta + \int_{S_O} h \,\mathrm{d}(\omega - \varDelta), \tag{3.24}$$

where the first and third integrals can simultaneously take the value of $-\infty$, and the remaining integrals are finite. By Statement II we have $\varDelta \simeq_{\mathrm{har}(S_O)} \omega$. Hence the *last integral* in (3.24) *vanishes* according to (1.8). Using Fubini's Theorem on repeated integrals, in view of the symmetry property of the kernel K_{d-2} in (1.2K), we have

$$\int \mathrm{pt}_{\mu'} \,\mathrm{d}\varDelta = \int\int K_{d-2}(y,x) \,\mathrm{d}\mu'(y) \,\mathrm{d}\varDelta(x)$$
$$= \int\int K_{d-2}(x,y) \,\mathrm{d}\varDelta(x) \,\mathrm{d}\mu'(y) \overset{(3.21)}{=} \int_{\operatorname{clos} O'} \mathrm{pt}_{\varDelta} \,\mathrm{d}\varDelta_u, \tag{3.25}$$

and in the same way

$$\int \mathrm{pt}_{\mu'} \,\mathrm{d}\omega = \int\int K_{d-2}(y,x) \,\mathrm{d}\mu'(y) \,\mathrm{d}\omega(x)$$
$$= \int\int K_{d-2}(x,y) \,\mathrm{d}\omega(x) \,\mathrm{d}\mu'(y) \overset{(3.21)}{=} \int_{\operatorname{clos} O'} \mathrm{pt}_{\omega} \,\mathrm{d}\varDelta_u \tag{3.26}$$

even if the integral on the left side of equalities (3.26) takes the value $-\infty$ because the integrand $K_{d-2}(\cdot,\cdot)$ is bounded from above on the compact set $\operatorname{clos} O' \times \operatorname{clos} O'$ [22, Theorem 3.5]. Hence equality (3.24) can be rewritten as

$$\int u \,\mathrm{d}\omega - \int u \,\mathrm{d}\varDelta = \int_{\operatorname{clos} O'} \mathrm{pt}_{\omega} \,\mathrm{d}\varDelta_u - \int_{\operatorname{clos} O'} \mathrm{pt}_{\varDelta} \,\mathrm{d}\varDelta_u$$

or in the form

$$\int u \,\mathrm{d}\omega + \int_{\operatorname{clos} O'} \mathrm{pt}_{\varDelta} \,\mathrm{d}\varDelta_u = \int u \,\mathrm{d}\varDelta + \int_{\operatorname{clos} O'} \mathrm{pt}_{\omega} \,\mathrm{d}\varDelta_u. \tag{3.27}$$

But Statement III, we have

$$\mathrm{pt}_{\omega} \overset{\mathrm{III}}{=} \mathrm{pt}_{\varDelta} \quad \text{on } \mathbb{R}^d \setminus S_O \supset \operatorname{clos} O' \setminus B.$$

Hence, by equality (3.27), we obtain equality (2.2f) in the case (3.20).

If condition (3.20) is not fulfilled, then from representation (3.23$\varDelta$) it follows that the integral on the left-hand side of (3.25) also takes the value $-\infty$. Equalities (3.25) is still true [22, Theorem 3.5]. Hence, the second integral on the right side of the formula (2.2f) also takes the value $-\infty$ and this formula (2.2f) remains true.

V$\overset{proof}{\Longrightarrow}$VI. Let $q \in \mathrm{sbh}_*(S_O)$ be a function with the Riesz measure $\varDelta_q = \varDelta$. Then there is a function $h \in \mathrm{har}(O)$ such that $q = \mathrm{pt}_{\varDelta} + h$ on O. By Statement V, we have (2.2f) for $B = S_O$. If we set $p := \mathrm{pt}_{\omega} + h$, then $\varDelta_p = \omega$, and (2.3) follows from (2.2f) with $B = S_O$.

VI$\overset{proof}{\Longrightarrow}$VII. We set $q := \mathrm{pt}_{\varDelta} \in \mathrm{sbh}_*(\mathbb{R}^d)$ with $\varDelta_q = \varDelta$. By Statement VI, there is a function $p \in \mathrm{sbh}_*(\mathbb{R}^d)$ with $\varDelta_p = \omega$ such that we have (2.3). In particular, the equality in (2.3) is true for each special subharmonic function $u_x \colon y \underset{y \in \mathbb{R}^d}{\longmapsto} K_{d-2}(y,x)$, $x \in \mathbb{R}^d$, and we obtain Statement VII.

VII$\overset{proof}{\Longrightarrow}$IV. Each special function u_x in Statement VII is subharmonic on $\mathbb{R}^d$ with Riesz measure δ_x. If $x\in O\setminus S$, where $S_O\subset S\Subset O$, then $S_O\cap\operatorname{supp}\delta_x=\varnothing$. Thus,

$$\int_{S_O} p\,\mathrm{d}\delta_x=\int_{S_O} q\,\mathrm{d}\delta_x=0\quad\text{for each } x\in O\setminus S. \tag{3.28}$$

Hence, by (2.3) with u_x instead of u, we obtain

$$\mathrm{pt}_{\varDelta}(x)\;=\;\int_{S_O}K(y,x)\,\mathrm{d}\varDelta(y)\;=\;\int_{S_O}u_x\,\mathrm{d}\varDelta\;\overset{(2.3),(3.28)}{=}\;\int_{S_O}u_x\,\mathrm{d}\omega\;=\;\int_{S_O}K(y,x)\,\mathrm{d}\omega(y)\;=\;\mathrm{pt}_{\omega}(x)$$

for each $x\in O\setminus S$. Thus, we obtain Statement IV for $q:=\mathrm{pt}_{\varDelta}$ and $p:=\mathrm{pt}_{\omega}$.

The Main Lemma is proved.

4 Proof of the main theorem

[I]$\overset{proof}{\Longrightarrow}$[II]. Without loss of generality, we can assume that $S=S_O$ in (2.4S). Then Statement [II] with (2.5) follows from Theorem 3.1 with (3.2)–(3.3).

[II]$\overset{proof}{\Longrightarrow}$[III]. By the equality $\mathrm{pt}_{\varDelta_q}\overset{(2.5)}{=}\mathrm{pt}_{\varDelta_p}$ on $\mathbb{R}^d\setminus S_O$, we have Statement III of the Main Lemma for $\varDelta:=\varDelta_q\in\operatorname{Meas}^+(S)$ and $\omega:=\varDelta_p\in\operatorname{Meas}^+(S)$. By implication III$\Rightarrow$V of the Main Lemma, we obtain

$$\int_{\operatorname{supp}\varDelta_q}u\,\mathrm{d}\varDelta_q+\int_B\mathrm{pt}_{\varDelta_p}\,\mathrm{d}\varDelta_u\overset{(2.2)}{=}\int_{\operatorname{supp}\varDelta_p}u\,\mathrm{d}\varDelta_p+\int_B\mathrm{pt}_{\varDelta_q}\,\mathrm{d}\varDelta_u \tag{4.1}$$

for each $B\in\operatorname{Bor}(\mathbb{R}^d)$ under $S_O\subset B\Subset O$ and for each $u\in\operatorname{sbh}_*(\operatorname{clos}B)$, where we returned to the separate notation $S\subset S_O:=\text{in-fill}\,S$. Obviously,

$$\int_B h\,\mathrm{d}\varDelta_u=\int_B h\,\mathrm{d}\varDelta_u\quad\text{for each } u\in\operatorname{sbh}_*(\operatorname{clos}B)\text{ and } h\in\operatorname{har}(O). \tag{4.2}$$

Adding (4.1) and (4.2), according to representations (2.5) of q and p, we obtain

$$\int_S u\,\mathrm{d}\varDelta_q+\int_B p\,\mathrm{d}\varDelta_u\overset{(2.2)}{=}\int_S u\,\mathrm{d}\varDelta_p+\int_B q\,\mathrm{d}\varDelta_u, \tag{4.3}$$

where B can be replaced with $B\cap S_{\neq}$. This proves (2.6f) already for a set B and functions u of form (2.6B). Thus, we obtain Statement [III].

[III]$\overset{proof}{\Longrightarrow}$[IV]. All functions u_x in Statement [IV] are subharmonic on $\mathbb{R}^d\supset O$.

[IV]$\overset{proof}{\Longrightarrow}$[I]. The Riesz measure of u_x is the Dirac measure δ_x, and, by Statement [IV],

$$\int_S u_x\,\mathrm{d}\varDelta_q+\int_{B_j}p\,\mathrm{d}\delta_x\overset{(2.6f)}{=}\int_S u_x\,\mathrm{d}\varDelta_p+\int_{B_j}q\,\mathrm{d}\delta_x\quad\text{for each } j\in\mathbb{N}\text{ and } x\in O. \tag{4.4}$$

If $j=0$ and $x\notin S_O=B_0$, then $\operatorname{supp}\delta_x=x\notin S_O$ and

$$\int_{S_O}p\,\mathrm{d}\delta_x\overset{(4.4)}{=}\int_{S_O}q\,\mathrm{d}\delta_x=0$$

These equalities do not depend on $j\in\mathbb{N}_0$ for points $x\notin S_O$. Hence

$$\int_S u_x\,\mathrm{d}\varDelta_q\overset{(4.4)}{=}\int_S u_x\,\mathrm{d}\varDelta_p\quad\text{for each } j\in\mathbb{N}_0\text{ and } x\notin S_O\supset S.$$

Therefore, it is follows from (4.4) that

$$\int_{B_j}p\,\mathrm{d}\delta_x\overset{(4.4)}{=}\int_{B_j}q\,\mathrm{d}\delta_x\quad\text{for each } j\in\mathbb{N}_0\text{ and } x\notin S_O,$$

i.e., $p(x)=q(x)$ for each $j\in\mathbb{N}_0$ and for every $x\in B_j\setminus S_O$. Thus, $p(x)=q(x)$ for each point $x\in\bigcup_{j\in\mathbb{N}_0}B_j\setminus S_O=O\setminus S_O$, and Statement [I] is established.

5 Duality theorems for balayage

Part of some equivalences of the Main Lemma and the Main Theorem allows us to give an internal dual description for the potentials of measures obtained through the balayage processes. Such descriptions in particular cases of Arens – Singer and Jensen measures and their potentials have already found important applications in the study of various problems of function theory [15, Chapter 3 etc.], [1], [24], [25], [26], [27], [28], [29], [44], [30], [31], [33], [34], [51], [32], [35], [36], [37], [7], [41], [42], [39].

Duality Theorem 1 (for $\mathrm{har}(O)$-balayage)**.** *Let* $\varDelta \in \mathrm{Meas}^+_{\mathrm{cmp}}(O)$.

If a measure $\omega \in \mathrm{Meas}^+_{\mathrm{cmp}}(O)$ *is a* $\mathrm{har}(O)$*-balayage of* $\varDelta$*, then* (cf. (2.11))

$$\mathrm{pt}_\omega \in \mathrm{sbh}_*(\mathbb{R}^d) \cap \mathrm{har}(\mathbb{R}^d \setminus \mathrm{supp}\,\omega), \tag{5.1p}$$

$$\mathrm{pt}_\omega = \mathrm{pt}_\varDelta \ \text{ on } \mathbb{R}^d \setminus \text{in-fill}_O(\mathrm{supp}\,\varDelta \cup \mathrm{supp}\,\omega). \tag{5.1=}$$

Conversely, suppose that there are a compact subset $S \Subset O$ *and a function* p *such that*

$$p \overset{cf.(5.1p)}{\in} \mathrm{sbh}(O) \cap \mathrm{har}(O \setminus S), \tag{5.2p}$$

$$p \overset{cf.(5.1=)}{=} \mathrm{pt}_\varDelta \quad \text{on } O \setminus S. \tag{5.2=}$$

Then the Riesz measure

$$\omega \overset{(1.6)}{:=} c_d \triangle p \overset{(5.2p)}{\in} \mathrm{Meas}^+(S) \subset \mathrm{Meas}^+_{\mathrm{cmp}}(O) \tag{5.3}$$

of this function p *is a* $\mathrm{har}(O)$*-balayage of the measure* $\varDelta$.

Proof. Properties (5.1) for $\omega \succeq_{\mathrm{har}(O)} \varDelta$ directly follow from the implication I⇒III of the Main Lemma. In the opposite direction, we can use the implication IV⇒I of the Main Lemma with p from (5.2) and $q := \mathrm{pt}_\varDelta$. □

Duality Theorem 2 (for $\mathrm{sbh}(O)$-balayage)**.** *Let* $\varDelta \in \mathrm{Meas}^+_{\mathrm{cmp}}(O)$. *If* $\omega \succeq_{\mathrm{sbh}(O)} \varDelta$*, then we have* (5.1) *and* $\mathrm{pt}_\omega \geq \mathrm{pt}_\varDelta$ *on* $\mathbb{R}^d$. *Conversely, suppose that there are a compact subset* S *in* O *containing* $\mathrm{supp}\,\varDelta$*, and a function* p *satisfying* (5.2) *such that*

$$p \geq \mathrm{pt}_\varDelta \qquad \text{on } S_O := \text{in-fill}(S). \tag{5.4}$$

Then the Riesz measure (5.3) *of this function* p *is a* $\mathrm{sbh}(O)$*-balayage of* $\varDelta$.

Proof. If $\varDelta \preceq_{\mathrm{sbh}(O)} \omega$, then $\varDelta \preceq_{\mathrm{har}(O)} \omega$, which was noted earlier in Subsection 1.2 (Property 4), and, by Duality Theorem 1, we obtain (5.1). Besides, functions $y \underset{y \in \mathbb{R}^d}{\longmapsto} K_{d-2}(y,x)$ are subharmonic on $\mathbb{R}^d$ for each $x \in \mathbb{R}^d$, and

$$\mathrm{pt}_\varDelta(x) = \int K_{d-2}(y,x)\,\mathrm{d}\varDelta(y) \overset{(1.7)}{\leq} \int K_{d-2}(y,x)\,\mathrm{d}\omega(y) = \mathrm{pt}_\omega(x) \quad \text{for each } x \in \mathbb{R}^d.$$

In the opposite direction, we set $q := \mathrm{pt}_\varDelta \in \mathrm{sbh}_*(\mathbb{R}^d) \cap \mathrm{har}(O \setminus S)$. By Duality Theorem 1, the Riesz measure $\varDelta_p \overset{(5.3)}{=} \omega \in \mathrm{Meas}^+_{\mathrm{cmp}}(O)$ of the function p is a $\mathrm{har}(O)$-balayage of $\varDelta$. By condition (5.2=) in notation (5.4), we have the equality $p = q$ on $O \setminus S_O \subset O \setminus S$, and, by condition (5.2p), the functions p and q are harmonic on $O \setminus S$. Thus, Statement [I] of the Main Theorem is fulfilled. By the implication [I]⇒[III] of the Main Theorem, using full symmetric Poisson – Jensen formula (2.6f) with $B \overset{(2.6B)}{:=} S_O$, we get

$$\int_S u\,\mathrm{d}\varDelta_q + \int_{S_O} p\,\mathrm{d}\varDelta_u \overset{(2.6f)}{=} \int_S u\,\mathrm{d}\varDelta_p + \int_{S_O} q\,\mathrm{d}\varDelta_u \quad \text{for each } u \overset{(2.6B)}{\in} \mathrm{sbh}_*(O). \tag{5.5}$$

Hence, by the condition $p \overset{(5.4)}{\geq} \mathrm{pt}_{\varDelta} = q$ on S_O, we obtain

$$\int_O u\,\mathrm{d}\varDelta + \int_{S_O} q\,\mathrm{d}\varDelta_u = \int_S u\,\mathrm{d}\varDelta_q + \int_{S_O} q\,\mathrm{d}\varDelta_u \leq \int_S u\,\mathrm{d}\varDelta_q + \int_{S_O} p\,\mathrm{d}\varDelta_u$$
$$\overset{(5.5)}{=} \int_S u\,\mathrm{d}\varDelta_p + \int_{S_O} q\,\mathrm{d}\varDelta_u = \int_O u\,\mathrm{d}\omega + \int_{S_O} q\,\mathrm{d}\varDelta_u \quad \text{for each } u \in \mathrm{sbh}_*(O). \tag{5.6}$$

In particular, if $u \in \mathrm{sbh}_*(O) \cap C^\infty(O)$, then the function q is $\varDelta_u$-integrable on S_O, and it is follows from (5.6) that

$$\int_O u\,\mathrm{d}\varDelta \overset{(5.6)}{\leq} \int_O u\,\mathrm{d}\omega \quad \text{for each } u \in \mathrm{sbh}_*(O) \cap C^\infty(O).$$

Hence, by Subsection 1.2 (Property 5), we obtain $\varDelta \preceq_{\mathrm{sbh}(O)} \omega$. □

The following long-known result for Arens – Singer and Jensen measures and their potentials on domains in $\mathbb{R}^d$ with $d \geq 2$ has found numerous applications in the theory of functions of one and several complex variables, which is partially reflected in the bibliographic sources listed at the beginning of Section 5. The proof of this result immediately follows from Duality Theorems 1–2, but already for open sets $O \subset \mathbb{R}^d$.

Duality Theorem 3 ([35, Proposition 1.4, Duality Theorem]). *Let $x \in O \subset \mathbb{R}^d$, $d \in \mathbb{N}$. The map*

$$\mathcal{P}_x \colon \omega \overset{(1.9)}{\longmapsto} \mathrm{pt}_{\omega - \delta_x} \tag{5.7}$$

defines an affine bijection from $AS_x(O)$ onto $ASP_x(O)$, as well as from $J_x(O)$ onto $JP_x(O)$ (see also, in addition, (2.11)) *with the inverse map*

$$\mathcal{P}_x^{-1} \colon V \overset{(1.6)}{\longmapsto} c_d \triangle V \,\Big|_{\mathbb{R}^d \setminus x} + \left(1 - \limsup_{x \neq y \to x} \frac{V(y)}{-K_{d-2}(x,y)}\right) \cdot \delta_x. \tag{5.8}$$

Remark 2. Theorems 1 and 2 can also be formulated in a form close to Theorem 3, using some affine bijection of type (5.7)–(5.8) and definitions of the generalized Arens – Singer and Jensen potentials. But such formulations require some development of the theory of δ-subharmonic functions [3], [4], [40, Section 3.1] and a delicate approach to upper/lower integrals (1.5) with values in $\overline{\mathbb{R}}$. We will not discuss similar interpretations of Theorems 1 and 2 here.

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