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E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 562
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 75th birthday)

Tynysbek Sharipovich Kal'menov was born in the village of Koksaek of the Tolebi district of the Turkestan region (earlier it was the Lenger district of the South-Kazakhstan region of the Kazakh SSR). Although "according to the passport" his birthday was recorded on May 5, his real date of birth is April 6, 1946.

Tynysbek Kal'menov is a graduate of the Novosibirsk State University (1969), and a representative of the school of A.V. Bitsadze, an outstanding scientist, corresponding member of the Academy of Sciences of the USSR. In 1972, he completed his postgraduate studies at the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR. In 1983, he defended his doctoral thesis at the M.V. Lomonosov Moscow State University. Since1989, he is a corresponding member of the Academy of Sciences of the Kazakh SSR, and since 2003, he is an academician of the National Academy of Sciences of the Republic of Kazakhstan.

Tynysbek Kal'menov worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (1972-1985). From 1986 to 1991, he was the dean of the Faculty of Mathematics of Al-Farabi Kazakh State University. From 1991 to 1997, he was the rector of the Kazakh Chemical-Technological University (Shymkent).

From 2004 to 2019, Tynysbek Kal'menov was the General Director of the Institute of Mathematics and Mathematical Modeling. He made it one of the leading scientific centers in the country and the best research institute in Kazakhstan. It suffices to say, that in terms of the number of scientific publications (2015-2018) in international rating journals indexed in the Web of Science, the Institute of Mathematics and Mathematical Modeling was ranked fourth among all Kazakhstani organizations, behind only three large universities: the Nazarbaev University, Al-Farabi National University and L.N. Gumilyov Eurasian National University.

Since 2019, Tynysbek Kal'menov has been working as the head of the Department of Differential Equations of the Institute of Mathematics and Mathematical Modeling. He is a member of the National Scientific Council "Scientific Research in the Field of Natural Sciences", which is the main Kazakhstan council that determines the development of science in the country.

T.Sh. Kal'menov was repeatedly elected to maslikhats of various levels, was a member of the Presidium of the Committee for Supervision and Attestation in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan. He is a Laureate of Lenin Komsomol Prize of the Kazakh SSR (1978), an Honored Worker of Science and Technology of Kazakhstan (1996), awarded with the order "Kurmet" (2008 Pi.) and jubilee medals.

In 2013, he was awarded the State Prize of the Republic of Kazakhstan in the field of science and technology for the series of works "To the theory of initial- boundary value problems for differential equations".

The main areas of scientific interests of academician Tynysbek Kal'menov are differential equations, mathematical physics and operator theory. He has obtained fundamental scientific results, many of which led to the creation of new scientific directions in mathematics.

Tynysbek Kal'menov, using a new maximum principle for an equation of mixed type (Kal'menov's maximum principle), was the first to prove that the Tricomi problem has an eigenfunction, thus he solved the famous problem of the Italian mathematician Francesco Tricomi, set in 1923 This marked the beginning of a new promising direction, that is, the spectral theory of equations of mixed type.

He established necessary and sufficient conditions for the well-posed solvability of the classical Darboux and Goursat problems for strongly degenerate hyperbolic equations.

Tynysbek Kal'menov solved the problem of completeness of the system of root functions of the nonlocal Bitsadze-Samarskii problem for a wide class of multidimensional elliptic equations. This result is final and has been widely recognized by the entire mathematical community.

He developed a new effective method for studying ill-posed problems using spectral expansion of differential operators with deviating argument. On the basis of this method, he found necessary and sufficient conditions for the solvability of the mixed Cauchy problem for the Laplace equation.

Tynysbek Kal'menov was the first to construct boundary conditions of the classical Newton potential. That is a fundamental result at the level of a classical one. Prior to the research of Kal'menov T.Sh., it was believed that the Newton potential gives only a particular solution of an inhomogeneous equation and does not satisfy any boundary conditions. Thanks for these results, for the first time, it was possible to construct the spectral theory of the classical Newton potential.

He developed a new effective method for constructing Green's function for a wide class of boundary value problems. Using this method, Green's function of the Dirichlet problem was first constructed explicitly for a multidimensional polyharmonic equation.

From 1989 to 1993, Tynysbek Kal'menov was the chairman of the Inter- Republican (Kazakhstan, Uzbekistan, Kyrgyzstan, Turkmenistan, Tajikistan) Dissertation Council. He is a member of the International Mathematical Society and he repeatedly has been a member of organizing committee of many international conferences. He carries out a lot of organizational work in training of highly qualified personnel for the Republic of Kazakhstan and preparing international conferences. Under his direct guidance, the First Congress of Mathematicians of Kazakhstan was held. He presented his reports in Germany, Poland, Great Britain, Sweden, France, Spain, Japan, Turkey, China, Iran, India, Malaysia, Australia, Portugal and countries of CIS.

In terms of the number of articles in scientific journals with the impact- factor Web of Science, in the research direction of "Mathematics", the Institute of Mathematics and Mathematical Modeling is on one row with leading mathematical institutes of the Russian Federation, and is ahead of all mathematical institutes in other CIS countries in this indicator.

Tynysbek Kal'menov is one of the few scientists who managed to leave an imprint of their individuality almost in all branches of mathematics in which he has been engaged.

Tynysbek Kal'menov has trained 11 doctors and more than 60 candidate of sciences and PhD, has founded a large scientific school on equations of mixed type and differential operators recognized all over the world. Many of his disciples are now independent scientists recognized in the world of mathematics.

He has published over 150 scientific articles, most of which are published in international mathematical journals, including 14 articles published in "Doklady AN SSSR/ Doklady Mathematics". In the last 5 years alone (2016-2020), he has published more than 30 articles in scientific journals indexed in the Web of Science database. To date, academician Tynysbek Kal'menov has a Hirsch index of 18 in the Web of Science and Scopus databases, which is the highest indicator among all Kazakhstan mathematicians.

Outstanding personal qualities of academician Tynysbek Kalmenov, his high professional level, adherence to principles of purity of science, high exactingness towards himself and his colleagues, all these are the foundations of the enormous authority that he has among Kazakhstan scientists and mathematicians of many countries.

Academician Tynysbek Sharipovich Kalmenov meets his 75th birthday in the prime of his life, and the mathematical community, many of his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal heartily congratulate him on his jubilee and wish him good health, happiness and new successes in mathematics and mathematical education, family well-being and long years of fruitful life.

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DETERMINATION OF DENSITY OF ELLIPTIC POTENTIAL

T.Sh. Kalmenov, A.K. Les, U.A. Iskakova

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Key words: Helmholtz potential, fundamental solution of Helmholtz equation, potential density, potential boundary condition, inverse problem.

AMS Mathematics Subject Classification: 47F05, 35P10.

Abstract. In this paper, using techniques of finding boundary conditions for the volume (Newton) potential, we obtain the boundary conditions for the volume potential

$$u(x) = \int_{\Omega} \varepsilon(x,\xi)\rho(\xi)\,d\xi,$$

where $\varepsilon(x,\xi)$ is the fundamental solution of the following elliptic equation

$$L(x,D)\varepsilon(x,\xi) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} \varepsilon(x,\xi) + a(x)\varepsilon(x,\xi) = \delta(x,\xi).$$

Using the explicit boundary conditions for the potential $u(x)$, the density $\rho(x)$ of this potential is uniquely determined. Also, the inverse Sommerfeld problem for the Helmholtz equation is considered.

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1 Introduction

It is known that the continuous distribution of masses and charge in a bounded domain $\Omega \subset \mathbb{R}^n$ generates the linear Newton potential [8] by the formula

$$u(x) = \int_{\Omega} \varepsilon(x-\xi)\rho(\xi)\,d\xi, \tag{1.1}$$

where $\varepsilon(x-\xi)$ is the fundamental solution of the Laplace equation

$$-\Delta_x \varepsilon(x) = \delta(x).$$

The problem of uniquely finding the density $\rho(x)$ from a given potential $u(x)$ is ill-posed, since for the necessary smoothness $u(x)$, it is still necessary to determine the unknown boundary condition of the volume potential, i.e. boundary condition of the integral (1.1).

In the work of T.Sh. Kal'menov and D. Suragan [8], the boundary conditions of the volume (Newton) potential were found for the first time, and in the works of I.V. Bezmenov [1], T.Sh. Kal'menov, D. Suragan [9], T.Sh. Kal'menov, M. Otelbaev, G. Arepova [7], the boundary conditions of the Helmholtz potential were studied, in [1], the boundary conditions were given in an approximate form, and in [9, 7] the boundary conditions of the Helmholtz operator were written out in an explicit form. These boundary conditions have the property that stationary waves arriving at the boundary $\partial\Omega$ from Ω pass through $\partial\Omega$ without reflection, i.e. boundary conditions are transparent.

2 Preliminaries

In a domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega \in \mathrm{C}^2$, we consider the following elliptic potential

$$u(x) = \int\limits_{\Omega} \varepsilon(x,\xi)\rho(\xi)d\xi, \tag{2.1}$$

where $\varepsilon(x,\xi)$ is the fundamental solution of the second order linear elliptic equation, i.e.

$$L(x,D)u(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u(x) + a(x)u = \rho(x), \tag{2.2}$$

where

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2, \quad \delta > 0,$$

$|\xi|^2 = \sum\limits_{i,j=1}^{n} \xi_i^2$, $a_{ij}(x) \in C^3(\bar{\Omega})$, $a(x) \in C^2(\bar{\Omega})$, $a(x) \geq 0$.

Now we give a brief scheme for constructing the fundamental solution of equation (2.2) according to the scheme proposed by A.V. Bitsadze [2].

Denote by A_{ij} the division ratios of the algebraic complement of the elements a_{ij} of the matrix $\|a_{ij}\|$ of the leading coefficients of equation (2.2) in the determinant $a = \det \|a_{ij}\|$.

We introduce the function:

$$\sigma(x,\xi) = \sum_{i,j,\xi}^{n} A_{ij}(x)(x_i - \xi_i)(x_j - \xi_j),$$

where x and ξ are arbitrary points in Ω.

Suppose $a_{ij}(x) \in C^3(\bar{\Omega})$, $a(x) \in C^1(\bar{\Omega})$. It is known that (2.1) is uniformly elliptic and there are positive constants k_0 and k_1 such that

$$k_0|x-\xi|^2 \leq \sigma(x,\xi) \leq k_1|x-\xi|^2.$$

For $x \neq \xi$ we define the function

$$\psi(x,\xi) = \begin{cases} \sigma_0(\xi)\sigma(x,\xi)^{\frac{2-n}{2}}, \ n > 2, \\ -\frac{1}{2\pi\sigma_0\sqrt{a(\xi)}} \ln \sigma(x,\xi), \ n = 2, \end{cases} \tag{2.3}$$

where for $n > 2$ $\sigma_0(\xi) = \omega_n(n-2)(\sqrt{|a(\xi)|})^{-1}$, ω_n is the area of n-dimensional unit sphere, and $a(\xi)$ is the determinant of the system $\{a_{i,j}\}$.

Define the generalized potential with the density $\mu(x)$ as follows

$$W(x) = \int_{\Omega_0} \psi(x,\xi)\mu(\xi)d\xi,$$

where $\Omega_0 \subset \Omega$.

Using the properties of $\psi(x,\xi)$, it is easy to show that

$$LW(x) = -\mu(x) + \int_{\Omega_0} L\psi(x,\xi)\mu(\xi)d\xi, \tag{2.4}$$

where the second term in the right-hand side is an improper integral.

The solution of the equation

$$Lu = -\sum_{i,j}^{n} \frac{\partial}{\partial x_j} a_{ij}(x) \frac{\partial}{\partial x_i} + a(x)u = \rho$$

in Ω_0 has the following form

$$u(x) = \omega(x) + \int_{\Omega_0} \psi(x,\xi)\mu(\xi)d\xi, \tag{2.5}$$

where $\omega(x)$ is a differentiable function and $\mu(x)$ is the unknown function to be determined.

Applying the operator L to formula (2.5) and taking into account (2.1), we have

$$-\mu(x) + \int_{\Omega_0} k(x,\xi)\mu(\xi)d\xi = F(x), \tag{2.6}$$

where $k(x,\xi) = L\psi(x,\xi)$ and $F(x) = -L\omega(x) + f(x)$.

Integral equation (2.6) is a second-kind Fredholm integral equation, and for small Ω_0 it has a unique solution. Define $\varepsilon(x,y)$ as follows

$$\varepsilon(x,y) = \psi(x,y) + \int_{\Omega} \psi(x,\xi)\mu(\xi,y)d\xi, \tag{2.7}$$

where $\psi(x,\xi)$ is the function defined by equation (2.2), function $\mu(x)$ is a solution of equation (2.6), and $\omega(x) \equiv W(x), f(x) = 0$.

Let us check that $\varepsilon(x,y)$ is a fundamental solution of equation (2.2). Suppose that

$$u(x) = \int_{\Omega_0} \varepsilon(x,y)f(y)dy.$$

Then using the property of $\psi(x,y)$, we have

$$\begin{aligned} Lu =& L\int_{\Omega_0} \varepsilon(x,y)f(y)dy \\ =& L\int_{\Omega_0} \left[\psi(x,y) + \int_{\Omega_0} \psi(x,\xi)\mu(\xi,y)d\xi\right] f(y)dy \\ =& f(x) + \int_{\Omega_0} \left[L\psi(x,y) - \mu(x,y) + \int_{\Omega_0} L\psi(x,\xi)\mu(\xi,y)d\xi\right] f(y)dy. \end{aligned} \tag{2.8}$$

Further, we choose $\mu(x,y)$ as the solution of equation

$$L\psi(x,y) - \mu(x,y) + \int_{\Omega_0} L\psi(x,\xi)\mu(\xi,y)d\xi = 0,$$

which gives

$$Lu = L\int_{\Omega_0} \varepsilon(x,y)f(y)dy = f(x). \tag{2.9}$$

Thus, the fundamental solution of equation $Lu = p(x)$ can be represented by formula (2.7). In this case, the main properties of the fundamental solution coincide with the properties of the function $\psi(x,y)$ given by formula (2.3).

3 Main results

Problem N: Find the density $\rho(x)$ of volume elliptic potential (2.1).

It should be noted that equation (2.1) for finding $\rho(x)$ is an integral equation of the first kind, therefore, in general the inverse problem **N** is an ill-posed problem.

Theorem 3.1. *Let $\rho \in L_2(\Omega)$, then the elliptic potential defined by formula* (2.1) *satisfies the following boundary condition*

$$-\frac{u(x)}{2} + \int_{\partial\Omega} u(\xi) \sum_{i,j=1}^{n} n_i a_{ij}(\xi) \frac{\partial}{\partial \xi_j} \varepsilon(x,\xi) d\xi - \int_{\partial\Omega} \varepsilon(x,\xi) \sum_{i,j=1}^{n} n_i a_{ij}(\xi) \frac{\partial}{\partial \xi_j} u(\xi) d\xi = 0, \tag{3.1}$$

where $x \in \partial\Omega$.

Conversely, if a function $u \in W_2^2(\Omega)$ satisfies equation (2.2) and potential boundary condition (3.1), then $u(x)$ coincides with elliptic potential (2.1).

Proof. Substituting the differential expression

$$L(\xi, D)u(\xi) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(\xi) \frac{\partial}{\partial x_j} u(\xi) + a(\xi)u,$$

instead of the function $\rho(\xi)$ into equation (2.1), we get

$$\begin{aligned} u(x) &= \int_{\Omega} \varepsilon(x,\xi) L(\xi, D) u(\xi) d\xi \\ &= \int_{\Omega} \varepsilon(x,\xi) \Big(-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(\xi) \frac{\partial}{\partial x_j} + a \Big) u(\xi) d\xi \\ &= -\int_{\partial\Omega} \varepsilon(x,\xi) \sum_{i,j=1}^{n} n_i a_{ij}(\xi) \frac{\partial}{\partial \xi_j} u(\xi) d\xi + \int_{\partial\Omega} u(\xi) \sum_{i,j=1}^{n} n_j a_{ij}(\xi) \frac{\partial}{\partial \xi_i} \varepsilon(x,\xi) d\xi + \\ &\quad + \lim_{r \to 0} \Big(\int_{U_r(x)} u(\xi) L(\xi, D) \varepsilon(x,\xi) d\xi + \int_{\Omega / U_r(x)} u(\xi) L(\xi, D) \varepsilon(x,\xi) d\xi \Big), \quad x \in \Omega, \end{aligned} \tag{3.2}$$

where $U_r(x) = \{x \in \Omega, \ \| x - \xi \| < r\}$, and n_j are the directional cosines of the normal to the boundary $\partial\Omega$.

Since $x \neq \xi$, it is easy to check that $L(\xi, D)\varepsilon(x,\xi) \equiv 0$ and $\lim\limits_{x \to \xi} \int_{U_r} u(\xi) L(\xi, D) \varepsilon(x,\xi) d\xi = u(x)$. From (3.2), we get

$$u(x) = \int_{\partial\Omega} \Big(\frac{\partial \varepsilon(x,\xi)}{\partial n_\xi} u(\xi) - \varepsilon(x,\xi) \frac{\partial u(\xi)}{\partial n_\xi} \Big) d\xi + u(x) = 0, \ x \in \Omega. \tag{3.3}$$

Using the properties of the fundamental solution $\varepsilon(x,\xi)$, from (2.3) we have

$$\lim_{x \to \partial\Omega} \int_{\partial\Omega} u(\xi) \frac{\partial \varepsilon(x,\xi)}{\partial n_\xi} d\xi = -\frac{u(x)}{2} + \int_{\partial\Omega} u(\xi) \frac{\partial \varepsilon(x,\xi)}{\partial n_\xi}, \tag{3.4}$$

$$\lim_{x \to \partial\Omega} \int_{\partial\Omega} \varepsilon(x,\xi) \frac{\partial u(\xi)}{\partial n_\xi} d\xi = \int_{\partial\Omega} \varepsilon(x,\xi) \frac{\partial u(\xi)}{\partial n_\xi} d\xi. \tag{3.5}$$

From (3.3) -(3.5) it follows

$$N[u] = -\frac{u(x)}{2} + \int_{\partial\Omega} \left(\frac{\partial \varepsilon(x,\xi)}{\partial n_\xi} u(\xi) - \varepsilon(x,\xi) \frac{\partial u(\xi)}{\partial n_\xi} \right) d\xi = 0, \;\; x \in \partial\Omega. \tag{3.6}$$

This proves that elliptic potential (2.1) satisfies boundary condition (3.6).

Conversely, if $u(x)$ is a solution of second-order elliptic equation (2.2) and satisfies potential boundary condition (3.6), then $u(x)$ coincides with elliptic potential (2.1).

Let $\vartheta \in W_2^2(\Omega)$ be an arbitrary regular solution of problem (2.2) and (3.6), and let $u(x)$ be the elliptic potential given by formula (2.1).

We denote $\omega(x) = \vartheta(x) - u(x)$. Obviously, the function $\omega(x)$ satisfies the second-order homogeneous elliptic equation

$$L(x,D)\omega(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} \omega(x) + a\omega(x) = 0. \tag{3.7}$$

By a direct calculation, we verify

$$\begin{aligned} 0 &= \int_{\Omega} L\omega(\xi)\varepsilon(x,\xi) d\xi \\ &= \omega(x) - \frac{\omega(x)}{2} + \int_{\partial\Omega} \left(\frac{\partial \varepsilon(x,\xi)}{\partial n_\xi} \omega(\xi) - \varepsilon(x,\xi) \frac{\partial \omega(\xi)}{\partial n_\xi} \right) d\xi \\ &= \omega(x)|_{x\in\partial\Omega} + N[\omega]|_{x\in\partial\Omega}. \end{aligned} \tag{3.8}$$

Since $\omega(x) = \vartheta(x) - u(x)$ and $N[\vartheta]|_{x\in\partial\Omega} = N[u]|_{x\in\partial\Omega} = 0$, then $N[\omega]|_{x\in\partial\Omega} = 0$. Taking into account this fact, from (3.8) it follows that

$$\omega(x)|_{x\in\partial\Omega} = 0.$$

Since $a(x) \geq 0$, then by the virtue of the uniqueness of a solution of the Dirichlet problem, we obtain $\omega(x) = \vartheta(x) - u(x) \equiv 0$, i.e., $\vartheta(x) = u(x)$, i.e., it coincides with the elliptic potential.

□

It follows by Theorem 3.1 that for any $\rho \in L_2(\Omega)$ the elliptic potential defined by formula (2.1) satisfies boundary condition (3.1).

Theorem 3.2. *Necessary and sufficient conditions for the unique solvability of an elliptic potential with respect to density* $\rho \in L_2(\Omega)$ *are the condition* $u \in W_2^2(\Omega)$ *and fulfillment of the relation*

$$N[u] = -\frac{u(x)}{2} + \int_{\partial\Omega} \left(\frac{\partial \varepsilon(x,\xi)}{\partial n_\xi} u(\xi) - \varepsilon(x,\xi) \frac{\partial u(\xi)}{\partial n_\xi} \right) d\xi = 0, \;\; x \in \partial\Omega. \tag{3.9}$$

Under the conditions of Theorem 3.2, i.e. conditions (3.9), the function $\rho(x)$ is defined by the formula

$$\rho(x) = L(x,D)u(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u(x) + a(x)u. \tag{3.10}$$

The lateral boundary conditions for the wave and heat potentials were found in [10, 11], that satisfy potential boundary condition (3.6). Note that some classes of inverse and spectral problems for the Poisson and parabolic equations were studied in [3, 4, 5, 6, 12, 13].

The study of the propagation of stationary waves in the entire space $\mathbb{R}^n$ is reduced to the Sommerfeld radiation condition for the three-dimensional Helmholtz equation (see [1, 14]).

In this section, we study the Sommerfeld problem and the inverse Sommerfeld problem for the multidimensional Helmholtz equation in $\mathbb{R}^n, n \geq 3, n = 2s+3, s = 0, 1, ...$

Let $\varepsilon_n(x, ik)$ be the fundamental solutions of the Helmholtz equation

$$(-\Delta_x - k^2)\varepsilon_n(x) = \delta(x), k = 0, 1, 2, ... \tag{3.11}$$

In the work of T.Sh. Kalmenov, M. Otelbaev and G. Arepova [7], using the method of descent from the multidimensional heat equation to the multidimensional Helmholtz equation, an explicit form of the fundamental solution $\varepsilon_n(x, ik)$ was found in the following form:

$$\varepsilon_n(x, ik) = \varepsilon_\Delta(x)\frac{(n-2)}{\Gamma(\frac{n}{2})}\frac{(ik|x|)^{\frac{(n-2)}{2}}}{2} \cdot K_{\frac{n-2}{2}}(ik|x|), \tag{3.12}$$

where $\varepsilon_\Delta(x) = \frac{1}{(n-2)\omega_n|x|^{n-2}}$ is the fundamental solution of Laplace equation, ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Here the Macdonald function $K_\nu(z)$ is defined as follows

$$\begin{aligned} K_\nu(z) &= \frac{\pi}{2}\frac{I_\nu(z) - I_{-\nu}(z)}{\sin \nu\pi}, \quad \text{if } \nu \text{ is non-integer} \\ K_\nu(z) &= \lim_{\mu \to \nu} K_\mu(z), \quad \text{if } \nu \text{ is integer} \\ I_\nu(z) &= e^{-\frac{\nu\pi i}{2}} J_\nu(e^{\frac{-i\nu\pi}{2}} z), \end{aligned} \tag{3.13}$$

where $J_\nu(z)$ is the Bessel function.

In the case of odd $n = 2s+1$, s=0,1,2

$$\begin{aligned} K_{s+\frac{1}{2}}(ik|x|) &= \sqrt{\frac{2}{\pi}}\frac{ie^{-ik|x|}}{\sqrt{ik|x|}}(ik|x|)^{s+\frac{1}{2}}, \\ \sum_{m=0}^{s}(s+\frac{1}{2}, m)\frac{1}{(2ik(x))^m} &= \sqrt{\frac{2}{\pi}}(ik(x))^s e^{-k|x|}\sum_{m=0}^{s}\frac{\Gamma(1+s+m)}{m!\Gamma(s+1-m)}\frac{1}{(ik|x|)^m}. \end{aligned} \tag{3.14}$$

Taking into account equation (3.14) and that $n = 2s+3, s = 0, 1, 2, ...$ it follows from (3.13) that

$$\begin{aligned} \varepsilon_n(x, ik) &= \varepsilon_\Delta(x)\frac{(n-2)}{\Gamma(\frac{n}{2})}(\frac{ik|x|}{2})^{\frac{(n-2)}{2}}K_{\frac{n-2}{2}}(ik|x|) \\ &= \frac{1}{(n-2)\omega_n|x|^{n-2}}\frac{(n-2)}{\Gamma(\frac{n}{2})} \cdot K_{\frac{n-2}{2}}(ik|x|) \\ &= \frac{1}{\Gamma(\frac{2s+3}{2})}\sqrt{\frac{2}{\pi}} \cdot \frac{(ik|x|)^s}{|x|^{2s+1}}\sum_{m=0}^{s}\frac{\Gamma(s+1-m)}{m!\Gamma(s+1-m)} \cdot \frac{1}{(2ik|x|)^m} \\ &= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\Gamma(\frac{2s+3}{2})}(ik)^s \cdot \sum_{m=0}^{s}\frac{e^{-ik|x|}}{|x|^{s+m+1}}\frac{1}{(2ik)^m}, \quad n = 2s+3. \end{aligned} \tag{3.15}$$

In a bounded domain $\Omega \subset \mathbb{R}^n, \partial\Omega \in C$ we define the Helmholtz potential in the following from

$$\begin{aligned} u(x) &= \int_\Omega \varepsilon_n(x-y, ik)f(y)dy \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{\omega_n \cdot \Gamma(\frac{2s+3}{2})} \cdot (ik)^s\sum_{m=0}^{s}\int_\Omega \frac{e^{ik|x-y|}}{(2ik)^m|x-y|^{s+1+m}}f(y)dy. \end{aligned} \tag{3.16}$$

Theorem 3.3. *Let $n = 2s+3, s = 0,1,2,...,$ then the Helmholtz potential $u \in W_2^2(\Omega)$ determined by formula (3.16) satisfies the following Sommerfeld radiation conditions at $R \to \infty$, that is there exist $r,\ c > 0$ such that for all $R \geq r$*

$$|u(x)|_{|x|=R} \leq c\frac{\| f \|_{L_2(\Omega)}}{R^{s+1}},$$
$$\left|\frac{\partial u}{\partial |x|} - iku\right|_{|x|=R} \leq c\frac{\| f \|_{L_2(\Omega)}}{R^{s+2}}. \tag{3.17}$$

Proof. Since Ω is bounded domain, there exists R_0 such that $|y| \leq R_0,\ \ \forall y \in \overline{\Omega}$. By choosing R large enough, in particular $R > r = 10R_0$, since $k \geq 1$ and following obvious inequality

$$\frac{1}{|x-y|} \leq \frac{1}{(|x|-|y|)} = \frac{1}{|x|}\cdot\left(\frac{1}{1-\frac{|y|}{|x|}}\right) \leq \frac{1}{R}\cdot\left(\frac{1}{1-\frac{R_0}{10R_0}}\right) = \frac{10}{9}\cdot\frac{1}{R},$$

from (3.16) by a direct calculation we can find that for some $c_1,\ c_2 > 0$.

$$\begin{aligned}
|u(x)|_{|x|=R} &\leq c_1\left|\sum_{m=0}^{s}\int_\Omega \frac{e^{ik|x-y|}}{(2ik)^m|x-y|^{m+s+1}}f(y)dy\right| \\
&\leq c_1\sum_{m=0}^{s}\left(\int_\Omega \frac{|e^{ik|x-y|}|}{|x-y|^{2(m+s+1)}}dy\right)^{\frac{1}{2}} \| f \|_{L_2(\Omega)} \\
&\leq c_1\sum_{m=0}^{s}\left(\int_\Omega \frac{dy}{(|x|-|y|)^{2(m+s+1)}}\right)^{\frac{1}{2}} \| f \|_{L_2(\Omega)} \\
&\leq \frac{c_1(s+1)}{R^{s+1}}\left(\int_\Omega \frac{dy}{(1-\frac{|y|}{R})^{2(2s+1)}}\right)^{\frac{1}{2}} \| f \|_{L_2(\Omega)} \\
&\leq \frac{c_2}{R^{s+1}} \| f \|_{L_2(\Omega)}\ .
\end{aligned} \tag{3.18}$$

It is easy to check that

$$\begin{aligned}
\frac{\partial}{\partial|x|}\frac{e^{ik|x-y|}}{|x-y|^{m+s+1}} - \frac{ike^{ik|x-y|}}{|x-y|^{m+s+1}} &= \frac{ike^{ik|x-y|}}{|x-y|^{m+s+1}}\left(\frac{\partial}{\partial|x|}|x-y|-1\right)+ \\
&+e^{ik|x-y|}\frac{m+s+1}{|x-y|^{m+s+1}}\cdot\frac{\partial}{\partial|x|}|x-y|.
\end{aligned} \tag{3.19}$$

Since

$$|x-y| = \left(|x|^2+|y|^2-2(\cos\gamma)|x||y|\right)^{\frac{1}{2}},$$

where $\cos\gamma = \cos(\widehat{x,y})$, then

$$\begin{aligned}
\frac{\partial}{\partial|x|}|x-y| &= \frac{|x|-\cos\gamma|y|}{|x-y|} = \frac{|x|(1-\frac{|y|\cos\gamma}{|x|})}{|x-y|} \\
&\leq \frac{|x|(1-\frac{y\cos\gamma}{|x|})}{|x|-|y|} \leq \frac{|x|(1+\frac{|y|}{|x|})}{|x|(1-\frac{|y|}{|x|})} \leq \frac{1+\frac{|y|}{|x|}}{1-\frac{|y|}{|x|}}.
\end{aligned} \tag{3.20}$$

It follows that

$$(\frac{\partial}{\partial|x|}|x-y|-1) \leq \frac{1+\frac{|y|}{|x|}}{1-\frac{|y|}{|x|}}-1=\frac{2\frac{|y|}{|x|}}{(1-\frac{|y|}{|x|})} \leq \frac{20R_0}{9}\cdot\frac{1}{|x|}. \tag{3.21}$$

Taking into account (3.20) and (3.21), from formula (3.16) as above we obtain that for some $c_3,\ c_4,\ c_5,\ c_6>0$ for $R>r$

$$\begin{aligned}\left|\frac{\partial u}{\partial|x|}-iku\right| \leq & c_3\int\limits_{\Omega}\left|\frac{f(y)}{|x-y|^{s+2}}\right|dy+c_4\frac{1}{|x|}\int\limits_{\Omega}\left|\frac{f(y)}{|x-y|^{s+1}}\right|dy \\ \leq & \frac{c_4}{R^{s+2}}\left(\int\limits_{\Omega}\frac{dy}{\left(1-\frac{|y|}{R}\right)^{2(s+2)}}\right)^{\frac{1}{2}}\parallel f\parallel_{L_2(\Omega)} \\ & +\frac{c_5}{R^{s+2}}\left(\int\limits_{\Omega}\frac{dy}{\left(1-\frac{|y|}{R}\right)^{2(s+1)}}\right)^{\frac{1}{2}}\parallel f\parallel_{L_2(\Omega)} \\ \leq & \frac{c_6}{R^{s+2}}\parallel f\parallel_{L_2(\Omega)}.\end{aligned} \tag{3.22}$$

This gives (3.17) by assuming $c=\max\{c_2,\ c_6\}$. □

In the case of $\Omega=\{|x|^2\leq R^2\}$, $(-\Delta_x^2-k^2)u=0$ (see. [14]) it is easy to show that

$$\begin{aligned}&\int_{\Omega}\varepsilon_{2s+3}(ik|x-y|)(-\Delta_y-k^2)u(y)dy \\ =&\int_{\partial\Omega}\frac{\partial}{\partial\eta_y}\varepsilon_{2s+3}(ik|x-y|)u(y)dy-\int_{\partial\Omega}\varepsilon_{2s+3}(ik|x-y|)\frac{\partial u}{\partial|y|}(y)dy \\ =&\int_{|y|=R}\frac{\partial}{\partial|y|}\cdot\varepsilon_{2s+3}(ik|x-y|)u(y)dy-\int_{|y|=R}\varepsilon_{2s+3}(ik|x-y|)\frac{\partial u}{\partial|y|}(y)dy \\ =&u(x),\quad x\in\Omega.\end{aligned} \tag{3.23}$$

This implies the following lemma.

Lemma 3.1. *Let $u\in W_2^2(\mathbb{R}^n)$ be a solution of the homogeneous Helmholtz equation*

$$(-\Delta_x^2-k^2)u=0$$

satisfying the inequalities

$$|u(x)|\Big|_{|x|=R}\leq\frac{d}{R^{s+1}},\quad\left|\frac{\partial u}{\partial|x|}-iku\right|_{|x|=R}\leq\frac{d}{R^{s+2}},\quad R\to\infty, \tag{3.24}$$

then $u(x)\equiv 0, x\in\Omega$.

Grouping in formula (3.23) the terms containing $\frac{\partial u}{\partial|x|}-iku$, and $u(x)$, using estimates (3.24), Lemma 3.1 is proved in the same way as Theorem 3.3.

Sommerfeld problem. Find solutions $u(x)\in W_2^2(\mathbb{R}^n)$ of the Helmholtz equation

$$\begin{aligned}-\Delta_x u-k^2u=\rho(x), x\in\Omega,\\ -\Delta_x u-k^2u=0,\ x\in\mathbb{R}^n\setminus\Omega,\end{aligned} \tag{3.25}$$

satisfying the Sommerfeld radiation condition

$$|u(x)|\Big|_{|x|=R}\leq\frac{d}{R^{s+1}},\quad\left|\frac{\partial}{\partial|x|}u(x)-iku\right|_{|x|=R}\leq\frac{d}{R^{s+2}},\quad R\to\infty. \tag{3.26}$$

Theorem 3.4. *Let $n = 2s + 3, s = 0, 1, ...,$ then the solution of Sommerfeld problem* (3.25) - (3.26) *coincides with the Helmholtz potential*

$$u(x) = \int_{\Omega} \varepsilon_n(ik|x-y|)\rho(y)dy \tag{3.27}$$

and on $\partial\Omega$ satisfies the potential boundary condition

$$N[u] = -\frac{u(x)}{2} + \int_{\partial\Omega} \left(\frac{\partial\varepsilon}{\partial n_y}(|x-y|ik)u(y) - \varepsilon(ik|x-y|) \cdot \frac{\partial u}{\partial n_y}(y) \right) dy = 0. \tag{3.28}$$

According to Theorem 3.3, the Helmholtz potential also satisfies the Sommerfeld radiation condition and equation (3.25).

Conversely, if $\vartheta(x)$ is a solution of Sommerfeld problem (3.25) - (3.26), then the function $\omega = u(x) - \vartheta(x)$ satisfies the homogeneous problem (3.25) - (3.26) , i.e.

$$-\Delta\omega - k^2\omega = 0, \quad |\omega|_{|x|=R} \leq \frac{d}{R^{s+1}}, \quad \left| \frac{\partial\omega}{\partial|x|} - ik\omega \right|_{|x|=R} \leq \frac{d}{R^{s+2}}. \tag{3.29}$$

By virtue of Lemma 3.1, the function $\omega(x) \equiv 0$, i.e. $u = \vartheta(x)$

Thus, the solutions of the Sommerfeld problem and the Helmholtz potential coincide, and potential boundary condition(3.28) and the Sommerfeld radiation conditions (3.26) are equivalent to each other.

The following theorem is proved similarly to Section 1.

Theorem 3.5. *Let $u \in W_2^2(\Omega)$ be given by formula* (3.27) *and satisfy condition* (3.29). *Then there exists only one $\rho \in L_2(\Omega)$ such that*

$$Lu = (-\Delta_x - k^2)u = \rho(x). \tag{3.30}$$

If the external source is the Sommerfeld problem $\rho(x) \equiv 0$ outside a bounded domain $\Omega \subset \mathbb{R}^n$, then according to Theorem 3.5, the density of the Sommerfeld potential $u(x)$ is defined by formula (3.30).

Using the properties of the fundamental solution of uniform elliptic equation (2.2), as indicated above, one can find the density of an elliptic potential generated by the Sommerfeld condition.

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Tynysbek Sharipovich Kalmenov, Aidana Kasimkyzy Les, Ulzada Asilovna Iskakova
Institute of Mathematics and Mathematical Modeling
125 Pushkin St,
050010 Almaty, Kazakhstan
E-mails: kalmenov.t@mail.ru, a.les@math.kz, iskakova@math.kz

Aidana Kasimkyzy Les
Al-Farabi Kazakh National University
71 Al-Farabi Av,
050010 Almaty, Kazakhstan
E-mail: a.les@math.kz

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