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TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 75th birthday)

Tynysbek Sharipovich Kal'menov was born in the village of Koksaek of the Tolebi district of the Turkestan region (earlier it was the Lenger district of the South-Kazakhstan region of the Kazakh SSR). Although "according to the passport" his birthday was recorded on May 5, his real date of birth is April 6, 1946.

Tynysbek Kal'menov is a graduate of the Novosibirsk State University (1969), and a representative of the school of A.V. Bitsadze, an outstanding scientist, corresponding member of the Academy of Sciences of the USSR. In 1972, he completed his postgraduate studies at the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR. In 1983, he defended his doctoral thesis at the M.V. Lomonosov Moscow State University. Since1989, he is a corresponding member of the Academy of Sciences of the Kazakh SSR, and since 2003, he is an academician of the National Academy of Sciences of the Republic of Kazakhstan.

Tynysbek Kal'menov worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (1972-1985). From 1986 to 1991, he was the dean of the Faculty of Mathematics of Al-Farabi Kazakh State University. From 1991 to 1997, he was the rector of the Kazakh Chemical-Technological University (Shymkent).

From 2004 to 2019, Tynysbek Kal'menov was the General Director of the Institute of Mathematics and Mathematical Modeling. He made it one of the leading scientific centers in the country and the best research institute in Kazakhstan. It suffices to say, that in terms of the number of scientific publications (2015-2018) in international rating journals indexed in the Web of Science, the Institute of Mathematics and Mathematical Modeling was ranked fourth among all Kazakhstani organizations, behind only three large universities: the Nazarbaev University, Al-Farabi National University and L.N. Gumilyov Eurasian National University.

Since 2019, Tynysbek Kal'menov has been working as the head of the Department of Differential Equations of the Institute of Mathematics and Mathematical Modeling. He is a member of the National Scientific Council "Scientific Research in the Field of Natural Sciences", which is the main Kazakhstan council that determines the development of science in the country.

T.Sh. Kal'menov was repeatedly elected to maslikhats of various levels, was a member of the Presidium of the Committee for Supervision and Attestation in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan. He is a Laureate of Lenin Komsomol Prize of the Kazakh SSR (1978), an Honored Worker of Science and Technology of Kazakhstan (1996), awarded with the order "Kurmet" (2008 Pi.) and jubilee medals.

In 2013, he was awarded the State Prize of the Republic of Kazakhstan in the field of science and technology for the series of works "To the theory of initial- boundary value problems for differential equations".

The main areas of scientific interests of academician Tynysbek Kal'menov are differential equations, mathematical physics and operator theory. He has obtained fundamental scientific results, many of which led to the creation of new scientific directions in mathematics.

Tynysbek Kal'menov, using a new maximum principle for an equation of mixed type (Kal'menov's maximum principle), was the first to prove that the Tricomi problem has an eigenfunction, thus he solved the famous problem of the Italian mathematician Francesco Tricomi, set in 1923 This marked the beginning of a new promising direction, that is, the spectral theory of equations of mixed type.

He established necessary and sufficient conditions for the well-posed solvability of the classical Darboux and Goursat problems for strongly degenerate hyperbolic equations.

Tynysbek Kal'menov solved the problem of completeness of the system of root functions of the nonlocal Bitsadze-Samarskii problem for a wide class of multidimensional elliptic equations. This result is final and has been widely recognized by the entire mathematical community.

He developed a new effective method for studying ill-posed problems using spectral expansion of differential operators with deviating argument. On the basis of this method, he found necessary and sufficient conditions for the solvability of the mixed Cauchy problem for the Laplace equation.

Tynysbek Kal'menov was the first to construct boundary conditions of the classical Newton potential. That is a fundamental result at the level of a classical one. Prior to the research of Kal'menov T.Sh., it was believed that the Newton potential gives only a particular solution of an inhomogeneous equation and does not satisfy any boundary conditions. Thanks for these results, for the first time, it was possible to construct the spectral theory of the classical Newton potential.

He developed a new effective method for constructing Green's function for a wide class of boundary value problems. Using this method, Green's function of the Dirichlet problem was first constructed explicitly for a multidimensional polyharmonic equation.

From 1989 to 1993, Tynysbek Kal'menov was the chairman of the Inter- Republican (Kazakhstan, Uzbekistan, Kyrgyzstan, Turkmenistan, Tajikistan) Dissertation Council. He is a member of the International Mathematical Society and he repeatedly has been a member of organizing committee of many international conferences. He carries out a lot of organizational work in training of highly qualified personnel for the Republic of Kazakhstan and preparing international conferences. Under his direct guidance, the First Congress of Mathematicians of Kazakhstan was held. He presented his reports in Germany, Poland, Great Britain, Sweden, France, Spain, Japan, Turkey, China, Iran, India, Malaysia, Australia, Portugal and countries of CIS.

In terms of the number of articles in scientific journals with the impact- factor Web of Science, in the research direction of "Mathematics", the Institute of Mathematics and Mathematical Modeling is on one row with leading mathematical institutes of the Russian Federation, and is ahead of all mathematical institutes in other CIS countries in this indicator.

Tynysbek Kal'menov is one of the few scientists who managed to leave an imprint of their individuality almost in all branches of mathematics in which he has been engaged.

Tynysbek Kal'menov has trained 11 doctors and more than 60 candidate of sciences and PhD, has founded a large scientific school on equations of mixed type and differential operators recognized all over the world. Many of his disciples are now independent scientists recognized in the world of mathematics.

He has published over 150 scientific articles, most of which are published in international mathematical journals, including 14 articles published in "Doklady AN SSSR/ Doklady Mathematics". In the last 5 years alone (2016-2020), he has published more than 30 articles in scientific journals indexed in the Web of Science database. To date, academician Tynysbek Kal'menov has a Hirsch index of 18 in the Web of Science and Scopus databases, which is the highest indicator among all Kazakhstan mathematicians.

Outstanding personal qualities of academician Tynysbek Kalmenov, his high professional level, adherence to principles of purity of science, high exactingness towards himself and his colleagues, all these are the foundations of the enormous authority that he has among Kazakhstan scientists and mathematicians of many countries.

Academician Tynysbek Sharipovich Kalmenov meets his 75th birthday in the prime of his life, and the mathematical community, many of his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal heartily congratulate him on his jubilee and wish him good health, happiness and new successes in mathematics and mathematical education, family well-being and long years of fruitful life.

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ON THE RELATION BETWEEN TWO APPROACHES TO EXTERIOR PENALTY METHOD FOR CONSTRAINED OPTIMAL CONTROL PROBLEMS

A. Hammoudi, M. Benharrat

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Key words: optimal control, control-state constraints, penalty function, nonlinear systems.

AMS Mathematics Subject Classification: Primary 49J15, 93C10; Secondary 49J30.

Abstract. The purpose of this paper is to discuss, via the exterior penalty functions method, a class of nonlinear optimal control problems with additional equality and inequality state and control constraints. Two different kinds of penalties are given, in the first the state and control constrained optimal control problem is replaced by a sequence of unconstrained control problems, while the second type transforms the constrained optimal control problem into a sequence of truly unconstrained optimization problems. Two convergence theorems are given to obtain approximate and, in the limit, exact solution of the given constrained optimal control problem. In particular, we show how the necessary conditions of optimality of these two methods yield the familiar Lagrange multipliers of the original constrained optimal control problem in the limit.

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1 Introduction

In the past few years, and more recently, considerable attention has been given to the study of the optimal control problems with mixed vector-valued state-control constraints [7], [10], [20], [23], [39], [40], [45]. Such interest is explained by their importance in the modeling of real-life phenomena in various physical, chemical and biological processes, in economics and mechanics and various other fields, see [4], [5], [24] and the references therein.

In general, it is difficult to solve a constrained optimal control problem by the standard techniques such as the Pontryagin maximum principle and the Hamilton-Jacobi-Bellman approach, results of this direction was be obtained more than half a century ago, see for instance [30], [11], [29] and more recent [9], [15], [32]; for a good survey of the maximum principles for optimal control problems with state constraints, see [12]. To attempt to overcome these difficulties, several strategies have been proposed. The most common method used to handle constraints is the penalty functions method. This method is based on developing an auxiliary function such that, by appropriate choice of parameters, the original constrained problem might be solved by unconstrained problems. The main problem is to justify the convergence of the sequence of the optimal solutions of the unconstrained problems to the optimal solution of the constrained one.

The application of penalty methods to optimal control problems has received much attention since Courant's work [8]. In the literature, we can distinguish two ways of formulating the unconstrained optimal control problems via the penalty methods. The first common formulation is as follows:

- Removing troublesome intermediate control and/or state constraints via a penalty function and then utilizing the associated necessary conditions for optimality to solve the transformed problem, still an optimal control problem. Investigations in this direction may be found in [16], [18], [27], [31], [34], [37], [25], [26], [46], [47], [48], [38] and [2].

The second alternative, which in general is stronger, is as follows:

- A system differential equations viewed as equality constraints is also penalized together with control-state constraints to obtain an abstract infinite-dimensional optimization problem. This approach was proposed for the first time by A.V. Balakrishnan [1], for unconstrained optimal control problems and generalized later for the constrained optimal control problems by many authors [3], [21], [35], [43], [50].

In this article we use the two different kinds of penalties cited above for solving the constrained control problems aiming to generalize some previous results. Note that this treatment yields certain useful properties to ensure the strong compactness in the Lebesgue spaces and new techniques for studies of many problems in the literature. We treat here a fixed endpoint optimal control problem with specified intermediate state and control constraints. We assume these constraints are given by finite systems of inequality and equality constraints.

The paper consists of two parts. In the first part, we discuss two different kinds of penalties. In the first case, equality and inequality state-control constraints are penalized in a way that guarantees the exteriority of the approaching solutions. This property allows one to produce a sequence of optimal control problems (without constraints) and, under reasonable assumptions, we generate a sequence of minimizing points which converge to the solution of the original control problem.

In the second, we develop a theoretical framework for constrained optimal control problems with not well-posed differential equation. We use the penalty method in which the control and the state function are at the same level and the state equation becomes a general equality type constraint. The penalty function used, in this case, gives a sequence of truly unconstrained optimization problems. It turns out that the obtained sequence of minimizing points offer a minimizing point as well as a solution of the system differential equations (as equality constraints) for the original problem. As such, this approach is a generalization of previous works on penalty methods in optimal control theory given in [1], [3], [21], [35], [43] and [50] by taking hypotheses ensure the lower semi-continuity.

In the second part of the paper, we give the necessary conditions of optimality of the solutions of the sequence of penalized problems in a Hilbert case. Under smoothness assumptions, we show that the sequence given by the Pontryagin maximum principle approximate the Lagrange multipliers of the initial problem.

The paper is organized as follows. In Section 2, we present the problem, notations, basic definitions and assumptions. In Section 3 we investigate the two different approaches to the exterior penalty method for solving the constrained optimal control problem considered. First, we give convergence results of the penalty method in the case whence also the equality and inequality state-control constraints are handled. Secondly, a system differential equations viewed as equality constraints is also penalized together with the control-state constraints to give a convergence theorem which guarantees that any accumulation point of the sequence of minimizing points for the penalized problems is also an optimal solution of the constrained optimal control problem. In Section 4, we discuss a relationship between the necessary conditions of optimality of the original problem and those of its penalized problems in a Hilbert spaces case. We show that if the optimal control for the non-dynamic unconstrained problem converges pointwise to an admissible control, the latter is actually an optimal control, and the sequence of the maximum principle associated to the sequence of the dynamic unconstrained problem yields the familiar Lagrange multipliers of the original constrained optimal control problem in the limit.

2 Statement of the problem and preliminary results

We consider the following constrained optimal control problem:

$$(P) \qquad \min \int_0^T l\left(t, x\left(t\right), u\left(t\right)\right) dt \tag{2.1}$$

subject to

$$\dot{x}\left(t\right) = f\left(t, x\left(t\right), u\left(t\right)\right) \qquad \text{for almost all } t \in [0, T] \tag{2.2}$$

$$x\left(0\right) = x_0 \tag{2.3}$$

and satisfying

$$g_i\left(t, x\left(t\right), u\left(t\right)\right) \leq 0 \qquad i = 1 \dots s \qquad \text{for almost all } t \in [0, T] \tag{2.4}$$

$$g_i\left(t, x\left(t\right), u\left(t\right)\right) = 0 \qquad i = s+1 \dots r \qquad \text{for almost all } t \in [0, T] \tag{2.5}$$

$$x\left(T\right) = 0 \tag{2.6}$$

$$x(.) \in AC\left([0, T], \mathbb{R}^{\mathbf{n}}\right) \quad \text{and} \quad u \in \mathcal{U}.$$

The data of the problem are the distributed cost $l : \mathbb{R} \times \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^m \to \mathbb{R}$, dynamics $f : \mathbb{R} \times \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^m \to \mathbb{R}^{\mathbf{n}}$, mixed control-state constraints $g_i : \mathbb{R} \times \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^m \to \mathbb{R}$ for $i = 1 \dots r$, a subset $\mathcal{U}$, (fixed) final time $T > 0$, and (fixed) final initial condition $x_0 \in \mathbb{R}^{\mathbf{n}}$.

$AC\left([0, T], \mathbb{R}^{\mathbf{n}}\right)$ stands for the space of absolutely continuous maps from $[0, T]$ to $\mathbb{R}^{\mathbf{n}}$. For $1 \leq p < \infty$ ($p = \infty$, respectively), $L^p\left([0, T], \mathbb{R}^m\right)$ denotes the Lebesgue space of all measurable functions u such that $\|u\|_{L^p} = \left(\int_0^T \|u\left(t\right)\|^p dt\right)^{\frac{1}{p}} < \infty$ ($\|u\|_{L^\infty} = \sup ess \|u\left(t\right)\| < \infty$, respectively). The Banach space of all vector-valued continuous functions x is denoted by $C\left([0, T], \mathbb{R}^{\mathbf{n}}\right)$ with standard norm $\|x\|_\infty = \sup_{t \in [0,T]} \|x\left(t\right)\|$.

We refer to a measurable function $u : [0, T] \longrightarrow \mathbb{R}^m$ as a control function or simply control. In the sequel, $\mathcal{U}$ will be the set of all control functions u such that $u(t) \in \mathbb{U}$ for almost all t, where $\mathbb{U}$ is a given nonempty bounded and closed subset of $\mathbb{R}^m$.

A trajectory or control process

$$\mathcal{T} = \{(x(t), u(t)) : t \in [0, T]\}$$

is said to be admissible if $x(.)$ is absolutely continuous, $u(.) \in \mathcal{U}$ and the pair of functions $(x(t), u(t))$ satisfies (2.2), (2.3) and (2.6) on the interval $I = [0, T]$. The component $x(.)$ will be called the state trajectory. In the sequel, we assume that there exists a control from a given set for which relations (2.2) - (2.6) are satisfied.

We make the following assumptions on the data:

(A-1) (i) The mapping l is a Carathéodory mapping, i.e., l is continuous in (x, u) for almost all $t \in [0, T]$, and is measurable in t for every $(x, u) \in \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^m$;

(ii) there exist $\theta \in L^1([0, T], \mathbb{R})$ and $\rho \in L^\infty([0, T], \mathbb{R}_+)$ such that for almost all $t \in [0, T]$,

$$l(t, x, u) \geq \theta(t) - \rho(t)(\|x\| + \|u\|).$$

(A-2) (i) The mappings f is a Carathéodory mapping and there exist $\theta_1 \in L^\infty\left([0, T], \mathbb{R}_+\right)$ and $\rho_1 \in L^\infty\left([0, T], \mathbb{R}_+\right)$ such that for almost all $t \in [0, T]$,

$$\|f\left(t, x, u\right)\| \leq \theta_1\left(t\right) + \rho_1\left(t\right)\left(\|x\| + \|u\|\right);$$

(ii) there exist $K > 0$ such that for almost all $t \in [0, T]$,

$$\|f(t, x_1, u) - f(t, x_2, u)\| \leq K \|x_1 - x_2\|,$$

for all x_1, x_2 in some bounded set of $\mathbb{R}^{\mathbf{n}}$ and for all u.

(A-3) For all $i = 1, \cdots, r$, the function g_i ia a Carathéodory mapping and there exists $\theta_2^i \in L^1([0, T], \mathbb{R})$ and $\rho_2^i \in L^\infty([0, T], \mathbb{R}_+)$ such that for almost all $t \in [0, T]$,

$$g_i(t, x, u) \geq \theta_2^i(t) - \rho_2^i(t)(\|x\| + \|u\|).$$

Let us give some examples to illustrate the above hypotheses **(A-1)**-**(A-3)**.

Example 1. Consider the optimal control problem

$$\min \int_0^1 (3t - \frac{1}{2}|x| - |u|)dt$$

subject to

$$\begin{array}{rcll} \dot{x}(t) & = & t \sin x + \sqrt{|u|} & \text{for almost all } t \in [0, 1] \\ x(0) & = & x_0 & \end{array}$$

and satisfying

$$\begin{array}{rcll} 1 + t^2 - |x| \sin t - |u| \cos x & \leq & 0 & \text{for almost all } t \in [0, 1] \\ x(1) & = & 0 & \\ |u| & \leq & 1. & \end{array}$$

We have

$$l(t, x, u) = 3t - \frac{1}{2}|x| - |u| \geq \theta(t) - \rho(t)(|x| + |u|),$$

where $\theta(t) = 3t$ and $\rho(t) = 1$, so **(A-1)** holds. On the other hand,

$$|f(t, x, u)| = \left| t \sin x + \sqrt{|u|} \right| \leq |x| + \sqrt{|u|},$$

and

$$|f(t, x_1, u) - f(t, x_2, u)| \leq t |\sin x_1 - \sin x_2| \leq |x_1 - x_2|,$$

for all x_1, x_2 in some bounded set of $\mathbb{R}$ and for all u such that $|u| \leq 1$. Thus, **(A-2)** holds. Also,

$$g(t, x, u) = 1 + t^2 - |x| \sin t - |u| \cos x \geq (1 + t^2) - (|x| + |u|).$$

Hence, **(A-3)** holds.

Example 2. Consider the optimal control problem

$$\min \int_0^1 (t^2 |u| + x^2 + y^2)dt$$

subject to

$$\begin{array}{rcll} \dot{x}(t) & = & (x + y)u & \\ \dot{y}(t) & = & u & \text{for almost all } t \in [0, 1] \\ (x(0), y(0)) & = & (x_0, y_0) & \end{array}$$

and satisfying

$$\begin{aligned} 3t+2-u-\sqrt{|xy|} &\leq 0 \qquad\qquad \text{for almost all } t\in[0,1] \\ (x(1),y(1)) &= (0,0) \\ |u| &\leq 1. \end{aligned}$$

We have

$$l(t,x,y,u)=t^2|u|+x^2+y^2\geq\theta(t)-\rho(t)(\|(x,y)\|+|u|).$$

with $\theta(t)=0$ and $\rho(t)=\max\{1,t^2\}=1$, so **(A-1)** holds.

$$\|f(t,x,y,u)\|=\sqrt{(x+y)^2u^2+u^2}\leq\sqrt{2}(\|(x,y)\|+|u|),$$

and

$$\|f(t,x_1,y_1,u)-f(t,x_2,y_2,u)\|=\|((x_1+y_1-x_2-y_2)u,0)\|\leq\|((x_1,y_1)-(x_2,y_2))\|,$$

for all u such that $|u|\leq 1$. Hence, **(A-2)** holds. Also,

$$g(t,x,y,u)=3t+2-u-\sqrt{|xy|}\geq 3t+2-\frac{1}{\sqrt{2}}\|(x,y)\|-|u|\geq 3t+2-(\|(x,y)\|+|u|).$$

Thus, **(A-3)** holds.

Now, let

$$\Omega=\mathcal{T}\cap\Omega_{in}\cap\Omega_{eq},$$

where

$$\Omega_{in}=\{(x,u)\text{ satisfying inequality constraints (2.4)}\},$$

and

$$\Omega_{eq}=\{(x,u)\text{ satisfying equality constraints (2.5)}\}.$$

Then the constrained optimal control problem is to find $(\overline{x},\overline{u})\in\Omega$ such that

$$F(\overline{x},\overline{u})=\min_{(x,u)\in\Omega}F(x,u) \tag{2.7}$$

where

$$F(x,u)=\int_0^T l(t,x(t),u(t))\,dt. \tag{2.8}$$

By [22, Theorem 4.1], we have

Proposition 2.1. *Suppose* (**A-1**) *holds and the functional* $F(.,.)$ *is not identical to* $+\infty$*, then* $F(.,.)$ *is lower-semicontinuous in* $L^1(I;\mathbb{R}^{\mathbf{n}})\times L^1(I;\mathbb{R}^m)$ *and*

$$F(x,u)>-\infty \quad \text{for all} \quad (x,u)\in L^1(I;\mathbb{R}^{\mathbf{n}})\times L^1(I;\mathbb{R}^m).$$

Note that assumptions (**A-2**) guarantees local existence and uniqueness of the solution of differential equations (2.2- 2.3) for a given control $u(.)$ defined in the whole interval $[0,T]$. In fact, since $u(.)$ is only assumed to be measurable and bounded, the right-hand side of equation (2.2) is continuous in x but only measurable and bounded in t for each x. Therefore, solutions are understood to be absolutely continuous functions,

$$x(t)=x_0+\int_0^t f(\tau,x(\tau),u(\tau))\,d\tau \qquad \text{for all } t\in[0,T']$$

for T' small enough. Now by growth condition (**A-2**)-(i), we have

$$\begin{aligned}\|x(t)\| &\leq \|x_0\| + \int_0^t \|f(\tau, x(\tau), u(\tau))\| \, d\tau \\ &\leq \|x_0\| + \|\theta_1(.)\|_{L^1} + \int_0^t \rho_1(\tau) \|x(\tau)\| \, d\tau + \|\rho_1(.)\|_{L^\infty} \|u(.)\| .\end{aligned}$$

Applying Gronwall's lemma we obtain

$$\|x(t)\| \leq M, \text{ for all } t \in [0, T'],$$

where $M = (\|x_0\| + \|\theta_1(.)\|_{L^1} + \|\rho_1(.)\|_{L^\infty} \|u(.)\|) \exp\left(\int_0^{T'} \rho_1(\tau) \, d\tau\right)$.

We see that the solution remains in a bounded fix set, independently of T' and u. It is known that in this case the solution has a maximum extension in the interval $[0, T]$. Further, this extension is unique by Lipschitz condition (**A-2**)-(ii). In the sequel, the unique solution of (2.2) such that (2.3) for a given $u(.)$ will be called the response to $u(.)$; and we denote it by

$$x_u(t) = x(t; x_0, u(.)) = x_0 + \int_0^t f(\tau, x_u(\tau), u(\tau)) \, d\tau \qquad t \in [0, T] .$$

We have

1. $\|x_u(t)\| \leq M,$ for all $t \in [0, T]$ and $u \in \mathbb{U}$,
2. $\|\dot{x}_u(t)\| \leq \max\{\|f(t, x, u)\| : 0 \leq t \leq T; u \in \mathbb{U}, \text{ and } \|x\| \leq M\}$.

Furthermore,

Proposition 2.2. *Under condition* (**A-2**)*, the mapping* $u \longrightarrow x_u$ *is a continuous function from* $\mathcal{U} \subset L^1(I; \mathbb{R}^m)$ *to* $C([0, T], \mathbb{R}^{\mathbf{n}})$.

Proof. Let $u(.), v(.)$ be two controls in $\mathcal{U}$ and $x_u(.), x_v(.)$ be the corresponding responses. We show that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\|u(.) - v(.)\|_{L^1} < \delta$ then $\|x_u(.) - x_v(.)\|_\infty < \epsilon$. Let $\epsilon_0 > 0$, define

$$B_{\epsilon_0} = \{x \in \mathbb{R}^{\mathbf{n}} : \inf_{0 \leq t \leq T} \|x - x_u(t))\| \leq \epsilon_0\}.$$

Also, f is continuous on the compact set $B_{\epsilon_0} \times \overline{\mathbb{U}}$ for almost all $t \in [0, T]$, then there exists a constant $M' > 0$ and a Lebesgue measurable set I_0 such that

$$\|f(t, x, u)\| \leq M' \qquad \text{for all } (t, x, u) \in B = (I - I_0) \times B_{\epsilon_0} \times \overline{\mathbb{U}},$$

with $meas(I_0) = 0$, where $meas(E)$ denote the Lebesgue measure of a Lebesgue measurable set E. Let $\epsilon' = \min\{\epsilon, \epsilon_0\}$, and $\sigma > 0$ be a number such that $\sigma(T + 2M') \exp(\kappa T) < \epsilon'$. Because of the uniform continuity of f on the compact set B, there exist some $\eta > 0$ such that $\|f(t, x, u) - f(t, x, v)\| < \sigma$ for all $(t, x, u), (t, x, v) \in B$ with $\|u - v\| < \eta$. Let $\delta = \sigma\eta$, if $u \in \mathbb{U}$ and

$$\int_0^T \|u(t) - v(t)\| \, dt < \delta = \sigma\eta,$$

let I_1 be the set of all $t \in [0,T]$ where $\|u(t)-v(t)\| > \eta$. Then

$$\eta(meas(I_1)) \leq \int_{I_1} \|u(t)-v(t)\|\, dt \leq \int_0^T \|u(t)-v(t)\|\, dt < \delta = \sigma\eta,$$

and hence $meas(I_1) < \sigma$. If $I_2 = I - I_1$, then $\|u(t)-v(t)\| < \eta$ for all $t \in I_2$. For every $t \in [0,T]$ let $E_t = [0,t] \cap I_2$ and $E'_t = [0,t] \cap I_1$. Thus, for all $t \geq 0$ of at least a right neighborhood of 0 we have

$$\begin{aligned}
\|x_u(t) - x_v(t)\| &\leq \int_0^t \|f(\tau, x_u(\tau), u(\tau)) - f(\tau, x_v(\tau), v(\tau))\| d\tau \\
&\leq \int_0^t \|f(\tau, x_u(\tau), u(\tau)) - f(\tau, x_v(\tau), u(\tau))\| d\tau \\
&+ \int_{E_t} \|f(\tau, x_v(\tau), u(\tau)) - f(\tau, x_v(\tau), v(\tau))\| d\tau \\
&+ \int_{E'_t} \|f(\tau, x_v(\tau), v(\tau))\| d\tau + \int_{E'_t} \|f(\tau, x_v(\tau), u(\tau))\| d\tau \\
&\leq \kappa \int_0^t \|x_u(\tau) - x_v(\tau)\| d\tau + \sigma(meas(E_t)) + 2M'(meas(E'_t)).
\end{aligned}$$

Since $meas(E_t) \leq T$ and $meas(E'_t) \leq meas(I_1) < \sigma$, we have

$$\|x_u(t) - x_v(t)\| \leq \kappa \int_0^t \|x_u(\tau) - x_v(\tau)\|\, d\tau + \sigma(T + 2M').$$

Applying Gronwall's lemma we obtain

$$\|x_u(t) - x_v(t)\| \leq \sigma(T + 2M')\exp(\kappa T) < \epsilon' = \min\{\epsilon, \epsilon_0\},$$

for all $t \in I$. □

Remark 1. Without of the boudedness of the set of controls, we can find two controls $u(.)$ and $v(.)$ in $L^1(I;\mathbb{R}^m)$ very close in norm but $f(t,x,u)$ and $f(t,x,v)$ may be quite different. For instance, for $f(t,x,u) = u^2$, $t \in [0,1]$, if we take $u(t) = 0$ and $v(t) = \varepsilon t^{-1/2}$, we have $\|u(.) - v(.)\|_{L^1} = 2\varepsilon$, $f(t,x,0) = 0$ and $f(t,x,v) = \varepsilon^2 t^{-1}$ is not in L^1–integrable in any neighborhood of $t = 0$. This example shows also there are no AC solutions passing through the point $(t, x_0) = (0,0)$ with $v(t) = \varepsilon t^{-1/2}$ (see [14], p. 506).

3 Approximate constrained optimal control problem via the exterior penalty method

3.1 Approximate constrained problem via a sequence of unconstrained optimal control problems

Using the penalty function method we consider the following sequence of unconstrained optimal control corresponding to the problem (P),

$$(P_n) \qquad \min_{(x,u)\in\mathcal{T}} F_n(x,u) = F(x,u) + c_n G(x,u),$$

where (c_n) is an increasing sequence of positive real numbers, and

$$G(x,u) = \sum_{i=1}^{s} \int_0^T \max(0, g_i(t, x(t), u(t)))dt + \sum_{i=s+1}^{r} \int_0^T |g_i(t, x(t), u(t))|\, dt.$$

Under the above assumptions it is clear by Proposition 2.1 that the functional F is lower semicontinous from $L^1(I;\mathbb{R}^{\mathbf{n}}) \times L^1(I;\mathbb{R}^m)$ to $\mathbb{R}_+$, and G is lower semicontinous from $L^1(I;\mathbb{R}^{\mathbf{n}}) \times L^1(I;\mathbb{R}^m)$ to $\mathbb{R}_+$, so $F_n(x,u)$ is also lower semicontinuous. Further, we have

$$G(x,u) \geq 0 \quad \text{for all } (x,u) \in L^1(I;\mathbb{R}^{\mathbf{n}}) \times L^1(I;\mathbb{R}^m),$$

and

$$\Omega_{in} \cap \Omega_{eq} = \left\{(x,u) \in AC(I;\mathbb{R}^n) \times L^1(I;\mathbb{R}^m) : G(x,u) = 0\right\}.$$

It is well known that the penalty function methods are very effective techniques to prove the existence of optimal solution for constrained optimization problems via unconstrained problems. The main question is the convergence of the sequence of the solutions of the unconstrained optimal control problems to the solution of the constrained one, and the nonlinearity of the state equation causes more difficulties. To overcome this, we consider the following set of admissible controls.

Definition 1. A subset $\Gamma \subset L^p((a,b)\,;\mathbb{R}^m)$, $p \in [1,+\infty)$, is said to be **L^p-equicontinuous**, if

$$\lim_{h\to 0} \int_a^{b-h} \|u(t+h) - u(t)\|^p \, dt = 0, \quad \text{uniformly} \quad \text{for } u \in \Gamma.$$

Let

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T,$$

be an arbitrary partition of the interval $[0,T]$, which divides the interval into N subintervals (subdivisions) (t_i, t_{i+1}). Let Γ_i a subset of $L^p([t_i,t_{i+1}];\mathbb{R}^m)$, $i = 0,\ldots,N-1$.

The set of admissible controls is given by

$$\mathcal{U}_{ad} = \{u : u(t) = v_i(t), t \in [t_i,t_{i+1}), v_i \in \Gamma_i \text{ and } \Gamma_i \text{ is } L^p - \text{equicontinuous for all } i\}.$$

Proposition 3.1. *If for all i, Γ_i is bounded in $L^p([t_i,t_{i+1}];\mathbb{R}^m)$, then $\mathcal{U}_{ad}$ is a totally bounded set of $L^p(I,\mathbb{R}^m)$, $1 \leq p < +\infty$.*

Proof. By the celebrated Theorem of Riesz-Fréchet-Kolmogorov (see e.g. [6, Theorem IV.26] Γ_i is totally bounded, for all i. Now, $\mathcal{U}_{ad}$ is totally bounded as a finite sum of totally bounded sets. □

The following are examples of classes of controls which are in $\mathcal{U}_{ad}$:

1. If there exists a finite number of constants k_i (independent of $v_i(.)$) such that $v_i(.)$ is k_i-Lipschitz in the subinterval (t_i,t_{i+1}) for all $v_i(.) \in \Gamma_i$ (see [2, Proposition 3]).

2. Recall that an integrable function v on $[0,T]$ is of bounded variation if it has finite essential or total variation, that is, if

$$Var(v) = \sup \sum_{i=0}^{N-1} \|v(t_{i+1}) - v(t_i)\| < \infty,$$

   where the supremum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, such that each t_i is a point of approximate continuity of v (that is, $meas\{t : |t-t_i| < \delta, \|v(t) - v(t_i)\| \geq \varepsilon\} \to 0$ as $\delta \to 0$). Let

$$\mathcal{F} = \{u : Var(u) \leq C\}.$$

   If $\mathcal{F}$ is bounded in $L^1(I,\mathbb{R}^m)$; then is a relatively compact set in $L^1(I,\mathbb{R}^m)$.

3. For a given integer N, the set of controls which are piecewise constant at most N points of discontinuity $t_0, \dots t_{N-1}$, on $[0,T]$. In this case, the control is approximated by a piecewise constant function as follows:

$$u_N(t) = \sum_{i=0}^{N-1} \sigma_i \chi_{[t_i,t_{i-1})}(t),$$

where $\sigma_i \in \mathbb{R}^m$ are fixed, $i = 0, \dots, N-1$ and χ_E is the indicator function of a subset E of $\mathbb{R}$, that is, $\chi_E(t) = 1$ if $t \in E$ and $\chi_E(t) = 0$ otherwise.

Now, if assume that $\mathcal{U}_{\mathbf{ad}} = \mathcal{U}_{ad} \cap \mathcal{U}$, that is all controls u such that $u(t) \in \mathbb{U}$ for almost all $t \in [0,T]$ and $u(t) = v_i(t), t \in [t_i, t_{i+1})$, $v_i \in \Gamma_i$ with Γ_i is L^1- equicontinuous on $L^1((t_i, t_{i+1}); \mathbb{R}^m)$ for , $i = 0, \dots, N-1$. Then by Proposition 3.1, $\mathcal{U}_{\mathbf{ad}}$ is a compact subset of $L^1(I, \mathbb{R}^m)$.

The following result gives an existence theorem of an optimal solution for problem (P).

Theorem 3.1. *Suppose that hypotheses* (**A-1**), (**A-2**) *and* (**A-3**) *hold and the functional $F(.,.)$ is not identical to $+\infty$. If the controls lie in $\mathcal{U}_{\mathbf{ad}}$, then there exists an optimal solution for problem (P).*

Proof. Denote by $\mathcal{V}(P)$ the value of (P). By Proposition 2.1, we can assert that problem (P) has a finite value $\mathcal{V}(P)$. Consequently, there exists a minimizing sequence $(x_k(.), u_k(.)) \in \mathcal{T}$ and $u_k(.) \in \mathcal{U}_{\mathbf{ad}}$ such that

$$F(x_k(.), u_k(.)) \leq \mathcal{V}(P) + \frac{1}{k}.$$

Since $u_k(.) \in \mathcal{U}_{\mathbf{ad}}$, by Proposition 3.1 we then conclude that $(u_k(.))_k$ contains a subsequence noted again by $(u_k(.))_k$ which converges strongly to v in $L^1(I, \mathbb{R}^m)$. By [6, Théoréme IV.9], the sequence $(u_k(.))_k$ contains a subsequence, noted again $(u_k(.))_k$, such that

- $\bar{u}_k(t) \longrightarrow v(t)$ for almost all $t \in I$, and
- there exists $h(.) \in L^1(I, \mathbb{R}^m)$ such that $\|u_k(t)\| \leq h(t)$ for all k and almost all $t \in I$.

Now, if we denote by $x_k(.)$ the response of $u_k(.)$, then by Proposition 2.2, the continuity of the input-output maps $u(.) \longrightarrow x_u(.)$ assert that there exists $y(.) \in C(I, \mathbb{R}^{\mathbf{n}})$ such that $x_k(.) \longrightarrow y(.) \in C(I, \mathbb{R}^n)$ strongly with $y(.)$ is the response of $v(.)$, this means $(y(.), v(.)) \in \mathcal{T}$. Under assumption (**A-3**) the mappings g_i are Carathéodory mapping, thus

$$g_i(t, x_k(t), u_k(t)) \longrightarrow g_i(t, y(t), v(t))$$

for almost all $t \in [0,T]$ and for every $i = 1 \dots r$, with

$$\begin{array}{rclll} g_i(t, y(t), v(t)) & \leq & 0 & i = 1 \dots s & \text{for almost all } t \in [0,T] \\ g_i(t, y(t), v(t)) & = & 0 & i = s+1 \dots r & \text{for almost all } t \in [0,T]. \end{array}$$

Consequently $(y, v) \in \Omega$. Now, by Proposition 2.1 F is lower semicontinous, then

$$F(y(.), v(.)) \leq \liminf_{k \longrightarrow +\infty} F(x_k(.), u_k(.)) \leq \mathcal{V}(P).$$

Hence $F(y(.), v(.)) = \mathcal{V}(P)$, this implies that $(y(.), v(.))$ is an optimal solution of (P). □

Now the first convergence theorem of the partially penalty method reads as follows.

Theorem 3.2. *Suppose that hypotheses* (**A-1**), (**A-2**) *and* (**A-3**) *hold, the functional $F(.,.)$ is not identical to $+\infty$ and the controls lie in $\mathcal{U}_{\mathbf{ad}}$. Then*

1. *The problem* $(\mathcal{P}_n)$ *is solvable for every* $c_n > 0$.

2. *The sequence* $(\bar{x}_n(.), \bar{u}_n(.))_n$ *of optimal solutions of the problem* $(\mathcal{P}_n)$ *contains a convergent subsequence* $(\bar{x}_k(.), \bar{u}_k(.))_k$*; such that*

   - $\bar{x}_k \longrightarrow \bar{x}(.)$ strongly in $C(I, \mathbb{R}^{\mathbf{n}})$,
   - $\dot{\bar{x}}_k(.) \longrightarrow \dot{\bar{x}}(.)$ weakly in $L^1(I, \mathbb{R}^{\mathbf{n}})$,
   - $\bar{u}_k(.) \longrightarrow \bar{u}(.)$ strongly in $L^1(I, \mathbb{R}^m)$,

3. *The limit* $(\bar{x}(.), \bar{u}(.))$ *is an optimal solution of original problem* (P).

Proof. (1) We put

$$\mathcal{V}(P_n) = \inf \{F_n(x,u), (x,u) \in \mathcal{T}, u \in \mathcal{U}_{\mathbf{ad}}\}$$

$\mathcal{V}(P_n)$ is finite; in fact, let $(x(.), u(.)) \in \mathcal{T}$ and $u(.) \in \mathcal{U}_{\mathbf{ad}}$, we have

$$F(x(.), u(.)) \leq F_n(x(.), u(.)).$$

By Proposition 2.1, F is bounded below, it follows that F_n is also bounded below. Consequently, there exists a minimizing sequence $(x_k(.), u_k(.)) \in \mathcal{T}$ and $u_k(.) \in \mathcal{U}_{\mathbf{ad}}$ such that

$$F_n(x_k(.), u_k(.)) \leq \mathcal{V}(P_n) + \frac{1}{k}.$$

Since $u_k(.) \in \mathcal{U}_{ad}$, by Proposition 3.1 we then conclude that $(u_k(.))_k$ contains a subsequence noted again by $(u_k(.))_k$ which converges strongly to v in $L^1(I, \mathbb{R}^m)$. Now, if we denote by $x_k(.)$ the response of $u_k(.)$, then by Proposition 2.2, the continuity of the input-output maps $u(.) \longrightarrow x_u(.)$ ensures that there exists $y(.) \in C(I, \mathbb{R}^{\mathbf{n}})$ such that $x_k(.) \longrightarrow y(.) \in C(I, \mathbb{R}^{\mathbf{n}})$ strongly, where $y(.)$ is the response of $v(.)$, this means $(y(.), v(.)) \in \mathcal{T}$. Now, F_n is lower semicontinous, then

$$F_n(y(.), v(.)) \leq \liminf_{k \longrightarrow +\infty} F_n(x_k(.), u_k(.)) \leq \mathcal{V}(P_n).$$

Hence $F_n(y(.), v(.)) = \mathcal{V}(P_n)$, this implies that $(y(.), v(.))$ is an optimal solution of (P_n), noted in the sequel by $(\bar{x}_n(.), \bar{u}_n(.))$.

(2) We have $\Omega \subset \mathcal{T}$, then $\mathcal{V}(P_n) \leq \mathcal{V}(P)$ since $G(x(.), u(.)) = 0$ in $\Omega_{in} \cap \Omega_{eq}$.

On other hand, we have that $G(.,.)$ is nonnegative and hence

$$F(\bar{x}_n(.), \bar{u}_n(.)) \leq F_n(\bar{x}_n(.), \bar{u}_n(.)) \leq \mathcal{V}(P).$$

Again since $(\bar{u}_n(.))_n \subset \mathcal{U}_{\mathbf{ad}}$, by Proposition 3.1, the sequence $(\bar{u}_n(.))_n$ contains a converging subsequence $(\bar{u}_k(.))_k$ to $\bar{u}(.)$ strongly on $L^1(I, \mathbb{R}^m)$. The map $u \longrightarrow x_u$ is continuous (Proposition 2.2), which implies that $\bar{x}_k(.) \longrightarrow \bar{x}(.)$ in $C(I, \mathbb{R}^{\mathbf{n}})$ strongly and $\bar{x}_k(T) \longrightarrow \bar{x}(T) = 0$. This implies $(\bar{x}(.), \bar{u}(.)) \in \mathcal{T}$.

(3) To complete the proof we prove that $(\bar{x}(.), \bar{u}(.))$ is the optimal solution of (P). First, we have that $G(.,.)$ is lower semicontinous, then

$$0 \leq G(\bar{x}(.), \bar{u}(.)) \leq \liminf_{k \longrightarrow +\infty} G(\bar{x}_k(.), \bar{u}_k(.)). \tag{3.1}$$

On other hand, $\lim_{k \longrightarrow +\infty} G(\bar{x}_k(.), \bar{u}_k(.))) = 0$, in fact

$$0 \leq G(\bar{x}_k(.), \bar{u}_k(.)) \leq \frac{1}{c_k} (\mathcal{V}(P) - F(\bar{x}_k(.), \bar{u}_k(.))). \tag{3.2}$$

From this point, we can distinguish two cases:

Case 1: There exist k_0 such that $F(\bar{x}_{k_0}(.), \bar{u}_{k_0}(.)) = \mathcal{V}(P)$. In this case we have $\mathcal{V}(\mathcal{P}_k) = \mathcal{V}(P)$ and $G(\bar{x}_k(.), \bar{u}_k(.)) = 0$ for all $k \geq k_0$, so $G(\bar{x}_{k_0}(.), \bar{u}_{k_0}(.)) = 0$, this implies that $(\bar{x}(.), \bar{u}(.)) = (\bar{x}_{k_0}(.), \bar{u}_{k_0}(.)) \in \Omega_{in} \cap \Omega_{eq}$.

Case 2: $F(\bar{x}_k(.), \bar{u}_k(.)) \neq \mathcal{V}(P)$ for all k. In this case by virtue (3.1) and (3.2) by letting k to $+\infty$, we deduce that $G(\bar{x}(.), \bar{u}(.)) = 0$.

We conclude that $(\bar{x}(.), \bar{u}(.))$ is an admissible point, and $\mathcal{V}(P) \leq F(\bar{x}(.), \bar{u}(.))$ in the two cases. Now, since $F(.,.)$ is lower semicontinous, we obtain

$$F(\bar{x}(.), \bar{u}(.)) \leq \liminf_{k \longrightarrow +\infty} F(\bar{x}_k(.), \bar{u}_k(.)) \leq \mathcal{V}(P),$$

that is $F(\bar{x}(.), \bar{u}(.)) = \mathcal{V}(P)$, this implies that the pair $(\bar{x}(.), \bar{u}(.))$ is an optimal solution of the problem (P). □

Remark 2. Without the boudedness of the set of controls, i.e. $u \in \mathcal{U}_{ad}$, we can derive a similar convergence theorem to Theorem 3.2, by assuming further that the mapping f is Lipschitz in u and l is coercive on u, i.e. there exists $K_x > 0$ and $K_u > 0$ such that

$$\|f(t, x_1, u_1) - f(t, x_2, u_2)\| \leq K_x \|x_1 - x_2\| + K_u \|u_1 - u_2\|,$$

for almost all $t \in [0, T]$, and

(A-4) there exists $\beta > 0$ such that

$$l(t, x, u) \geq \beta \|u\|,$$

for almost all $t \in [0, T]$ and for all $x \in \mathbb{R}^{\mathbf{n}}$. See Theorem 1 of [2].

3.2 Approximate constrained control problem via an unconstrained optimization problems

Using the exterior penalty function method we consider the following sequence of infinite-dimensional unconstrained optimization problems corresponding to problem (P),

$$(\mathcal{P}_n) \qquad \min_{(x,u) \in AC_0 \times \mathcal{U}_{\mathbf{ad}}} \Phi_n(x, u) = F(x, u) + c_n H(x, u) + c_n G(x, u),$$

where,

- AC_0 is the space of absolutely continuous maps from $[0, T]$ to $\mathbb{R}^{\mathbf{n}}$ such that $x(0) = x_0$ and $x(T) = 0$,
- (c_n) is an increasing sequence of positive real numbers,
- $$H(x, u) = \int_0^T \|\dot{x}(t) - f(t, x(t), u(t))\| \, dt,$$
  and
- $$G(x, u) = \sum_{i=1}^{s} \int_0^T \max(0, g_i(t, x(t), u(t)))dt + \sum_{i=s+1}^{r} \int_0^T |g_i(t, x(t), u(t))| \, dt.$$

Under assumption (**A-2**)-(i) and the continuity of the norm, we can assert that $H(.,.)$ is lower semicontinuous form $AC([0,T],\mathbb{R}^{\mathbf{n}}) \times L^1([0,T],\mathbb{R}^m)$ to $\mathbb{R}_+$. So then $\Phi_n(x,u)$ is also bonded below and lower semicontinuous from $AC([0,T],\mathbb{R}^{\mathbf{n}}) \times L^1([0,T],\mathbb{R}^m)$ to $\mathbb{R}_+$. The next result is the second convergence theorem for the totally penalty method.

Theorem 3.3. *Suppose that hypotheses* (**A-1**), (**A-2**)-(i), *and* (**A-3**) *hold, the functional* $F(.,.)$ *is not identical to* $+\infty$ *and the controls lie in* $\mathcal{U}_{\mathbf{ad}}$, *then we have:*

1. *Problem* $(\mathcal{P}_n)$ *is solvable for every* $c_n > 0$.
2. *The sequence* $(\bar{x}_n(.), \bar{u}_n(.))_n$ *of optimal solutions of problem* $(\mathcal{P}_n)$ *contains a convergent subsequence* $(\bar{x}_k(.), \bar{u}_k(.))_k$; *such that*
   - $\bar{x}_k \longrightarrow \bar{x}(.)$ strongly in $C(I,\mathbb{R}^{\mathbf{n}})$,
   - $\dot{\bar{x}}_k(.) \longrightarrow \dot{\bar{x}}(.)$ weakly in $L^1(I,\mathbb{R}^{\mathbf{n}})$,
   - $\bar{u}_k(.) \longrightarrow \bar{u}(.)$ strongly in $L^1(I,\mathbb{R}^m)$.
3. *The limit* $(\bar{x}(.), \bar{u}(.))$ *is an optimal solution of original problem* (P).

Proof. Denote by $\mathcal{V}(\mathcal{P}_n)$ the value of $(\mathcal{P}_n)$. Since the sequence (c_n) is non-decreasing then $(\mathcal{V}(\mathcal{P}_n))_n$ is a decreasing sequence bounded from above by $\mathcal{V}(P)$ the value of problem (P) which is finite. $\Phi_n(x,u)$ is bounded below, so there exists a minimizing sequence $(x_k(.), u_k(.)) \in AC_0 \times \mathcal{U}_{\mathbf{ad}}$ such that

$$\Phi_n(x_k(.), u_k(.)) \leq \mathcal{V}(\mathcal{P}_n) + \frac{1}{k}. \tag{3.3}$$

Since $(u_k(.))_k \subset \mathcal{U}_{\mathbf{ad}}$, by Proposition 3.1 we conclude that $(u_k(.))_k$ contains a subsequence, noted again, by $(u_k(.))_k$ which converges strongly to $w \in \mathcal{U}_{\mathbf{ad}}$ in L^1 and there is a constant $\mu > 0$ such that $\|u_k(.)\| \leq \mu$. On the other hand, the functions $x_k(.)$ are uniformly bounded, in fact, if we observe that

$$\langle \dot{x}_k(t), x_k(t)\rangle = \langle z_k(t), x_k(t)\rangle + \langle f(t, x_k(t), u_k(t)), x_k(t)\rangle,$$

where $z_k(t) = \dot{x}_k(t) - f(t, x_k(t), u_k(t))$, then we have

$$\begin{aligned}
\langle \dot{x}_k(t), x_k(t)\rangle &\leq |\langle z_k(t), x_k(t)\rangle| + |\langle f(t, x_k(t), u_k(t)), x_k(t)\rangle|| \\
&\leq \|z_k(t)\|\|x_k(t)\| + \theta_1(t)\|x_k(t)\| + \rho_1(t)\|u_k(t)\|\|x_k(t)\| + \rho_1(t)\|x_k(t)\|^2 \\
&\leq (\|z_k(t)\| + \theta_1(t) + \rho_1(t)\|u_k(t)\|)\|x_k(t)\| + \rho_1(t)\|x_k(t)\|^2.
\end{aligned}$$

On the other hand, we have

$$\langle \dot{x}_k(t), x_k(t)\rangle = \frac{1}{2}\frac{d}{dt}\|x_k(t)\|^2 = \|x_k(t)\|\frac{d}{dt}\|x_k(t)\|.$$

Therefore,

$$\|x_k(t)\| \leq \|x_k(0)\| + \int_0^t (\|z_k(t)\| + \theta_1(\tau) + \rho_1(\tau)\|u_k(\tau)\|)d\tau + \int_0^t \rho_1(\tau)\|x_k(\tau)\|d\tau,$$

and

$$\|x_k(t)\| \leq \|x_0\| + (M_1 + \|\theta_1(.)\|_{L^\infty} + \|\rho_1(.)\|_{L^\infty}\|u_k(.)\|_{L^1}) + \int_0^t \rho_1(\tau)\|x_k(\tau)\|d\tau.$$

Here $\|z_k(.)\|_{L^1} \leq M_1$ for some $M_1 > 0$ which follows by (3.3). So, we have

$$\|x_k(t)\| \leq \|x_0\| + (M_1 + \|\theta_1(.)\|_{L^\infty} + \|\rho_1(.)\|_{L^\infty}\|u_k(.)\|_{L^1}) + \int_0^t \rho_1(\tau)\|x_k(\tau)\|d\tau.$$

Applying Gronwall's lemma we obtain

$$\|x_k(t)\| \leq \mathcal{K}, \qquad \text{for all } t \in [0,T],$$

for k sufficiently large, with $\mathcal{K} = (\|x_0\| + M_1 + \|\theta_1(.)\|_{L^\infty} + \|\rho_1(.)\|_{L^\infty}\mu)\exp(\|\rho_1(.)\|_{L^\infty})$. This proves the boundedness of $(x_k(.))_k$ independently of k and t. We shall now show that $(x_k(.))_k$ is also equicontinuous. To this end we must first show that $(\dot{x}_k(.))_k$ is equi-integrable in $L^1(I,\mathbb{R}^{\mathbf{n}})$. If I_0 is any Lebesgue measurable subset of I, we have

$$\begin{aligned}\int_{I_0}\|\dot{x}_k(\tau)\|\,d\tau &\leq \int_{I_0}\|z_k(\tau)\|\,d\tau + \int_{I_0}\|f(\tau,x_k(\tau),u_k(\tau))\|\,d\tau\\ &\leq \int_{I_0}\|z_k(\tau)\|\,d\tau + \int_{I_0}\theta_1(\tau)\,d\tau + \int_{I_0}\rho_1(\tau)\,(\|x_k(\tau)\| + \|u_k(\tau)\|)d\tau\\ &\leq \int_{I_0}\|z_k(\tau)\|\,d\tau + \int_{I_0}\theta_1(\tau)\,d\tau + M_2\int_{I_0}\rho_1(\tau)\,d\tau.\end{aligned}$$

The right-hand side approaches zero as $\text{meas}(I_0) \longrightarrow 0$, so $(\dot{x}_k(.))_k$ is equi-integrable in $L^1(I,\mathbb{R}^{\mathbf{n}})$. By the Dunfords-Pettis theorem, we can extract a further subsequence from $(\dot{x}_k(.))_k$ such that $\dot{x}_k(.)$ converges weakly to, say, $\sigma(.)$ in L^1. On the other hand, since $(\dot{x}_k(.))_k$ is equi-integrable, the sequence $(x_k(.))_k$ is equicontinous, indeed, for all $t_1, t_2 \in [0,T]$ such that $|t_2 - t_1| \leq \delta$ we have

$$\|x_k(t_1) - x_k(t_2)\| = \left\|\int_{t_1}^{t_2}\dot{x}_k(\tau)d\tau\right\| \leq \int_{t_1}^{t_2}\|\dot{x}_k(\tau)\|\,d\tau \leq \epsilon, \qquad \text{for all } k.$$

Now, since the sequence $(x_k(.))_k$ is equibounded and equicontinous, by Arzela's theorem there exists a subsequence, again denoted by $(x_k(.))_k$ which converges uniformly on $[0,T]$ to a continuous function $v(.)$. The equality

$$x_k(t) = x_0 + \int_0^t \dot{x}_k(\tau)\,d\tau\ ;\ \text{for all } t \in\ I,$$

implies that

$$v(t) = x_0 + \int_0^t \sigma(\tau)d\tau\ ;\ \text{for all } t \in\ I,$$

and hence $\dot{v}(t) = \sigma(t)$ for almost all $t \in I$. Now, because the uniform convergence of $(x_k(.))_k$ to $v(.)$, the weak convergence of $\dot{x}_k(.)$ to $\dot{v}(.)$ in L^1 and the strong convergence of $(u_k(.))_k$ to w in L^1, we have $z_k(.)$ converge weakly to $z(.)$ in $L^1(I,\mathbb{R}^{\mathbf{n}})$ where $z(t) = \dot{v}(t) - f(t,v(t),w(t))$. Then,

$$\begin{aligned}H(v,w) &= \int_0^T \|\dot{v}(\tau) - f(\tau,v(\tau),w(\tau))\|\,d\tau\\ &\leq \liminf_{k\to\infty}\int_0^T \|\dot{x}_k(\tau) - f(\tau,x_k(\tau),u_k(\tau))\|\,d\tau.\end{aligned}\tag{3.4}$$

Furthermore,

$$F(v,w) \leq \liminf_{k\to\infty} F(x_k,u_k) \quad \text{and} \quad G(v,w) \leq \liminf_{k\to\infty} G(x_k,u_k).\tag{3.5}$$

Since $F(.,.)$ and $G(.,.)$ are lower semicontinuous, by virtue of (3.4), (3.5) and (3.3), it follows that

$$\Phi_n(v,w) \leq \liminf_{k\to\infty}\Phi_n(x_k,u_k) \leq \mathcal{V}(\mathcal{P}_n).$$

Hence, $\Phi_n(v(.),w(.)) = \mathcal{V}(\mathcal{P}_n)$ which implies that $(v(.),w(.))$ is an optimal solution of $(\mathcal{P}_n)$, noted in the sequel by $(\bar{x}_n(.),\bar{u}_n(.))$. We shall show that the sequence $(\bar{x}_n(.),\bar{u}_n(.))_n$ contains a convergent subsequence to the solution of the original problem (P). Let $\mathcal{V}(P)$ be the value of problem (P). Since $G(.,.) + H(.,.)$ is nonnegative, then

$$F(\bar{x}_n(.), \bar{u}_n(.)) \leq \Phi_n(\bar{x}_n(.), \bar{u}_n(.)) \leq \mathcal{V}(P).$$

Again, since $(\bar{x}_n(.), \bar{u}_n(.))_n \subset AC_0 \times \mathcal{U}_{\mathbf{ad}}$, by the same argument as before we show that the sequence $(\bar{u}_n(.))_n$ contains a convergent subsequence $(\bar{u}_k(.))_k$ to $\bar{u}(.)$ strongly on $L^1(I, \mathbb{R}^m)$ and the sequence $(\bar{x}_n(.))_n$ contains a convergent subsequence $(\bar{x}_k(.))_k$ to $\bar{x}(.)$ uniformly on $C(I, \mathbb{R}^{\mathbf{n}})$, which has a derivative $\dot{\bar{x}}(.)$ belonging to $L^1(I, \mathbb{R}^{\mathbf{n}})$, where $\bar{x}(0) = x_0$ and $\bar{x}(T) = 0$. This implies that $(\bar{x}(.), \bar{u}(.)) \in AC_0 \times \mathcal{U}_{\mathbf{ad}}$. We have $G(.,.) + H(.,.)$ is lower semicontinous, then

$$0 \leq G(\bar{x}(.), \bar{u}(.)) + H(\bar{x}(.), \bar{u}(.)) \leq \liminf_{k \longrightarrow +\infty} (G(\bar{x}_k(.), \bar{u}_k(.)) + H(\bar{x}_k(.), \bar{u}_k(.))). \tag{3.6}$$

On other hand, $\liminf_{k \longrightarrow +\infty} (G(\bar{x}_k(.), \bar{u}_k(.)) + H(\bar{x}_k(.), \bar{u}_k(.))) = 0$, in fact

$$0 \leq G(\bar{x}_k(.), \bar{u}_k(.) + H(\bar{x}_k(.), \bar{u}_k(.))) \leq \frac{1}{c_k} (\mathcal{V}(P) - F(\bar{x}_k(.), \bar{u}_k(.))). \tag{3.7}$$

Now we proceed as in the proof of Theorem 3.2 to prove that the pair $(\bar{x}(.), \bar{u}(.))$ is an optimal solution of problem (P). □

Remark 3. Note that Lipschitz condition **(A-2)**-(ii) is not assumed in the preceding theorem, so the uniqueness of the solution for equation (2.2) is not required.

4 First order necessary conditions for optimality

We shall next turn to first order necessary conditions for optimality. To see this, let us consider the control process in the Hilbert space $H^1(I, \mathbb{R}^{\mathbf{n}}) \times L^2(I, \mathbb{R}^m)$, where H^1 is the Sobolev space of all functions in $L^2(I, \mathbb{R}^{\mathbf{n}})$ such that its weak derivatives lies in $L^2(I, \mathbb{R}^{\mathbf{n}})$. We shall first obtain necessary conditions for optimality for the non-dynamic problem $(\mathcal{P}_n)$ and then show how under suitable limiting conditions they lead to the Pontryagin maximum principle for the sequence of the dynamic optimal control problem (P_n).

Suppose, for each $c_n > 0$, we can solve the following differentiable unconstrained optimization problem

$$(\mathcal{P}_n) \qquad \min_{(x,u) \in H_0^1 \times \mathcal{U}_{\mathbf{ad}}^2} \Phi_n(x, u) = F(x, u) + c_n H(x, u) + c_n G(x, u),$$

where,

- H_0^1 is the space of all absolutely continuous function x from $[0, T]$ to $\mathbb{R}^{\mathbf{n}}$ such that its derivatives lies in $L^2(I, \mathbb{R}^{\mathbf{n}})$ with $x(0) = x_0$ and $x(T) = 0$,
- $\mathcal{U}_{\mathbf{ad}}^2 = \mathcal{U}_{\mathbf{ad}} \cap L^2(I, \mathbb{R}^m)$,
- (c_n) is an increasing sequence of positive real numbers,
- $$H(x, u) = \frac{1}{2} \int_0^T \|\dot{x}(t) - f(t, x(t), u(t))\|^2 \, dt,$$

  and

- $$G(x,u)=\frac{1}{2}\sum_{i=1}^{s}\int_0^T \max(0,g_i\left(t,x\left(t\right),u\left(t\right)\right))^2dt+\frac{1}{2}\sum_{i=s+1}^{r}\int_0^T \left|g_i\left(t,x\left(t\right),u\left(t\right)\right)\right|^2dt.$$

To derive necessary conditions of optimality satisfied by an optimal control, we need to make the following assumptions on the data of the problem:

(H_1) (i) The mapping l is a C^1-Carathéodory mapping, i.e., l is C^1 in (x,u) for almost all $t\in[0,T]$ and is measurable in t for every $(x,u)\in\mathbb{R}^n\times\mathbb{R}^m$;

(ii) there exist $\theta\in L^2([0,T],\mathbb{R}_+)$, $\rho\in L^2([0,T],\mathbb{R}_+)$ and $\beta_1(\cdot)\in L^2(I,\mathbb{R}^+)$ such that for almost all $t\in[0,T]$,

$$|l(t,x,u)|\le\theta(t)+\rho(t)(\|x\|+\|u\|)$$

and

$$\|\nabla_x l(t,x,u)\|+\|\nabla_u l\left(t,x,u\right)\|\le\beta_1\left(t\right),$$

where $\nabla_j l$ is the partial derivative of l with respect to its jth argument, $j\in\{x,u\}$.

(H_2) (i) The mapping f is a C^1-Carathéodory mapping and there exist $\theta_1\in L^\infty\left([0,T],\mathbb{R}_+\right)$ and $\rho_1\in L^\infty\left([0,T],\mathbb{R}_+\right)$ such that for almost all $t\in[0,T]$,

$$\|f\left(t,x,u\right)\|\le\theta_1\left(t\right)+\rho_1\left(t\right)\left(\|x\|+\|u\|\right),$$

(ii) there exists $\alpha\left(.\right)\in L^2\left(I,\mathbb{R}^+\right)$ and $\gamma'\left(.\right)\in L^2\left(I,\mathbb{R}^+\right)$ such that

$$\|\nabla_x f\left(t,x,u\right)\|\le\alpha\left(t\right)$$

and

$$\|\nabla_u f\left(t,x,u\right)\|\le\gamma'\left(t\right),$$

for almost all $t\in[0,T]$.

(H_3) (i) For all i; the mapping g_i is a C^1-Carathéodory mapping and there exist $\theta_2\in L^2\left([0,T],\mathbb{R}_+\right)$ and $\rho_2\in L^2\left([0,T],\mathbb{R}_+\right)$ such that for almost all $t\in[0,T]$,

$$|g_i\left(t,x,u\right)|\le\theta_2\left(t\right)+\rho_2\left(t\right)\left(\|x\|+\|u\|\right),$$

(ii) there exist $\rho_3\in L^2\left([0,T],\mathbb{R}_+\right)$ such that for almost all $t\in[0,T]$,

$$\|\nabla_x g_i(t,x,u)\|+\|\nabla_u g_i\left(t,x,u\right)\|\le\rho_3\left(t\right),$$

for all i.

Let

$$g(t,x,u)=(g_1(t,x,u),\ldots,g_r(t,x,u)),$$

$$\nabla_x g(t,x,u)=(\frac{\partial}{\partial x_i}g_j(t,x,u))_{ij},\quad 1\le i\le n,1\le j\le r,$$

$$\nabla_u g(t,x,u)=(\frac{\partial}{\partial u_i}g_j(t,x,u))_{ij},\quad 1\le i\le m,1\le j\le r,$$

and $\rho(g)=\rho(g(t,x,u))$ the vector in $\mathbb{R}^r$ defined as follows,

- $\rho_i(g)=g_i\left(t,x,u\right)$ if $g_i\left(t,x,u\right)\ge0$ and $\rho_i(g)=0$ if $g_i\left(t,x,u\right)\le0$, for $i=1\ldots s$.

- $\rho_i(g) = g_i(t, x, u)$, for $i = s+1 \dots r$.

We see that for every fixed n and fixed control u, the problem $(\mathcal{P}_n)$ becomes a problem of the calculus of variations. We have the following necessary optimality conditions of Euler–Lagrange type.

Theorem 4.1. *If for every n $(\bar{x}_n(\cdot), \bar{u}_n(\cdot)) \in H_0^1 \times U_{\boldsymbol{ad}}$ is an optimal solution to $(\mathcal{P}_n)$, then*

$$\frac{d}{dt} z_n(t) = -\nabla_x f(t, \bar{x}_n(t), \bar{u}_n(t))^\top z_n(t) + \frac{1}{c_n} \nabla_x l(t, \bar{x}_n(t), \bar{u}_n(t)) + \nabla_x g(t, \bar{x}_n(t), \bar{u}_n(t)) \rho(g_n) \tag{4.1}$$

$$\nabla_u f(t, \bar{x}_n(t), \bar{u}_n(t))^\top z_n(t) = \frac{1}{c_n} \nabla_u l(t, \bar{x}_n(t), \bar{u}_n(t)) + \nabla_u g(t, \bar{x}_n(t), \bar{u}_n(t)) \rho(g_n) \tag{4.2}$$

$$z_n(T) = 0, \tag{4.3}$$

for almost all $t \in [0, T]$, where

$$z_n(t) = \dot{\bar{x}}_n(t) - f(t, \bar{x}_n(t), \bar{u}_n(t))$$

and

$$\rho(g_n) = \rho(g(t, \bar{x}_n(t), \bar{u}_n(t))).$$

Proof. Let $(\bar{x}_n(\cdot), \bar{u}_n(\cdot)) \in H_0^1 \times U_{\mathbf{ad}}$ be an optimal solution to $(\mathcal{P}_n)$ for every n. Let $v(t)$ and $w(t)$ be any $n \times 1$ and $m \times 1$ functions, respectively, in the Schwartz space of infinitely smooth functions vanishing outside compact subsets of $(0, T)$. By our assumptions on $l(.)$, $f(.)$, and $g_i(.)$ it follows that the Fréchet derivative of $\Phi_n(x, u)$ with respect to x and u, respectively, equals

$$\int_0^T \langle \nabla_x l(t, \bar{x}_n(t), \bar{u}_n(t)), v(t) \rangle \, dt + c_n \int_0^T \langle z_n(t), \dot{v}(t) - \nabla_x f(t, \bar{x}_n(t), \bar{u}_n(t))(v(t)) \rangle \, dt$$

$$+ c_n \int_0^T \langle \nabla_x g(t, \bar{x}_n(t), \bar{u}_n(t)) \rho(g_n), v(t) \rangle \, dt$$

and

$$\int_0^T \langle \nabla_u l(t, \bar{x}_n(t), \bar{u}_n(t)), w(t) \rangle \, dt + c_n \int_0^T \langle \nabla_u f(t, \bar{x}_n(t), \bar{u}_n(t)), w(t) \rangle \, dt$$

$$+ c_n \int_0^T \langle \nabla_u g(t, \bar{x}_n(t), \bar{u}_n(t)) \rho(g_n), w(t) \rangle \, dt.$$

Since $(\bar{x}_n(\cdot), \bar{u}_n(\cdot))$ minimize $\Phi_n(x, u)$, we get the desired system. Next, specializing $v(.)$ to any smooth function with $v(0) = 0$, $v(T)$ arbitrary and nonzero, it follows that $z_n(T) = 0$. □

The Lagrangian $\mathcal{L}$ from $H_0^1([0, T], \mathbb{R}^{\mathbf{n}}) \times L^2([0, T], \mathbb{R}^m) \times H^1([0, T], \mathbb{R}^{\mathbf{n}}) \times L^2([0, T], \mathbb{R}^s) \times L^2([0, T], \mathbb{R}^{r-s})$ to $\mathbb{R}$, for the initial problem (P) is given by

$$\mathcal{L}(x, u, \psi, \lambda, \mu) = \int_0^T l(t, x(t), u(t) dt + \int_0^T \langle \psi(t), \dot{x}(t) - f(t, x(t), u(t)) \rangle \, dt$$

$$+ \sum_{i=1}^{s} \int_0^T \langle \lambda_i(t), g_i(t, x(t), u(t)) \rangle \, dt + \sum_{i=s+1}^{r} \int_0^T \langle \mu_i(t), g_i(t, x(t), u(t)) \rangle \, dt$$

If $(\bar{x}(.), \bar{u}(.))$ is a local minimum of problem (P) and under suitable constraint qualification conditions, as Robinson's condition [33], there exist nontrivial multipliers $(\psi^*, \lambda^*, \mu^*)$ such that the following conditions are satisfied:

- $(\bar{x}(.), \bar{u}(.))$ is an optimal solution of $\min \mathcal{L}(x, u, \psi^*, \lambda^*, \mu^*)$.
- $\lambda_i^*(t) g_i(t, \bar{x}(t), \bar{u}(t)) = 0$ and $\lambda_i^*(t) \geq 0$, for almost all $t \in [0, T]$ and for every $i = 1 \ldots s$.

In the following statement we give a relation between the multipliers of problem (P) and the necessary optimality conditions given by Theorem 4.1.

Theorem 4.2. *Let $(\bar{x}_n(\cdot), \bar{u}_n(\cdot))$ be an optimal solution of problem $(\mathcal{P}_n)$. If we put*

$$\psi_n(t) = c_n z_n(t) \qquad \text{and} \qquad (\lambda_n, \mu_n) = c_n \rho(g_n).$$

Then $(\psi_n, \lambda_n, \mu_n)_n$ contains a subsequence $(\psi_k, \lambda_k, \mu_k)_k$ such that

- $\psi_k \longrightarrow \psi^*$ strongly in $C(I, \mathbb{R}^{\mathbf{n}})$,
- $\lambda_k \longrightarrow \lambda^*$ weakly in $L^2(I, \mathbb{R}^s)$,
- $\mu_k \longrightarrow \mu^*$ weakly in $L^2(I, \mathbb{R}^{r-s})$,

and $(\psi^*, \lambda^*, \mu^*)$ is a non-trivial Lagrange multiplier associated to the solution optimal of problem (P).

Proof. By Theorem 3.3, we know that the sequence $(\bar{x}_n(.), \bar{u}_n(.))_n$ of optimal solutions of problem $(\mathcal{P}_n)$ contains a convergent subsequence $(\bar{x}_k(.), \bar{u}_k(.))_k$ to the optimal solution $(\bar{x}(.), \bar{u}(.))$ of (P) with the properties:

- $\bar{x}_k(.) \longrightarrow \bar{x}(.)$ strongly in $C(I, \mathbb{R}^{\mathbf{n}})$,
- $\bar{u}_k(.) \longrightarrow \bar{u}(.)$ strongly in $L^2(I, \mathbb{R}^m)$,

By [6, Théoréme IV.9], the sequence $(\bar{u}_k(.))_k$ contains a subsequence, denoted again by $(\bar{u}_k(.))_k$, such that

- $\bar{u}_k(t) \longrightarrow \bar{u}(t)$ for almost all $t \in I$, and
- there exists $h(.) \in L^2(I, \mathbb{R}^m)$ such that $\|\bar{u}_k(t)\| \leq h(t)$ for all k and almost all $t \in I$.

We also have $\bar{x}_k(t) \longrightarrow \bar{x}(t)$, for all $t \in I$. This implies, by (H_3)-(i), that

$$g_i(t, \bar{x}_k(t), \bar{u}_k(t)) \longrightarrow g_i(t, \bar{x}(t), \bar{u}(t)), \qquad \text{for almost all } t \in I, \quad \text{for every } i = 1 \ldots r.$$

and

$$\begin{aligned} |g_i(t, \bar{x}_k(t), \bar{u}_k(t))| &\leq \theta_2(t) + \rho_2(t)(\|\bar{x}_k(t)\| + \|\bar{u}_k(t)\|) \\ &\leq \theta_2(t) + \rho_2(t)(\mathcal{K} + h(t)) = h_1(t), \end{aligned}$$

for all i and almost all $t \in I$, where $h_1(.) \in L^2(I, \mathbb{R})$. Now, Lebesgue's theorem assert that $g_i(., \bar{x}_k(.), \bar{u}_k(.)) \longrightarrow g_i(., \bar{x}(.), \bar{u}(.))$ in the strong topology of $L^2(I, \mathbb{R})$, for every $i = 1 \ldots r$. By definition, we have

$$c_k \max(0, g_i(t, \bar{x}_k(t), \bar{u}_k(t)))^2 = \lambda_{i,k}(t) g_i(t, \bar{x}_k(t), \bar{u}_k(t)), \qquad \text{for every } i = 1 \ldots s,$$

and

$$c_k |g_i(t, \bar{x}_k(t), \bar{u}_k(t))|^2 = \mu_{i,k}(t) g_i(t, \bar{x}_k(t), \bar{u}_k(t)), \qquad \text{for every } i = s+1 \ldots r.$$

The fact that

$$c_k \int_0^T \max(0, g_i\left(t, \bar{x}_k(t), \bar{u}_k(t)\right))^2 dt \longrightarrow 0, \qquad \text{for every } i = 1 \dots s,$$

and

$$c_k \int_0^T \left|g_i\left(t, \bar{x}_k(t), \bar{u}_k(t)\right)\right|^2 dt \longrightarrow 0, \qquad \text{for every } i = s+1 \dots r,$$

ensures that

$$\int_0^T \lambda_{i,k}(t) g_i\left(t, \bar{x}_k(t), \bar{u}_k(t)\right) dt \longrightarrow 0, \quad \text{for every } i = 1 \dots s, \tag{4.4}$$

and

$$\int_0^T \mu_{i,k}(t) g_i\left(t, \bar{x}_k(t), \bar{u}_k(t)\right) dt \longrightarrow 0, \quad \text{for every } i = s+1 \dots r.$$

Since $g_i\left(., \bar{x}_k(.), \bar{u}_k(.)\right) \longrightarrow g_i\left(., \bar{x}(.), \bar{u}(.)\right)$ strongly in $L^2(I, \mathbb{R})$, for every $i = 1 \dots r$, then the sequences $(\lambda_{i,k}(.))_k$ and $(\mu_{i,k}(.))_k$ are bounded in $L^2(I, \mathbb{R})$. This implies, by the Theorem of Banach-Alaoglu-Bourbaki, that there exist subsequences, denoted again by $(\lambda_{i,k}(.))_k$ and $(\mu_{i,k}(.))_k$ such that $\lambda_{i,k}(.) \longrightarrow \lambda_i(.)$ and $\mu_{i,k}(.) \longrightarrow \mu_i(.)$ weakly in $L^2(I, \mathbb{R})$, for every $i = 1 \dots r$. Also, since $\lambda_{i,k}(t) g_i\left(t, \bar{x}_k(t), \bar{u}_k(t)\right) \geq 0$, for every $i = 1 \dots s$. By (4.4)

$$\lambda_{i,k}(t) g_i\left(t, \bar{x}_k(t), \bar{u}_k(t)\right) \longrightarrow 0,$$

for almost all $t \in I$ and every $i = 1 \dots s$. On the other hand,

$$\lambda_{i,k}(t) g_i\left(t, \bar{x}_k(t), \bar{u}_k(t)\right) \longrightarrow \lambda_i(t) g_i\left(t, \bar{x}(t), \bar{u}(t)\right),$$

for almost all $t \in I$ and every $i = 1 \dots s$. Consequently, $\lambda_i(t) \geq 0$ and $\lambda_i(t) g_i\left(t, \bar{x}\left(t\right), \bar{u}\left(t\right)\right) = 0$ for almost all $t \in [0, T]$ and for every $i = 1 \dots s$. Now, from (4.1), which is a linear equation for $z_k(.)$ it follows that the $\lim_{k \to +\infty} c_k z_k(t)$ exists, and if we denote this limit by $\psi(t)$, it follows (by multiplying by c_k system (4.1)-(4.3) and taking limits) that

$$\begin{aligned}
&\dot{\psi}(t) = -\nabla_x f(t, \bar{x}\left(t\right), \bar{u}\left(t\right))^\top \psi(t) + \nabla_x l\left(t, \bar{x}\left(t\right), \bar{u}\left(t\right)\right) \\
&\qquad + (\lambda(t), \mu(t)) \nabla_x g(t, \bar{x}\left(t\right), \bar{u}\left(t\right)) \\
&\nabla_u f(t, \bar{x}\left(t\right), \bar{u}\left(t\right))^\top \psi(t) = \nabla_u l(t, \bar{x}\left(t\right), \bar{u}\left(t\right)) + (\lambda(t), \mu(t)) \nabla_u g(t, \bar{x}\left(t\right), \bar{u}\left(t\right)) \\
&\psi(T) = 0,
\end{aligned}$$

which, of course, means that $(\bar{x}(.), \bar{u}(.))$ is an optimal solution of $\min \mathcal{L}(x, u, \psi^*, \lambda^*, \mu^*)$. We conclude that $(\psi^*, \lambda^*, \mu^*)$ is a non-trivial Lagrange multiplier associated to the optimal solution of problem (P). □

Concluding remarks.

In this paper, we have constructively developed a theoretical framework how to solve an optimal control problem with state and control constraints. Under strong compact assumptions on the set of controls, the state and control constraints (the differential equation, respectively) are handled by defining an equivalent unconstrained control problem (infinite-dimensional optimization problem, respectively). This is done by defining a penalty function involving a non-decreasing sequence $(c_n)_{n \in \mathbb{N}}$. The two problems are equivalent if c_n is sufficiently large. A correspondence has been shown between this penalty functions and the duality for this class of constrained optimal control problems. Results bearing on computational aspects will be reported in a future work. It is to be noted that the functional obtained is non-smooth, but has interesting differential properties and modern non-differentiable optimization methods can be applied to solve these problems.

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