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TYNYSBEK SHARIPOVICH KAL'MENOV

(to the 75th birthday)

Tynysbek Sharipovich Kal'menov was born in the village of Koksaek of the Tolebi district of the Turkestan region (earlier it was the Lenger district of the South-Kazakhstan region of the Kazakh SSR). Although "according to the passport" his birthday was recorded on May 5, his real date of birth is April 6, 1946.

Tynysbek Kal'menov is a graduate of the Novosibirsk State University (1969), and a representative of the school of A.V. Bitsadze, an outstanding scientist, corresponding member of the Academy of Sciences of the USSR. In 1972, he completed his postgraduate studies at the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR. In 1983, he defended his doctoral thesis at the M.V. Lomonosov Moscow State University. Since1989, he is a corresponding member of the Academy of Sciences of the Kazakh SSR, and since 2003, he is an academician of the National Academy of Sciences of the Republic of Kazakhstan.

Tynysbek Kal'menov worked at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (1972-1985). From 1986 to 1991, he was the dean of the Faculty of Mathematics of Al-Farabi Kazakh State University. From 1991 to 1997, he was the rector of the Kazakh Chemical-Technological University (Shymkent).

From 2004 to 2019, Tynysbek Kal'menov was the General Director of the Institute of Mathematics and Mathematical Modeling. He made it one of the leading scientific centers in the country and the best research institute in Kazakhstan. It suffices to say, that in terms of the number of scientific publications (2015-2018) in international rating journals indexed in the Web of Science, the Institute of Mathematics and Mathematical Modeling was ranked fourth among all Kazakhstani organizations, behind only three large universities: the Nazarbaev University, Al-Farabi National University and L.N. Gumilyov Eurasian National University.

Since 2019, Tynysbek Kal'menov has been working as the head of the Department of Differential Equations of the Institute of Mathematics and Mathematical Modeling. He is a member of the National Scientific Council "Scientific Research in the Field of Natural Sciences", which is the main Kazakhstan council that determines the development of science in the country.

T.Sh. Kal'menov was repeatedly elected to maslikhats of various levels, was a member of the Presidium of the Committee for Supervision and Attestation in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan. He is a Laureate of Lenin Komsomol Prize of the Kazakh SSR (1978), an Honored Worker of Science and Technology of Kazakhstan (1996), awarded with the order "Kurmet" (2008 Pi.) and jubilee medals.

In 2013, he was awarded the State Prize of the Republic of Kazakhstan in the field of science and technology for the series of works "To the theory of initial- boundary value problems for differential equations".

The main areas of scientific interests of academician Tynysbek Kal'menov are differential equations, mathematical physics and operator theory. He has obtained fundamental scientific results, many of which led to the creation of new scientific directions in mathematics.

Tynysbek Kal'menov, using a new maximum principle for an equation of mixed type (Kal'menov's maximum principle), was the first to prove that the Tricomi problem has an eigenfunction, thus he solved the famous problem of the Italian mathematician Francesco Tricomi, set in 1923 This marked the beginning of a new promising direction, that is, the spectral theory of equations of mixed type.

He established necessary and sufficient conditions for the well-posed solvability of the classical Darboux and Goursat problems for strongly degenerate hyperbolic equations.

Tynysbek Kal'menov solved the problem of completeness of the system of root functions of the nonlocal Bitsadze-Samarskii problem for a wide class of multidimensional elliptic equations. This result is final and has been widely recognized by the entire mathematical community.

He developed a new effective method for studying ill-posed problems using spectral expansion of differential operators with deviating argument. On the basis of this method, he found necessary and sufficient conditions for the solvability of the mixed Cauchy problem for the Laplace equation.

Tynysbek Kal'menov was the first to construct boundary conditions of the classical Newton potential. That is a fundamental result at the level of a classical one. Prior to the research of Kal'menov T.Sh., it was believed that the Newton potential gives only a particular solution of an inhomogeneous equation and does not satisfy any boundary conditions. Thanks for these results, for the first time, it was possible to construct the spectral theory of the classical Newton potential.

He developed a new effective method for constructing Green's function for a wide class of boundary value problems. Using this method, Green's function of the Dirichlet problem was first constructed explicitly for a multidimensional polyharmonic equation.

From 1989 to 1993, Tynysbek Kal'menov was the chairman of the Inter- Republican (Kazakhstan, Uzbekistan, Kyrgyzstan, Turkmenistan, Tajikistan) Dissertation Council. He is a member of the International Mathematical Society and he repeatedly has been a member of organizing committee of many international conferences. He carries out a lot of organizational work in training of highly qualified personnel for the Republic of Kazakhstan and preparing international conferences. Under his direct guidance, the First Congress of Mathematicians of Kazakhstan was held. He presented his reports in Germany, Poland, Great Britain, Sweden, France, Spain, Japan, Turkey, China, Iran, India, Malaysia, Australia, Portugal and countries of CIS.

In terms of the number of articles in scientific journals with the impact- factor Web of Science, in the research direction of "Mathematics", the Institute of Mathematics and Mathematical Modeling is on one row with leading mathematical institutes of the Russian Federation, and is ahead of all mathematical institutes in other CIS countries in this indicator.

Tynysbek Kal'menov is one of the few scientists who managed to leave an imprint of their individuality almost in all branches of mathematics in which he has been engaged.

Tynysbek Kal'menov has trained 11 doctors and more than 60 candidate of sciences and PhD, has founded a large scientific school on equations of mixed type and differential operators recognized all over the world. Many of his disciples are now independent scientists recognized in the world of mathematics.

He has published over 150 scientific articles, most of which are published in international mathematical journals, including 14 articles published in "Doklady AN SSSR/ Doklady Mathematics". In the last 5 years alone (2016-2020), he has published more than 30 articles in scientific journals indexed in the Web of Science database. To date, academician Tynysbek Kal'menov has a Hirsch index of 18 in the Web of Science and Scopus databases, which is the highest indicator among all Kazakhstan mathematicians.

Outstanding personal qualities of academician Tynysbek Kalmenov, his high professional level, adherence to principles of purity of science, high exactingness towards himself and his colleagues, all these are the foundations of the enormous authority that he has among Kazakhstan scientists and mathematicians of many countries.

Academician Tynysbek Sharipovich Kalmenov meets his 75th birthday in the prime of his life, and the mathematical community, many of his friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal heartily congratulate him on his jubilee and wish him good health, happiness and new successes in mathematics and mathematical education, family well-being and long years of fruitful life.

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ON EXACT PENALTIES FOR CONSTRAINED OPTIMIZATION PROBLEMS IN METRIC SPACES

A.V. Arutyunov, S.E. Zhukovskiy

Communicated by V.I. Burenkov

Key words: exact penalty, coincidence point.

AMS Mathematics Subject Classification: 90C26.

Abstract. The problem of minimization of Lipschitz continuous functions over the set of coincidence points of mappings between metric spaces is considered. It is shown that under the assumptions of the known coincidence point theorems, the problem under consideration possesses the exact penalty property. For proving this fact, we obtain a modification of the exact penalization theorem.

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1 Introduction

In this paper, we study the following constrained minimization problem in metric spaces.

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $f_0 : X \to \mathbb{R}$ be a given function, $\Psi, \Phi : X \rightrightarrows Y$ be given set-valued mappings (i.e. mappings which assign non-empty closed subsets of Y to each point $x \in X$). For every $k \geq 0$, define a function $\varphi_k : X \to \mathbb{R}$ by the formula

$$\varphi_k(x) := f_0(x) + k \operatorname{dist}_Y(\Psi(x), \Phi(x)), \quad x \in X.$$

Here $\operatorname{dist}_Y(U, V) := \inf\{\rho_Y(u, v) : \, u \in U, \, v \in V\}$. Consider the problem

$$\begin{array}{l} \text{minimize} \quad f_0(x) \\ \text{over points} \quad x \in X \quad \text{satisfying} \quad \Psi(x) \cap \Phi(x) \neq \emptyset, \end{array} \tag{1.1}$$

and the family of unconstrained penalized problems

$$\text{minimize} \quad \varphi_k(x) \quad \text{over points} \quad x \in X \tag{1.2}$$

with the penalty parameter $k \geq 0$. Our goal is to show that under natural regularity assumptions problem (1.1) possesses the exact penalty property, i.e. for sufficiently large $k \geq 0$ the solutions to problems (1.1) and (1.2) coincide.

The notion of exact penalization was introduced in [16, 25]. Theorems on exact penalization play an important role in constrained optimization. In particular, they are applied in optimal control for the elimination of various constraints (see, for example, [4, 24]). For historical review of the literature on exact penalization see [12, 13, 15]. We also refer the reader to papers [9, 10], where the exact penalty property was proved for a wide class of problems with degenerate constraints, paper [26], where the exact penalty property was proved for approximate solutions to optimization problems with locally Lipschitzian constraints, and papers [22, 23], where the exact penalty property was considered for problems with constraints defined by set-valued mappings.

Consider the following particular case of problem (1.1)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{over points} & x \in X \quad \text{satisfying} \quad \widehat{y} \in \Psi(x). \end{array} \tag{1.3}$$

Here $\widehat{y} \in Y$ is a given point.

Assume that X and Y are normed spaces, the function f_0 is locally Lipschitz around a local solution $\widehat{x} \in X$ to problem (1.3) and the mapping Ψ is regular at the point $(\widehat{x}, \widehat{y})$, i.e. there exist $\gamma \geq 0$ and neighbourhoods U of $\widehat{x}$ and V of $\widehat{y}$ such that

$$\operatorname{dist}_X(x, \Psi^{-1}(y)) \leq \gamma \operatorname{dist}_Y(\Psi(x), y) \quad \forall x \in X, \quad \forall y \in Y$$

(here $\Psi^{-1}(y) = \{u \in X : y \in \Psi(u)\}$). It has been shown in [19, Theorem 10] that under these assumptions the function $x \mapsto f_0(x) + k \operatorname{dist}_Y(\widehat{y}, \Psi(x))$, $x \in X$ has a local minimum at the point $\widehat{x}$ for sufficiently large $k \geq 0$.

Problem (1.1) can be viewed as perturbed problem (1.3). Hence, there arises a natural question, if the exact penalty property is stable under small (in a certain sense) perturbations. In this paper, we provide an affirmative answer to this question (see Theorems 4.4 and 4.4 below). The stability of the exact penalty property under another type of perturbations of problem (1.1) when Ψ is a single-valued mapping was studied in [27].

Another reason for the investigation of the exact penalties for problem (1.1) is that under natural assumptions, the admissible set $\mathcal{D} = \{x \in X : \Psi(x) \cap \Phi(x) \neq \emptyset\}$ is not necessarily closed. This causes some difficulties in the application of the known exact penalization theorem (for various formulations of the exact penalization theorem see, for example, [24, Theorem 3.2.1], [14, Proposition 2.4.3], [11, Proposition 3.111]). In this paper, we prove a semi-local modification of the exact penalization theorem and apply it to the investigation of problem (1.1).

Another important tool for the investigation of problem (1.1) are coincidence point theorems. Recall that a point $x \in X$ is called a coincidence point of set-valued mappings Ψ and Φ if $\Psi(x) \cap \Phi(x) \neq \emptyset$ (for single-valued mappings Ψ and Φ, this relationship is equivalent to the equality $\Psi(x) = \Phi(x)$). In this paper, we study problem (1.1) under the assumptions of coincidence point theorems from [1, 2]. Note also that the area of applications of coincidence point results goes beyond optimization (see [5, 6, 7, 18] and the bibliography therein).

Note also that the problem of the minimization of a function f_0 on the set of coincidence points of two mappings is interesting in terms of applications. Problems of this type naturally arise in the study of mathematical models in economics. In these models, x represents a vector of certain parameters including prices and tax rates; the admissible set $\mathcal{D}$ represents the set of equilibrium points, i.e. points $x \in X$ for which the values of a single-valued supply function Ψ and a single-valued demand function Φ coincide; the value to maximize $-f_0(x)$ represents a total tax revenue. For more detail on these models see [17, 20]. Despite the possible interest of the considered problem in terms of applications, in this paper we focus on theoretical side of the problem only.

2 Exact penalization in metric spaces

Let (X, ρ_X), (Y, ρ_Y) be metric spaces. Throughout the paper $\mathbb{R}_+$ stands for non-negative reals; $B_X(x_0, r)$ stands for the closed ball centered at $x_0 \in X$ with radius $r \in \mathbb{R}_+ \cup \{+\infty\}$, i.e.

$$B_X(x_0, r) = \{x \in X : \rho_X(x_0, x) \leq r\} \quad \text{if} \quad r > 0, \quad B_X(x_0, +\infty) = X;$$

the symbol $+\infty$ satisfies the following properties: $c \cdot (+\infty) = +\infty$ for all $c > 0$, $+\infty > c$ for all $c \in \mathbb{R}$, if at least one of the sets $U \subset Y$ or $V \subset Y$ is empty then $\operatorname{dist}_Y(U, V) = +\infty$.

Given a function $f_0 : X \to \mathbb{R}$ and a set $\mathcal{D} \subset X$, consider the constrained minimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{over points} & x \in X \quad \text{satisfying} \quad x \in \mathcal{D}. \end{array} \tag{2.1}$$

Given a function $f : X \to \mathbb{R}_+$, denote

$$\varphi_k(x) := f_0(x) + kf(x), \quad x \in X, \quad k \geq 0. \tag{2.2}$$

The following proposition is the main result of this section. It provides conditions under which the solutions to problem (2.1) coincide with minima of the function φ_k.

Proposition 2.1. *Given a number $\gamma > 0$ and a set $\mathcal{B}$, assume that the function f_0 is Lipschitz continuous on $\mathcal{B}$ with the constant γ and*

$$\operatorname{dist}_X(x, \mathcal{D}) \leq f(x) \quad \forall\, x \in \mathcal{B}. \tag{2.3}$$

Let the function φ_k be defined by (2.2).

(a) *Assume that*

$$\mathcal{D} = \{x \in X : \, f(x) = 0\}, \tag{2.4}$$

Choose any $k > \gamma$ and nonzero $d \in \mathbb{R}_+ \cup \{+\infty\}$. If a point $\widehat{x} \in \mathcal{B}$ minimizes the function φ_k over the ball $B_X(\widehat{x}, d) \subset \mathcal{B}$ and $f(\widehat{x}) < d$, then the point $\widehat{x}$ minimizes the function f_0 over the set $\mathcal{D} \cap B_X(\widehat{x}, d)$.

(b) *Assume that*

$$\mathcal{D} \subset \{x \in X : \, f(x) = 0\}. \tag{2.5}$$

Choose any $k \geq \gamma$, nonzero $d \in \mathbb{R}_+ \cup \{+\infty\}$ and nonzero $\delta \in \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\text{either} \quad 2\delta < d < +\infty \quad \text{or} \quad \delta = d = +\infty. \tag{2.6}$$

If a point $\widehat{x} \in \mathcal{B}$ minimizes the function f_0 over the set $\mathcal{D} \cap B_X(\widehat{x}, d)$ and $B_X(\widehat{x}, d) \subset \mathcal{B}$, then the point $\widehat{x}$ minimizes the function φ_k over the ball $B_X(\widehat{x}, \delta)$.

Proof. (a): Assume that $k > \gamma$, a point $\widehat{x} \in \mathcal{B}$ minimizes φ_k over $B_X(\widehat{x}, d)$, $B_X(\widehat{x}, d) \subset \mathcal{B}$, and $f(\widehat{x}) < \mathcal{D}$.

First, we prove that $\widehat{x} \in \mathcal{D}$. Consider the contrary: $\widehat{x} \notin \mathcal{D}$. Then (2.4) implies that $f(\widehat{x}) > 0$. Take arbitrary $\varepsilon > 0$ such that

$$(k - \gamma)f(\widehat{x}) + \gamma\varepsilon > 0, \quad f(\widehat{x}) + \varepsilon \leq d. \tag{2.7}$$

The existence of such ε follows from the inequalities $k > \gamma$, $f(\widehat{x}) > 0$, $d > 0$. The definition of $\operatorname{dist}_X(\cdot)$ implies that there exists a point $x_\varepsilon \in X$ such that

$$\rho_X(\widehat{x}, x_\varepsilon) \leq \operatorname{dist}_X(\widehat{x}, \mathcal{D}) + \varepsilon \quad \text{and} \quad x_\varepsilon \in \mathcal{D}. \tag{2.8}$$

We have $x_\varepsilon \in B_X(\widehat{x}, d)$ since

$$\rho_X(\widehat{x}, x_\varepsilon) \leq \operatorname{dist}_X(\widehat{x}, \mathcal{D}) + \varepsilon \leq f(\widehat{x}) + \varepsilon \leq d. \tag{2.9}$$

Here the first inequality follows from (2.8), the second inequality follows from (2.3), the third inequality follows from (2.7). In particular, the inclusion $x_\varepsilon \in B_X(\widehat{x}, d)$ implies $\varphi_k(\widehat{x}) \leq \varphi_k(x_\varepsilon)$ since $\widehat{x}$ minimizes φ_k over $B_X(\widehat{x}, d)$. So, the following inequalities hold:

$$0 \leq \varphi_k(x_\varepsilon) - \varphi_k(\widehat{x}) = f_0(x_\varepsilon) - f_0(\widehat{x}) - kf(\widehat{x}) \leq \gamma\rho_X(x_\varepsilon, \widehat{x}) - kf(\widehat{x})$$

$$\leq \gamma \operatorname{dist}_X(\widehat{x}, \mathcal{D}) + \gamma\varepsilon - kf(\widehat{x}) \leq (\gamma - k)f(\widehat{x}) + \gamma\varepsilon.$$

Here the first inequality follows from the relationship $\varphi_k(\widehat{x}) \leq \varphi_k(x_\varepsilon)$; the first equality follows from (2.2), (2.4) and the relationship $x_\varepsilon \in \mathcal{D}$; the second inequality follows by the Lipschitz continuity of f_0 over $\mathcal{B}$ and the relationships $\widehat{x} \in \mathcal{B}$ and $x_\varepsilon \in B_X(\widehat{x}, d) \subset \mathcal{B}$; the third inequality follows from (2.8); the last inequality follows from (2.3). So, we have arrived at a contradiction to the first inequality in (2.7). The inclusion $\widehat{x} \in \mathcal{D}$ is proved.

Let us prove now that $f_0(\widehat{x}) \leq f_0(x)$ for every $x \in \mathcal{D} \cap B_X(\widehat{x}, d)$. The relations $\widehat{x} \in \mathcal{D}$, (2.2) and (2.4) imply that

$$f_0(\widehat{x}) = \varphi_k(\widehat{x}), \quad f_0(x) = \varphi_k(x) \quad \forall\, x \in \mathcal{D} \cap B_X(\widehat{x}, d).$$

Therefore, $f_0(\widehat{x}) \leq f_0(x)$ for every $x \in \mathcal{D} \cap B_X(\widehat{x}, d)$ since $\widehat{x}$ minimizes φ_k over $B_X(\widehat{x}, d)$.

(b): Assume that $d \in \mathbb{R}_+ \cup \{+\infty\}$, $\delta \in \mathbb{R}_+ \cup \{+\infty\}$, $\delta > 0$, (2.5) and (2.6) hold, $k \geq \gamma$, a point $\widehat{x} \in \mathcal{B}$ minimizes f_0 over $\mathcal{D} \cap B_X(\widehat{x}, d)$ (this, in particular, implies that $\widehat{x} \in \mathcal{D}$) and $B_X(\widehat{x}, d) \subset \mathcal{B}$.

Take an arbitrary $x \in B_X(\widehat{x}, \delta)$ and $\varepsilon > 0$ such that

$$2\delta + \varepsilon \leq d. \tag{2.10}$$

The existence of such ε follows from (2.6). The definition of $\operatorname{dist}_X(\cdot)$ implies that there exists a point $x_\varepsilon \in X$ such that

$$\rho_X(x, x_\varepsilon) \leq \operatorname{dist}_X(x, \mathcal{D}) + \varepsilon, \quad x_\varepsilon \in \mathcal{D}. \tag{2.11}$$

We have $x_\varepsilon \in B_X(\widehat{x}, d)$ since

$$\rho_X(\widehat{x}, x_\varepsilon) \leq \rho_X(\widehat{x}, x) + \rho_X(x, x_\varepsilon) \leq \rho_X(\widehat{x}, x) + \operatorname{dist}_X(x, \mathcal{D}) + \varepsilon \leq 2\rho_X(\widehat{x}, x) + \varepsilon \leq 2\delta + \varepsilon \leq d.$$

Here the second inequality follows from (2.11), the third inequality follows from the inclusion $\widehat{x} \in \mathcal{D}$, the second to the last inequality follows from the inclusion $x \in B_X(\widehat{x}, \delta)$, the last inequality follows from (2.10). Moreover,

$$f_0(\widehat{x}) \leq f_0(x_\varepsilon), \tag{2.12}$$

since $\widehat{x}$ minimizes the function f_0 over the set $\mathcal{D} \cap B_X(\widehat{x}, d)$. In addition,

$$f_0(x_\varepsilon) \leq f_0(x) + \gamma\rho(x, x_\varepsilon), \tag{2.13}$$

since f_0 is Lipschitz continuous on $\mathcal{B}$ with the constant γ, and the inclusion $x_\varepsilon \in B_X(\widehat{x}, d) \subset \mathcal{B}$ and $x \in B_X(\widehat{x}, \delta) \subset B_X(\widehat{x}, d) \subset \mathcal{B}$ hold.

Applying the above inequalities we obtain

$$\varphi_k(\widehat{x}) = f_0(\widehat{x}) + kf(\widehat{x}) = f_0(\widehat{x}) \leq f_0(x_\varepsilon) \leq f_0(x) + \gamma\rho(x, x_\varepsilon)$$

$$\leq f_0(x) + k\rho(x, x_\varepsilon) \leq f_0(x) + k(1+\varepsilon)\operatorname{dist}_X(x, \mathcal{D}) \leq \varphi_k(x) + k\varepsilon\operatorname{dist}_X(x, \mathcal{D}).$$

Here the first equality follows from (2.2), the second equality follows from (2.5) and the inclusion $\widehat{x} \in \mathcal{D}$, the first inequality follows from (2.12), the second inequality follows from (2.13), the third inequality follows from the inequality $k \geq \gamma$, the second to the last inequality follows from (2.11), the last inequality follows from (2.2) and (2.3). Therefore, $\varphi_k(\widehat{x}) \leq \varphi_k(x)$, since ε was chosen arbitrarily. Thus, $\widehat{x}$ minimizes φ_k over $B_X(\widehat{x}, \delta)$. $\square$

3 Discussion

Proposition 2.1 is a semi-local analogue of the known exact penalization theorems. It can be used to obtain exact penalty assertions both for local and global minima. Moreover, it provides explicit estimates on the radii of neighbourhoods in which the functions φ_k, ψ_k and f_0 attain their minima.

The set $\mathcal{B}$ represents the domain where f_0 is Lipschitz continuous and property (2.3) holds. It is not assumed that $\mathcal{D} \subset \mathcal{B}$. Thus, if we put $\mathcal{B} = X$ in Proposition 2.1, then we obtain a weaker assertion.

Some formulations of the exact penalization theorem deal with the function

$$\psi_k(x) := f_0(x) + k\mathrm{dist}_X(x, \mathcal{D}), \quad x \in X, \tag{3.1}$$

instead of φ_k (see, for example, [24, Theorem 3.2.1], [14, Proposition 2.4.3]). In this regard, the question arises, whether Proposition 2.1 remains valid if we replace φ_k by ψ_k in its formulation. To answer this question let us deduce the following corollary of Proposition 2.1.

Corollary 3.1. *Given a number $\gamma > 0$ and a set $\mathcal{B}$, assume that the function f_0 is Lipschitz continuous on $\mathcal{B}$ with the constant γ. Let the function φ_k be defined by* (2.2)*.*

(a′) *Assume that the set $\mathcal{D}$ is closed. Choose any $k > \gamma$ and nonzero $d \in \mathbb{R}_+ \cup \{+\infty\}$. If a point $\widehat{x} \in \mathcal{B}$ minimizes the function ψ_k over the ball $B_X(\widehat{x}, d) \subset \mathcal{B}$, then the point $\widehat{x}$ minimizes the function f_0 over the set $\mathcal{D} \cap B_X(\widehat{x}, d)$.*

(b′) *Choose any $k \geq \gamma$, nonzero $d \in \mathbb{R}_+ \cup \{+\infty\}$ and nonzero $\delta \in \mathbb{R}_+ \cup \{+\infty\}$ such that relationship* (2.6) *holds. If a point $\widehat{x} \in \mathcal{B}$ minimizes the function f_0 over the set $\mathcal{D} \cap B_X(\widehat{x}, d)$ and $B_X(\widehat{x}, d) \subset \mathcal{B}$, then the point $\widehat{x}$ minimizes the function ψ_k over the ball $B_X(\widehat{x}, \delta)$.*

Proof. Put $f(x) := \mathrm{dist}_X(x, \mathcal{D})$, $x \in X$. This choice of f implies that the functions φ_k and ψ_k defined by formulae (2.2) and (3.1) coincide. This function f satisfies (2.3) and (2.5). Moreover, if $\mathcal{D}$ is closed then (2.4) holds. Hence, **(a)** implies **(a′)**, and **(b)** implies **(b′)**. □

The following example shows that the closedness of $\mathcal{D}$ is essential in the assertion **(a′)** and cannot be omitted.

Example 1. *Let $X = \mathcal{B} = \mathcal{R}$, $\mathcal{D} = (-\infty, 0)$, $f_0(x) \equiv 0$, $\gamma = 0$, $k > 0$, $\widehat{x} = 0$, $r = d = +\infty$. Obviously, all the assumptions of Corollary* 3.1 *hold, except for the closedness of $\mathcal{D}$. We have $\psi_k(x) \equiv k\max\{0, x\}$. Hence, $\widehat{x}$ minimizes ψ_k over X. However, $\widehat{x}$ does not minimize f_0 over $\mathcal{D}$, since $\widehat{x} \notin \mathcal{D}$.*

The aforementioned bring us to the following conclusions. It is more appropriate to use function φ_k in Proposition 2.1, since for the function ψ_k this assertion is not valid without additional assumptions.

Let us compare Proposition 2.1 with the known exact penalization theorems. Under the additional assumptions that $\mathcal{B} = X$, $d = \delta = +\infty$, Proposition 2.1 generalizes [24, Theorem 3.2.1], assertions **(a′)** and **(b′)** coincide with [24, Theorem 3.2.1]. Under the additional assumptions that X is a subset of a normed space, $\mathcal{B} = X$, $d = \delta = +\infty$, Proposition 2.1 generalizes [14, Proposition 2.4.3], assertions **(a′)** and **(b′)** coincide with [14, Proposition 2.4.3]. Under additional assumptions that X is a normed space, $\mathcal{B} = X$, $d = +\infty$, assertion **(b)** implies [11, Proposition 3.111].

To conclude this section let us deduce corollaries of Proposition 2.1 which we shall apply in the following section.

By setting $\mathcal{B} := X$, $d := +\infty$, $\delta := +\infty$ in Proposition 2.1, we obtain the following assertion.

Corollary 3.2. *Given a number $\gamma > 0$, assume that*

$$\mathrm{dist}_X(x, \mathcal{D}) \leq f(x) \quad \forall x \in X, \tag{3.2}$$

and the function f_0 is Lipschitz continuous with the constant γ. Let the function φ_k be defined by (2.2)*.*

Assume that (2.4) *holds. For every $k > \gamma$, if a point $\widehat{x} \in X$ minimizes the function φ_k over X, then the point $\widehat{x}$ minimizes the function f_0 over $\mathcal{D}$.*

Assume that (2.5) *holds. For every $k \geq \gamma$, if a point $\widehat{x} \in X$ minimizes the function f_0 over the set $\mathcal{D}$, then the point $\widehat{x}$ minimizes the function φ_k over X.*

4 Main results

Before formulating the main result of this section, we introduce the necessary notation and recall some coincidence point theorems.

4.1 Preliminaries. Coincidence point theorems

As before, assume that we are given metric spaces (X, ρ_X) and (Y, ρ_Y), set-valued mappings $\Psi, \Phi : X \rightrightarrows Y$ and numbers $\alpha > 0$ and $\beta \geq 0$.

Denote by $\mathrm{Coin}(\Psi, \Phi)$ the set of coincidence points of mappings Ψ and Φ, i.e.

$$\mathrm{Coin}(\Psi, \Phi) := \{x \in X : \Psi(x) \cap \Phi(x) \neq \emptyset\}.$$

For an arbitary set $U \subset X$ and a number $r \geq 0$, denote

$$B_X(U, r) := \bigcup_{u \in U} B_X(u, r).$$

Denote by $\Psi(U)$ the image of a set U under the mapping Ψ, i.e. $\Psi(U) := \bigcup\limits_{u \in U} \Psi(u)$.

Given a set $U \subset X$, the set-valued mapping $\Psi : X \rightrightarrows Y$ is said to be *α-covering with respect to the set U*, if

$$B_X(x, r) \subset U \;\Rightarrow\; B_Y(\Psi(x), \alpha r) \cap V \subset \Psi(B_X(x, r)) \quad \forall x \in X, \quad \forall r \geq 0.$$

The set-valued mapping Ψ is said to be *α-covering,* if it is α-covering with respect to X and Y.

Given sets $U \subset X$, $V \subset Y$, the set-valued mapping $\Phi : X \rightrightarrows Y$ is said to be *Lipschitz continuous with the constant β on the set U*, if

$$\Phi(u) \subset B_Y(\Phi(x), \beta \rho_X(x, u)) \quad \forall x, u \in U.$$

The set-valued mapping Φ is said to be *Lipschitz continuous with constant β,* if it is Lipschitz continuous with the constant β on X.

Define a metric ρ on $X \times Y$ as follows:

$$\rho\big((x_1, y_1), (x_2, y_2)\big) := \rho_X(x_1, x_2) + \rho_Y(y_1, y_2), \quad (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Denote by $\mathrm{gph}\Psi$ the graph of the set-valued mapping Ψ, i.e.

$$\mathrm{gph}\Psi := \{(x, y) \in X \times Y : y \in \Psi(x)\}.$$

Given a set $U \subset X$, the set-valued mapping $\Psi : X \rightrightarrows Y$ is said to be *closed on U*, if the set $\mathrm{gph}\Psi \cap (U \times Y)$ is closed in $X \times Y$. The set-valued mapping Ψ is said to be *closed* if it is closed on X.

Let us formulate the coincidence point theorem from [1].

Theorem 4.1. ([1, Theorem 2]) *Assume that*

(GA) *the set-valued mapping Ψ is α-covering and closed; the set-valued mapping Φ is Lipschitz continuous with the constant β; $\beta < \alpha$; at least one of the graphs* gphΦ *or* gphΨ *is a complete metric space.*

Then

$$\mathrm{Coin}(\Psi,\Phi) \neq \emptyset, \quad \text{and} \quad \mathrm{dist}_X\big(x, \mathrm{Coin}(\Psi,\Phi)\big) \leq \frac{\mathrm{dist}_Y\big(\Psi(x),\Phi(x)\big)}{\alpha-\beta} \quad \forall\, x \in X.$$

Let us formulate a local version of this result. Given $\widehat{x} \in X$ and $r > 0$, denote

$$R := \frac{(\alpha-\beta)r - \mathrm{dist}_X(\Psi(\widehat{x}),\Phi(\widehat{x}))}{\beta}. \tag{4.1}$$

The following theorem is a direct corollary of [2, Theorem 3.1].

Theorem 4.2. *Assume that*

(LA) *the set-valued mapping Ψ is α-covering with respect to the ball $B_X(\widehat{x},r)$ and closed on $B_X(\widehat{x},r)$; the set-valued mapping Φ is Lipschitz continuous with the constant β on $B_X(\widehat{x},r)$; $\beta < \alpha$; at least one of the graphs* gphΦ *or* gphΨ *is a complete metric space.*

If $\mathrm{dist}_Y\big(\Psi(\widehat{x}),\Phi(\widehat{x})\big) < (\alpha-\beta)r$, *then*

$$\mathrm{Coin}(\Psi,\Phi) \neq \emptyset, \quad \text{and} \quad \mathrm{dist}_X\big(x, \mathrm{Coin}(\Psi,\Phi)\big) \leq \frac{\mathrm{dist}_Y\big(\Psi(x),\Phi(x)\big)}{\alpha-\beta} \quad \forall\, x \in O_X(\widehat{x},R). \tag{4.2}$$

4.2 Exact penalty theorems for the problem of minimization on the coincidence set

Let us address to problem (1.1). Below we prove that under assumption **(GA)** this problem possesses the exact penalty property.

As before, assume that we are given metric spaces (X,ρ_X) and (Y,ρ_Y), a function $f_0 : X \to \mathbb{R}$, set-valued mappings $\Psi,\Phi : X \rightrightarrows Y$ and numbers $\alpha > 0$, $\beta \geq 0$ and $\gamma \geq 0$. For every $k > 0$, denote

$$\varphi_k(x) := f_0(x) + k\,\mathrm{dist}_Y(\Psi(x),\Phi(x)), \quad x \in X. \tag{4.3}$$

Theorem 4.3. *Assume that the function f_0 is Lipschitz continuous with the constant $\gamma \geq 0$, assumption* **(GA)** *holds, functions φ_k are defined by formula* (4.3).

(c) *Assume that for every $x \in X$, either $\Psi(x)$ is compact or $\Phi(x)$ is compact. Choose any $k > \dfrac{\gamma}{\alpha-\beta}$. If a point $\widehat{x} \in X$ minimizes the function φ_k over the entire space X, then the point $\widehat{x}$ minimizes the function f_0 over the set* $\mathrm{Coin}(\Psi,\Phi)$.

(d) *Choose any $k \geq \dfrac{\gamma}{\alpha-\beta}$. If a point $\widehat{x} \in X$ minimizes the function f_0 over the set* $\mathrm{Coin}(\Psi,\Phi)$, *then the point $\widehat{x} \in X$ minimizes the function φ_k over the entire space X.*

Proof. Denote

$$\mathcal{D} := \mathrm{Coin}(\Psi,\Phi), \quad f(x) := \frac{\mathrm{dist}(\Psi(x),\Phi(x))}{\alpha-\beta}, \quad x \in X.$$

(c): Let $k > \dfrac{\gamma}{\alpha-\beta}$, $\widehat{x}$ minimize φ_k over X. Theorem 4.1 implies that f satisfies (3.2). Let us show that (2.4) holds. Indeed, for every $x \in X$ such that $f(x) = 0$, we have $\mathrm{dist}_Y(\Psi(x), \Phi(x)) = 0$. So, $\Psi(x) \cap \Phi(x) \neq \emptyset$, since at least one of the sets $\Psi(x)$ or $\Phi(x)$ is compact. Hence, $x \in \mathrm{Coin}(\Psi, \Phi) = \mathcal{D}$. Thus, Corollary 3.2 implies that the point $\widehat{x}$ minimizes the function f_0 over $\mathcal{D}$.

(d): Let $k \geq \dfrac{\gamma}{\alpha-\beta}$, $\widehat{x}$ minimize φ_k over X. Theorem 4.1 implies that f satisfies (3.2). Moreover, (2.5) holds. Indeed, for every $x \in \mathcal{D}$, we have $\Psi(x) \cap \Phi(x) \neq \emptyset$. Therefore, $f(x) = \mathrm{dist}_Y(\Psi(x), \Phi(x)) = 0$. Thus, Corollary 3.2 implies that the point $\widehat{x}$ minimizes the function φ_k over X. □

Note that under the assumptions of Theorem 4.3 the set of admissible points in problem (1.1) (i.e. the set $\mathrm{Coin}(\Psi, \Phi)$) is not necessarily closed. The corresponding example is provided by the mappings $\Psi, \Phi : \mathbb{R} \to \mathbb{R}_+$,

$$\Phi(x) = [1, +\infty), \quad \Psi(x) = \left\{\frac{1}{x-n} - 1\right\}, \quad x \in (n, n+1], \quad n \in \mathbb{Z}$$

(here $\mathbb{Z}$ stands for integers). It is a straightforward task to ensure that Ψ is 1-covering and closed; Φ is Lipschitz continuous with the constant zero; both $\mathrm{gph}\Phi$ and $\mathrm{gph}\Psi$ are complete spaces; $\Psi(x)$ is compact for every $x \in \mathbb{R}$. However, the set $\mathrm{Coin}(\Psi, \Phi) = \bigcup_{n \in \mathbb{Z}} (n, n+1/2]$ is not closed.

Let us now obtain a local version of Theorem 4.3.

Theorem 4.4. *Given a point $\widehat{x} \in X$, assume that the function f_0 is Lipschitz continuous with the constant $\gamma \geq 0$ in a neighbourhood of $\widehat{x}$; the set-valued mapping Ψ is α-covering with respect to a neighbourhood of $\widehat{x}$ and closed on this neighbourhood; the set-valued mapping Φ is Lipschitz continuous with the constant β on a neighbourhood of $\widehat{x}$; $\beta < \alpha$; at least one of the graphs* $\mathrm{gph}\Phi$ *or* $\mathrm{gph}\Psi$ *is a complete metric space.*

Choose any $k \geq \dfrac{\gamma}{\alpha-\beta}$. If a point $\widehat{x} \in X$ locally minimizes the function f_0 over the set $\mathrm{Coin}(\Psi, \Phi)$, *then the point $\widehat{x} \in X$ locally minimizes the function φ_k over the entire space X.*

Proof. There exists $r > 0$ such that assumptions **(LA)** hold for Ψ and Φ. Let R be defined by (4.1). Since $\widehat{x} \in X$ minimizes f_0 over $\mathrm{Coin}(\Psi, \Phi)$, we have $\widehat{x} \in \mathrm{Coin}(\Psi, \Phi)$. Thus, $\mathrm{dist}(\Psi(\widehat{x}), \Phi(\widehat{x})) < (\alpha - \beta)r$. Hence, Theorem 4.2 implies (4.2).

Denote

$$\mathcal{D} := \mathrm{Coin}(\Psi, \Phi), \quad \mathcal{B} := B_X(\widehat{x}, \min\{r, R\}),$$

$$f(x) := \frac{\mathrm{dist}(\Psi(x), \Phi(x))}{\alpha - \beta}, \quad x \in X.$$

Relationship (4.2) implies that (2.3) holds. Moreover, (2.5) holds. Indeed, for every $x \in \mathcal{D}$, we have $\Psi(x) \cap \Phi(x) \neq \emptyset$. Therefore, $f(x) = \mathrm{dist}_Y(\Psi(x), \Phi(x)) = 0$. Thus, Proposition 2.1 implies that the point $\widehat{x}$ minimizes the function φ_k over X. □

Remark 1. *The formulation of Theorem* 4.4 *does not contain a proposition saying that local minima of functions φ_k are local constrained minima of the function f_0. This is caused due to the following reasons.*

If $\widehat{x} \in$ $\mathrm{Coin}(\Psi, \Phi)$, *then the following elementary proposition holds.* If $k \geq 0$, and a point $\widehat{x} \in X$ locally minimizes the function φ_k over X, then the point $\widehat{x} \in X$ locally minimizes the function f_0 over the set $\mathrm{Coin}(\Psi, \Phi)$.

In the case when $\widehat{x} \notin$ $\mathrm{Coin}(\Psi, \Phi)$, *the set* $\mathrm{Coin}(\Psi, \Phi)$ *may be empty. Hence, there may exist no minimizers of the function f_0 over* $\mathrm{Coin}(\Psi, \Phi)$.

As it can be seen in the proofs of Theorems 4.3 and 4.4, an important role in the proof of exact penalty property is due to the propositions on the existence of solutions to equations and inclusions and properties of the solutions. In this paper, we used the coincidence point theorems from [1, 2], guaranteeing the regularity of the constraints in problem (1.1) in the following sense. If a certain point $x \in X$ is "almost" admissible, i.e. $\operatorname{dist}(\Psi(x), \Phi(x))$ is sufficiently small, then there exists an admissible point $\xi \in X$ that is "close" to the point x. This property is strictly formulated by relationship (4.2). In this regard, it seems interesting to study problems in which the regularity property in the indicated sense does not hold. For example, some problems of this type were studied in [3, 8, 9, 21].

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