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GREEN FUNCTIONS, REFLECTIONS, AND PLANE PARQUETING

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Abstract. The harmonic Green and Neumann functions are constructed for a particular triangle in the complex plane, which by reflections across its sides provides a triangulation of the plane.

1 Construction of harmonic Green functions

A fundamental solution to the Laplace operator $\triangle = 4\partial_z\partial_{\overline{z}}$ for regular plane domains D adjusted to vanishing boundary values is called harmonic Green function. It exists for any domain for which the Dirichlet problem for harmonic functions with continuous boundary data is solvable. Thus in principle the existence of the harmonic Green function is guaranteed for a wide class of domains (in the complex plane). As the Green function provides the solution to the Dirichlet problem for the Poisson equation, it is important to know it explicitly. The conformal invariance of the Green function for the Laplace operator is a nice and useful tool to find the Green functions for a variety of domains. Because the Green function for the unit disc is known the Riemann mapping theorem serves to get the Green functions for a large class of simply connected domains. But this does not always give the Green function in explicit form. One example is the conformal map of a polygonal domain given by the Schwarz-Christoffel formula.

Another principle for constructing Green functions (not only for simply connected domains) is by reflection. To explain this principle let $\mathbb{D}$ be the unit disc in the complex plane. In order to alter the fundamental solution $\log|\zeta - z|$ for $z, \zeta \in \mathbb{D}$ let ζ be fixed and z vary. Reflecting the point ζ at the boundary $\partial\mathbb{D}$ provides the point $1/\overline{\zeta}$. The function $1 - z\overline{\zeta}$ is analytic in $z \in \mathbb{C}$ with a zero at $z = 1/\overline{\zeta}$. The difference $\log|1 - z\overline{\zeta}| - \log|\zeta - z|$ is a harmonic function in $z \in \mathbb{D}$ for any fixed ζ, $\zeta \in \mathbb{D}$, with the exceptional point $z = \zeta$. Moreover, it vanishes on the boundary $|z| = 1$. Thus $\log\left|\frac{1-z\overline{\zeta}}{\zeta - z}\right|$ is the harmonic Green function for $\mathbb{D}$.

Similarly, for the upper half plane $0 < \operatorname{Im} z$ the reflection of the fixed point ζ, $0 < \operatorname{Im}\zeta$, at the real axis gives $\overline{\zeta}$. The difference $\log|\overline{\zeta} - z| - \log|\zeta - z|$ is the Green

function for the upper half plane.

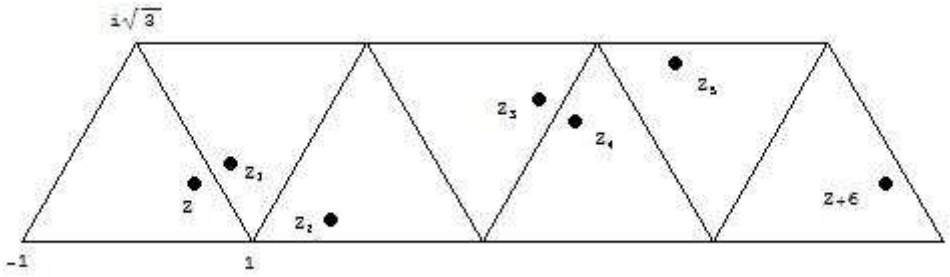
Continued reflections serve to determine the Green function for a circular concentric ring domain, see [4], [6].

In all these examples the original domain is mapped via these reflection processes onto the entire plane or even the Riemann sphere, occasionally with the exception of certain isolated points. The reflection method does work also for regular convex polygons providing a parqueting of the complex plane. A simple such parqueting is a triangulation of the plane. This case will be elaborated here.

For further examples of explicit formulas for harmonic Green functions for domains with corners see e.g. [1], [2], [3], [5].

2 Green function for a certain equilateral triangle

For getting the harmonic Green function $G(z,\zeta)$ for the open triangle T with the corner points -1, 1, $i\sqrt{3}$ the triangle is reflected at its three sides. The three resulting triangles are reflected at their sides, etc. This leads to a triangulation of the complex plane.



Reflecting $z \in T$ at the line from 1 to $i\sqrt{3}$ gives

$$z_1 = -\frac{1}{2}(1+i\sqrt{3})\overline{z} + \frac{\sqrt{3}}{2}(\sqrt{3}+i). \tag{1}$$

The triangle T itself is mapped onto the triangle T_1 with the corners 1, $2+i\sqrt{3}$, $i\sqrt{3}$. Reflecting T_1 at the line from 1 to $2+i\sqrt{3}$ maps z_1 onto

$$z_2 = -\frac{1}{2}(1+i\sqrt{3})z + \frac{\sqrt{3}}{2}(\sqrt{3}+i), \tag{2}$$

and T_1 onto the triangle T_2 with the corners 3, $2+i\sqrt{3}$, 1. Continuing reflecting in the same direction leads to

$$z_3 = \overline{z} + \sqrt{3}(\sqrt{3}+i) \tag{3}$$

and to T_3 with the corners at 3, $4+i\sqrt{3}$, $2+i\sqrt{3}$.

Then to

$$z_4 = -\frac{1}{2}(1-i\sqrt{3})z + \frac{\sqrt{3}}{2}(3\sqrt{3}+i) \tag{4}$$

within T_4 with the corners at 3, 5, $4+i\sqrt{3}$, and finally to

$$z_5 = -\frac{1}{2}(1-i\sqrt{3})\overline{z} + \frac{\sqrt{3}}{2}(3\sqrt{3}+i) \tag{5}$$

and the triangle T_5 with the corners at 5, $6+i\sqrt{3}$, $4+i\sqrt{3}$. The next reflection gives $z_6 = z+6$ and $T_6 = T+6$. Reflecting now at the sides on the real axis leads to the points $\overline{z}$, $\overline{z_1}$, $\overline{z_2}$, $\overline{z_3}$, $\overline{z_4}$, $\overline{z_5}$.

Further reflections on all sides of the twelve triangles attained so far result in the same periodic pattern with the basic periods 6 and $2i\sqrt{3}$. The entire complex plane is covered by a periodic pattern of translations of these twelve basic triangles performing a parqueting of the plane by triangles.

Endowing z with the label p, its direct reflections z_1 and $\overline{z}$ with the label n, any direct reflection of a point labeled p (or n) with the label n (or p, respectively) decomposes all these points into two classes. They are used to form an analytic function with simple zeroes at the $n-$points and simple poles at the $p-$points just by an infinite product. This product turns out to be convergent and obviously forms an elliptic function. The log of its absolute value appears to be the harmonic Green function for the triangle T.

Notation. Let for entire m and n, m, $n \in \mathbb{Z}$, denote

$$\Omega_{m,n} = 6m + 2i\sqrt{3}n$$

and

$$G_1(z,\zeta) = \log \prod_{m,n\in\mathbb{Z}} \left| \frac{\zeta-\overline{z}-\Omega_{m,n}}{\zeta-z-\Omega_{m,n}} \frac{\zeta-z_1-\Omega_{m,n}}{\zeta-\overline{z_1}-\Omega_{m,n}} \frac{\zeta-\overline{z_2}-\Omega_{m,n}}{\zeta-z_2-\Omega_{m,n}} \right.$$

$$\left. \times \frac{\zeta-z_3-\Omega_{m,n}}{\zeta-\overline{z_3}-\Omega_{m,n}} \frac{\zeta-\overline{z_4}-\Omega_{m,n}}{\zeta-z_4-\Omega_{m,n}} \frac{\zeta-z_5-\Omega_{m,n}}{\zeta-\overline{z_5}-\Omega_{m,n}} \right|^2 .$$

The periods Ω_{mn} obviously satisfy

$$\Omega_{-m,-n} = -\Omega_{m,n}, \quad \Omega_{-m,n} = -\overline{\Omega_{m,n}}, \quad \Omega_{m,-n} = \overline{\Omega_{m,n}}.$$

Theorem 1. *The function $G_1(z,\zeta)$ is the Green function for the triangle T, satisfying*

- *$G_1(\,\cdot\,,\zeta)$ is harmonic in $T \setminus \{\zeta\}$,*
- *$G_1(z,\zeta) + \log|\zeta - z|^2$ is harmonic in T,*
- *$\lim\limits_{z\to\partial T} G_1(z,\zeta) = 0$*

for any $\zeta \in T$.

The third property holds even in the three corner points of the triangle T.

The proof of Theorem 1 is given in the following two lemmas.

Lemma 1. *The double infinite products*

$$\prod_{m,n\in\mathbb{Z}}\left|\frac{\zeta-\overline{z_k}-\Omega_{m,n}}{\zeta-z_k-\Omega_{m,n}}\right|^2$$

converge for $0\le k\le 5$, *where* $z_0=z\in T$.

Proof. Let $a_{m,n}$ denote the factor of the above product

$$a_{m,n}=\left|\frac{\zeta-\overline{z_k}-\Omega_{m,n}}{\zeta-z_k-\Omega_{m,n}}\right|^2,$$

then

$$\prod_{m,n\in\mathbb{Z}}a_{m,n}=a_{0,0}\prod_{m=1}^{\infty}a_{-m,0}a_{m,0}\prod_{n=1}^{\infty}a_{0,-n}a_{0,n}\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}a_{-m,-n}a_{-m,n}a_{m,-n}a_{m,n}.$$

At first

$$\prod_{m=1}^{\infty}a_{-m,0}a_{m,0}=\prod_{m=1}^{\infty}\left|\frac{\zeta-\overline{z_k}+6m}{\zeta-z_k+6m}\frac{\zeta-\overline{z_k}-6m}{\zeta-z_k-6m}\right|^2=\prod_{m=1}^{\infty}\left|\frac{(\zeta-\overline{z_k})^2-36m^2}{(\zeta-z_k)^2-36m^2}\right|^2$$

converges as

$$\sum_{m=1}^{\infty}\left[\frac{36m^2-(\zeta-\overline{z_k})^2}{36m^2-(\zeta-z_k)^2}-1\right]=\sum_{m=1}^{\infty}\frac{(\zeta-z_k)^2-(\zeta-\overline{z_k})^2}{36m^2-(\zeta-z_k)^2}$$

is convergent.

Similarly

$$\prod_{n=1}^{\infty}a_{0,-n}a_{0,n}=\prod_{n=1}^{\infty}\left|\frac{\zeta-\overline{z_n}+2\sqrt{3}in}{\zeta-z_n+2\sqrt{3}in}\frac{\zeta-\overline{z_n}-2\sqrt{3}in}{\zeta-z_n-2\sqrt{3}in}\right|^2=\prod_{n=1}^{\infty}\left|\frac{(\zeta-\overline{z_n})^2+12n^2}{(\zeta-z_n)^2+12n^2}\right|^2$$

converges.

Finally

$$\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}a_{-m,-n}a_{-m,n}a_{m,-n}a_{m,n}$$

$$=\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left|\frac{\zeta-\overline{z_n}+\Omega_{m,n}}{\zeta-z_n+\Omega_{m,n}}\frac{\zeta-\overline{z_n}+\Omega_{m,-n}}{\zeta-z_n+\Omega_{m,-n}}\frac{\zeta-\overline{z_n}-\Omega_{m,-n}}{\zeta-z_n-\Omega_{m,-n}}\frac{\zeta-\overline{z_n}-\Omega_{m,n}}{\zeta-z_n-\Omega_{m,n}}\right|^2$$

$$=\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left|\frac{(\zeta-\overline{z_n})^2-\Omega_{m,n}^2}{(\zeta-z_n)^2-\Omega_{m,n}^2}\frac{(\zeta-\overline{z_n})^2-\Omega_{m,-n}^2}{(\zeta-z_n)^2-\Omega_{m,-n}^2}\right|^2$$

$$=\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\left|\frac{(\zeta-\overline{z_n})^4-24(3m^2-n^2)(\zeta-\overline{z_n})^2+144(3m^2+n^2)^2}{(\zeta-z_n)^4-24(3m^2-n^2)(\zeta-z_n)^2+144(3m^2+n^2)^2}\right|^2$$

is convergent as

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\zeta-\overline{z_n})^4-(\zeta-z_n)^4-24(3m^2-n^2)[(\zeta-\overline{z_n})^2-(\zeta-z_n)^2]}{144(3m^2+n^2)^2-24(3m^2-n^2)(\zeta-z_n)^2+(\zeta-z_n)^4}$$

converges. □

Lemma 2. *The function $G_1(\,\cdot\,,\zeta)$ has vanishing boundary values on ∂T, i.e. for $\zeta \in T$*

$$\lim_{\substack{z\to z_0\in\partial T,\\ z\in T}} G_1(z,\zeta) = 0.$$

Proof. The three parts $\partial_1 T$, $\partial_2 T$, $\partial_3 T$ of ∂T have to be investigated separately.

i. On the segment $\partial_1 T$ from 1 to $i\sqrt{3}$ where

$$z = z_1 = -\frac{1}{2}(1+i\sqrt{3})\overline{z} + \frac{\sqrt{3}}{2}(\sqrt{3}+i)$$

also

$$z_2 = \overline{z_2},\quad z_3 = z_4,\quad z_5 = z_2 + 3 + i\sqrt{3},\quad \overline{z_5} = z_5 - 2i\sqrt{3}$$

hold. Then

$$\frac{\zeta-\overline{z}-\Omega_{m,n}}{\zeta-z-\Omega_{m,n}}\frac{\zeta-z_1-\Omega_{m,n}}{\zeta-\overline{z_1}-\Omega_{m,n}} = 1,$$

$$\frac{\zeta-z_3-\Omega_{m,n}}{\zeta-\overline{z_3}-\Omega_{m,n}}\frac{\zeta-\overline{z_4}-\Omega_{m,n}}{\zeta-z_4-\Omega_{m,n}} = 1,$$

$$\frac{\zeta-\overline{z_2}-\Omega_{m,n}}{\zeta-z_2-\Omega_{m,n}} = 1,\quad \frac{\zeta-z_5-\Omega_{m,n}}{\zeta-\overline{z_5}-\Omega_{m,n}} = \frac{\zeta-z_5-\Omega_{m,n}}{\zeta-z_5-\Omega_{m,n-1}}.$$

Because

$$\prod_{m,n\in\mathbb{Z}}\left|\frac{\zeta-z_5-\Omega_{m,n}}{\zeta-\overline{z_5}-\Omega_{m,n}}\right|^2$$

is convergent

$$\prod_{n\in\mathbb{Z}}\left|\frac{\zeta-z_5-\Omega_{m,n}}{\zeta-z_5-\Omega_{m,n-1}}\right|^2 = \lim_{N\to\infty}\prod_{n=-N}^{N}\left|\frac{\zeta-z_5-\Omega_{m,n}}{\zeta-z_5-\Omega_{m,n-1}}\right|^2$$

$$= \lim_{N\to\infty}\left|\frac{\zeta-z_5-\Omega_{m,N}}{\zeta-z_5-\Omega_{m,-N-1}}\right|^2 = \lim_{N\to\infty}\left|\frac{\zeta-z_5-6m-2i\sqrt{3}N}{\zeta-z_5-6m+2i\sqrt{3}(N+1)}\right|^2 = 1.$$

Therefore $G_1(z,\zeta) = 0$ for $z = z_1$, $\zeta \in T$.

ii. On the segment $\partial_2 T$ from $i\sqrt{3}$ to -1 where

$$z = -\frac{1}{2}(1-i\sqrt{3})\overline{z} - \frac{\sqrt{3}}{2}(\sqrt{3}-i),$$

also the equalities

$$\overline{z_1} = z_1 - 2i\sqrt{3},\quad z_3 = z_2,\quad \overline{z_4} = z_4,\quad z_5 = z+6$$

are true. Then

$$\frac{\zeta-z_1-\Omega_{m,n}}{\zeta-\overline{z_1}-\Omega_{m,n}} = \frac{\zeta-z_1-\Omega_{m,n}}{\zeta-z_1-\Omega_{m,n-1}},$$

$$\frac{\zeta-\overline{z_2}-\Omega_{m,n}}{\zeta-z_2-\Omega_{m,n}}\frac{\zeta-z_3-\Omega_{m,n}}{\zeta-\overline{z_3}-\Omega_{m,n}}=1,$$
$$\frac{\zeta-\overline{z_4}-\Omega_{m,n}}{\zeta-z_4-\Omega_{m,n}}=1,$$
$$\frac{\zeta-z_5-\Omega_{m,n}}{\zeta-\overline{z_5}-\Omega_{m,n}}=\frac{\zeta-z-\Omega_{m+1,n}}{\zeta-\overline{z}-\Omega_{m+1,n}}.$$

Similarly as in case i

$$G_1(z,\zeta)=0 \text{ for } z=z_5+6,\ \ \zeta\in T,$$

is seen.

iii. On the segment $\partial_3 T$ on the real axis $z=\overline{z}$, then

$$z_1=z_2,\ \ \overline{z_3}=z_3-2i\sqrt{3},\ \ z_4=z_5.$$

Arguing as before $G_1(z,\zeta)=0$ is seen for $z=\overline{z}$, $\zeta\in T$. □

Lemma 2 shows $G_1(z,\zeta)$ to be the harmonic Green function for T. In fact, it satisfies all properties of a Green function formulated in Theorem 1. The uniqueness of the Green function in general follows easily form the maximum principle for harmonic functions, as is well known. Its symmetry $G_1(z,\zeta)=G_1(\zeta,z)$ follows from the basic properties of $G_1(z,\zeta)$ in general. But for a particular case the symmetry should be seen directly.

Lemma 3. *For z, $\zeta\in T$ the symmetry relation*

$$G_1(z,\zeta)=G_1(\zeta,z)$$

holds.

Proof. Introducing the relations

$$z_3=\overline{z}+3+i\sqrt{3},$$
$$z_4=\overline{z_1}+3+i\sqrt{3},$$
$$z_5=\overline{z_2}+3+i\sqrt{3},$$

the Green function can be expressed as

$$G_1(z,\zeta)=\log\prod_{m+n\in 2\mathbb{Z}}\left|\frac{\zeta-\overline{z}-\omega_{m,n}}{\zeta-z-\omega_{m,n}}\frac{\zeta-z_1-\omega_{m,n}}{\zeta-\overline{z_1}-\omega_{m,n}}\frac{\zeta-\overline{z_2}-\omega_{m,n}}{\zeta-z_2-\omega_{m,n}}\right|^2, \tag{6}$$

where now

$$\omega_{m,n}=3m+i\sqrt{3}n \text{ for } m+n\in 2\mathbb{Z}.$$

The single factors are

$$\left|\frac{\zeta-\overline{z}-\omega_{m,n}}{\zeta-z-\omega_{m,n}}\right|=\left|\frac{z-\overline{\zeta}-\omega_{-m,n}}{z-\zeta-\omega_{-m,-n}}\right|,$$

$$\left|\frac{\zeta - z_1 - \omega_{m,n}}{\zeta - \overline{z_1} - \omega_{m,n}}\right| = \left|\frac{\zeta + \frac{1}{2}(1+i\sqrt{3})\overline{z} - \frac{\sqrt{3}}{2}(\sqrt{3}+i) - \omega_{m,n}}{\overline{\zeta} + \frac{1}{2}(1+i\sqrt{3})\overline{z} - \frac{\sqrt{3}}{2}(\sqrt{3}+i) - \omega_{m,-n}}\right|$$

$$= \left|\frac{\frac{1}{2}(1-i\sqrt{3})\zeta + \overline{z} - \frac{\sqrt{3}}{2}(\sqrt{3}-i) - \frac{1}{2}(3(m+n) - i\sqrt{3}(3m-n))}{\frac{1}{2}(1-i\sqrt{3})\overline{\zeta} + \overline{z} - \frac{\sqrt{3}}{2}(\sqrt{3}-i) - \frac{1}{2}(3(m-n) - i\sqrt{3}(3m+n))}\right|$$

$$= \left|\frac{z - \zeta_1 - \omega_{\frac{m+n}{2},\frac{3m-n}{2}}}{z - \zeta_2 - \omega_{\frac{m-n}{2},\frac{3m+n}{2}}}\right|,$$

$$\left|\frac{\zeta - \overline{z_2} - \omega_{m,n}}{\zeta - z_2 - \omega_{m,n}}\right| = \left|\frac{\overline{\zeta} + \frac{1}{2}(1+i\sqrt{3})z - \frac{\sqrt{3}}{2}(\sqrt{3}+i) - \omega_{m,-n}}{\zeta + \frac{1}{2}(1+i\sqrt{3})z - \frac{\sqrt{3}}{2}(\sqrt{3}+i) - \omega_{m,n}}\right|$$

$$= \left|\frac{\frac{1}{2}(1-i\sqrt{3})\overline{\zeta} + z - \frac{\sqrt{3}}{2}(\sqrt{3}-i) - \frac{1}{2}(3(m-n) - i\sqrt{3}(3m+n))}{\frac{1}{2}(1-i\sqrt{3})\zeta + z - \frac{\sqrt{3}}{2}(\sqrt{3}-i) - \frac{1}{2}(3(m+n) - i\sqrt{3}(3m-n))}\right|$$

$$= \left|\frac{z - \overline{\zeta_2} - \omega_{\frac{m-n}{2},-\frac{3m+n}{2}}}{z - \overline{\zeta_1} - \omega_{\frac{m+n}{2},-\frac{3m-n}{2}}}\right|,$$

so that

$$G_1(z,\zeta) = \log \prod_{m+n\in 2\mathbb{Z}} \left|\frac{z - \overline{\zeta} - \omega_{-m,n}}{z - \zeta - \omega_{-m,-n}} \frac{z - \zeta_1 - \omega_{\frac{m+n}{2},\frac{3m-n}{2}}}{z - \overline{\zeta_1} - \omega_{\frac{m+n}{2},-\frac{3m-n}{2}}} \frac{z - \overline{\zeta_2} - \omega_{\frac{m-n}{2},-\frac{3m+n}{2}}}{z - \zeta_2 - \omega_{\frac{m-n}{2},\frac{3m+n}{2}}}\right|^2.$$

The system

$$m + n = 2k, \;\; 3m - n = 2l$$

is uniquely solvable in $\mathbb{Z}$ for given k, l satisfying $k + l \in 2\mathbb{Z}$ by

$$2m = k + l, \;\; 2n = k - l.$$

Multiplying

$$\prod_{m+n\in 2\mathbb{Z}} \left|\frac{z - \overline{\zeta} - \omega_{m,-n}}{z - \zeta - \omega_{m,n}}\right|^2$$

by

$$\prod_{m+n\in 2\mathbb{Z}} \left|\frac{z - \overline{\zeta} - \omega_{m,n}}{z - \overline{\zeta} - \omega_{m,-n}}\right|^2 = 1$$

finally

$$G_1(z,\zeta) = \log \prod_{m+n\in 2\mathbb{Z}} \left|\frac{z - \overline{\zeta} - \omega_{m,n}}{z - \zeta - \omega_{m,n}} \frac{z - \zeta_1 - \omega_{m,n}}{z - \overline{\zeta_1} - \omega_{m,n}} \frac{z - \overline{\zeta_2} - \omega_{m,n}}{z - \zeta_2 - \omega_{m,n}}\right|^2 = G_1(\zeta, z)$$

is obtained.

From (6) the boundary behavior of the Green function $G_1(z,\zeta)$ is easily seen. As well

i. for $z = z_1$ and the fact $\overline{z_2} = z_2$ as

ii. for $z = \overline{z}$ and the fact $z_1 = z_2$ the formula (6) immediately shows that the boundary values vanish.

iii. For the last side of T where

$$z = -\frac{1}{2}(1 - i\sqrt{3})\overline{z} - \frac{\sqrt{3}}{2}(\sqrt{3} - i)$$

implies

$$z_1 = \overline{z_1} + 2i\sqrt{3}, \quad z_2 = \overline{z} + 3 + i\sqrt{3}$$

the form (6) shows

$$G_1(z,\zeta) = \log \prod_{m,n\in 2\mathbb{Z}} \left| \frac{\zeta - \overline{z} - \omega_{m,n}}{\zeta - z - \omega_{m,n}} \frac{\zeta - z_1 - \omega_{m,n}}{\zeta - z_1 - \omega_{m,n-2}} \frac{\zeta - z - \omega_{m+1,n-1}}{\zeta - \overline{z} - \omega_{m+1,n+1}} \right|^2 = 0.$$

□

In the Green representation formula besides the Green function itself also the Poisson kernel is needed. It is given by the outward normal derivative of the Green function on the boundary

$$P(z,\zeta) = -\partial_{\nu_z} G_1(z,\zeta), \quad z \in \partial T, \quad \zeta \in T.$$

For $\partial_1 T$, where $z = z_1$, the outward normal derivative is

$$\partial_{\nu_z} = \frac{1}{2}(\sqrt{3} + i)\partial_z + \frac{1}{2}(\sqrt{3} - i)\partial_{\overline{z}}.$$

For $\partial_2 T$, where $z = z_5 - 6$, it is

$$\partial_{\nu_z} = \frac{1}{2}(\sqrt{3} - i)\partial_z + \frac{1}{2}(\sqrt{3} + i)\partial_{\overline{z}}.$$

For $\partial_3 T$, where $z = \overline{z}$, this is

$$\partial_{\nu_z} = -i\partial_z + i\partial_{\overline{z}}.$$

Theorem 2. *The Poisson kernel for T is given as*

$$P(z,\zeta) = \mathrm{Re} \sum_{m+n\in 2\mathbb{Z}} \begin{cases} 2(\sqrt{3}+i)\left[\frac{1}{z-\zeta-\omega_{m,n}} - \frac{1}{z-\overline{\zeta}-\omega_{m,n}} + \frac{1}{z-\overline{\zeta_1}-\omega_{m,n}}\right] & \text{on } \partial_1 T, \\ 2(\sqrt{3}-i)\left[\frac{1}{z-\zeta-\omega_{m,n}} - \frac{1}{z-\overline{\zeta}-\omega_{m,n}} + \frac{1}{z-\zeta_2-\omega_{m,n}}\right] & \text{on } \partial_2 T, \\ 2i\left[\frac{1}{z-\zeta-\omega_{m,n}} + \frac{1}{z-\overline{\zeta_1}-\omega_{m,n}} - \frac{1}{z-\overline{\zeta_2}-\omega_{m,n}}\right] & \text{on } \partial_3 T. \end{cases}$$

Proof. i. On $\partial_1 T$ the relations $z = z_1$ and $\overline{z_2} = z_2$ hold. $\partial_{\nu_z} G_1(z,\zeta) = \mathrm{Re}(\sqrt{3} + i)\partial_z G_1(z,\zeta)$. Moreover, for the Poisson integral formula

$$ds_z = -2dx, \quad dz = (1 - i\sqrt{3})dx, \quad dx = \frac{1}{4}(1 + i\sqrt{3})dz,$$

$$d\overline{z} = -\frac{1}{2}(1 - i\sqrt{3})dz,$$

are mentioned. From (6)

$$\partial_{\nu_z} G_1(z,\zeta) = \sum_{m+n\in 2\mathbb{Z}} \left[\frac{1}{\zeta - z - \omega_{m,n}} - \frac{1}{\overline{\zeta} - z - \overline{\omega_{m,n}}} \right.$$

$$\left. -\frac{1}{2}\frac{1-i\sqrt{3}}{\zeta - \overline{z_1} - \omega_{m,n}} - \frac{1}{2}\frac{1-i\sqrt{3}}{\overline{\zeta - z_1 - \omega_{m,n}}} - \frac{1}{2}\frac{1+i\sqrt{3}}{\zeta - z_2 - \omega_{m,n}} + \frac{1}{2}\frac{1+i\sqrt{3}}{\overline{\zeta} - z_2 - \overline{\omega_{m,n}}} \right],$$

i.e.

$$\partial_{\nu_z} G_1(z,\zeta) = \mathrm{Re} \sum_{m+n\in 2\mathbb{Z}} \left[\frac{\sqrt{3}+i}{\zeta - z - \omega_{m,n}} - \frac{\sqrt{3}+i}{\overline{\zeta} - z - \overline{\omega_{m,n}}} \right.$$

$$\left. -\frac{\sqrt{3}-i}{\zeta - \overline{z_1} - \omega_{m,n}} + \frac{\sqrt{3}-i}{\overline{\zeta - z_1 - \omega_{m,n}}} - \frac{2i}{\zeta - z_2 - \omega_{m,n}} + \frac{2i}{\overline{\zeta} - z_2 - \overline{\omega_{m,n}}} \right]$$

$$= \mathrm{Re} \sum_{m+n\in 2\mathbb{Z}} 2 \left[\frac{\sqrt{3}+i}{\zeta - z - \omega_{m,n}} - \frac{\sqrt{3}+i}{\overline{\zeta} - z - \omega_{m,n}} - \frac{2i}{\zeta - z_2 - \omega_{m,n}} \right]$$

follows. The last term in the sum can be rewritten as

$$\frac{2i}{\zeta - z_2 - \omega_{m,n}} = \frac{\sqrt{3}+i}{z - \overline{\zeta_1} - \omega_{\frac{m+n}{2}, \frac{3m-n}{2}}},$$

where

$$\zeta_1 = -\frac{1}{2}(1+i\sqrt{3})\overline{\zeta} + \frac{\sqrt{3}}{2}(\sqrt{3}+i).$$

Hence, on ∂T_1

$$\partial_{\nu_z} G_1(z,\zeta) = \mathrm{Re} \sum_{m+n\in 2\mathbb{Z}} 2(\sqrt{3}+i) \left[\frac{-1}{z - \zeta - \omega_{m,n}} + \frac{1}{z - \overline{\zeta} - \omega_{m,n}} - \frac{1}{z - \overline{\zeta_1} - \omega_{m,n}} \right].$$

Remark 1. *For any complex number a the sum*

$$\sum_{m+n\in 2\mathbb{Z}} \frac{1}{a - \omega_{m,n}}$$

is convergent. This is seen from

$$2 \sum_{m+n\in 2\mathbb{Z}} \frac{1}{a - \omega_{m,n}} = \sum_{m+n\in 2\mathbb{Z}} \left[\frac{1}{a - \omega_{m,n}} + \frac{1}{a + \omega_{m,n}} \right] = \sum_{m+n\in 2\mathbb{Z}} \frac{2a}{a^2 - \omega_{m,n}^2}.$$

ii. On $\partial_2 T$ besides $\overline{z_1} = z_1 - 2i\sqrt{3}$, $z_2 = \overline{z} + 3 + i\sqrt{3}$ holds

$$\partial_{\nu_z} G_1(z,\zeta) = \mathrm{Re}(\sqrt{3} - i)\partial_z G_1(z,\zeta).$$

Also

$$ds_z = -2dx, \quad dz = (1+i\sqrt{3})dx, \quad dx = \frac{1}{4}(1-i\sqrt{3})dz, \quad d\overline{z} = -\frac{1}{2}(1-i\sqrt{3})dz.$$

As before in step i

$$\partial_{\nu_z} G_1(z,\zeta) = \mathrm{Re} \sum_{m+n\in 2\mathbb{Z}} \left[\frac{\sqrt{3}-i}{\zeta - z - \omega_{m,n}} - \frac{\sqrt{3}-i}{\overline{\zeta} - z - \overline{\omega_{m,n}}} \right.$$

$$\left. + \frac{2i}{\zeta - \overline{z_1} - \omega_{m,n}} - \frac{2i}{\overline{\zeta - z_1 - \omega_{m,n}}} - \frac{\sqrt{3}+i}{\zeta - z_2 - \omega_{m,n}} + \frac{\sqrt{3}+i}{\overline{\zeta} - z_2 - \overline{\omega_{m,n}}} \right]$$

$$= \mathrm{Re} \sum_{m+n\in 2\mathbb{Z}} 2 \left[\frac{\sqrt{3}-i}{\zeta - z - \omega_{m,n}} - \frac{\sqrt{3}-i}{\overline{\zeta} - z - \overline{\omega_{m,n}}} + \frac{2i}{\zeta - \overline{z_1} - \omega_{m,n}} \right]$$

is seen. Here the last term is

$$\frac{2i}{\zeta - \overline{z_1} - \omega_{m,n}} = -\frac{\sqrt{3}-i}{z - \zeta_2 - \omega_{\frac{m-n}{2}, \frac{3m+n}{2}}},$$

where

$$\zeta_2 = -\frac{1}{2}(1 + i\sqrt{3})\zeta + \frac{\sqrt{3}}{2}(\sqrt{3} + i).$$

Thus on $\partial_2 T$

$$\partial_{\nu_z} G_1(z,\zeta) = \mathrm{Re} \sum_{m+n\in 2\mathbb{Z}} 2(\sqrt{3}-i) \left[\frac{-1}{z - \zeta - \omega_{m,n}} + \frac{1}{z - \overline{\zeta} - \omega_{m,n}} - \frac{1}{z - \zeta_2 - \omega_{m,n}} \right].$$

iii. On $\partial_3 T$ holds $\overline{z} = z$ so that $z_1 = z_2$ and

$$\partial_{\nu_z} G_1(z,\zeta) = 2\mathrm{Re}\, i\, \partial_z G_1(z,\zeta).$$

Here $ds_z = dx = dz = d\overline{z}$. Moreover,

$$\partial_{\nu_z} G_1(z,\zeta) = \mathrm{Re} \sum_{m+n\in 2\mathbb{Z}} \left[\frac{i}{\zeta - z - \omega_{m,n}} - \frac{i}{\overline{\zeta} - z - \overline{\omega_{m,n}}} \right.$$

$$\left. - \frac{1}{2}\frac{\sqrt{3}+i}{\zeta - \overline{z_1} - \omega_{m,n}} + \frac{1}{2}\frac{\sqrt{3}+i}{\overline{\zeta - z_1 - \omega_{m,n}}} + \frac{1}{2}\frac{\sqrt{3}-i}{\zeta - z_2 - \omega_{m,n}} - \frac{1}{2}\frac{\sqrt{3}-i}{\overline{\zeta} - z_2 - \overline{\omega_{m,n}}} \right].$$

Because

$$\frac{\sqrt{3}-i}{\zeta - z_1 - \omega_{m,n}} = -\frac{2i}{z - \overline{\zeta_1} - \omega_{\frac{m+n}{2}, \frac{n-3m}{2}}},$$

$$\frac{\sqrt{3}-i}{\overline{\zeta} - z_1 - \overline{\omega_{m,n}}} = -\frac{2i}{z - \overline{\zeta_2} - \omega_{\frac{m-n}{2}, -\frac{3m+n}{2}}},$$

then

$$\partial_{\nu_z} G_1(z,\zeta) = \mathrm{Re} \sum_{m+n\in 2\mathbb{Z}} 2i \left[\frac{-1}{z - \zeta - \omega_{m,n}} - \frac{1}{z - \overline{\zeta_1} - \omega_{m,n}} + \frac{1}{z - \overline{\zeta_2} - \omega_{m,n}} \right].$$

□

3 Neumann function for the triangle T

To our knowledge there is no theoretical method to get a harmonic Neumann function from the Green function. By experience it can be achieved by the following procedure. When the Green function is given by a certain quotient, the Neumann function turns out to be just given by multiplying all the factors of the quotient, see e.g. [3], [6]. However, sometimes a modification is needed in order to get convergence of the product in case when this is an infinite one.

Theorem 3. *A harmonic Neumann function for T is*

$$N_1(z,\zeta) = -\log\left|\prod_{k=0}^{5}(\zeta - z_k)(\zeta - \overline{z_k}) \prod_{\substack{m,n\in\mathbb{Z},\\ 0<m^2+n^2}} \prod_{k=0}^{5}\left(\frac{\zeta - z_k}{\Omega_{m,n}} - 1\right)\left(\frac{\zeta - \overline{z_k}}{\Omega_{m,n}} - 1\right)\right|^2$$

for z, $\zeta \in T$, $z \neq \zeta$, where $z_0 = z$.

Proof. At first

$$\prod_{\substack{m,n\in\mathbb{Z},\\ 0<m^2+n^2}} \left|\frac{\zeta - z_k}{\Omega_{m,n}} - 1\right|^2$$

will be shown to converge. Let the factor be denoted by $a_{m,n}$. Then

$$\prod_{\substack{m,n\in\mathbb{Z},\\ 0<m^2+n^2}} a_{m,n} = \prod_{m=1}^{\infty} a_{-m,0}a_{m,0} \prod_{n=1}^{\infty} a_{0,-n}a_{0,n} \prod_{m,n=1}^{\infty} a_{-m,-n}a_{-m,n}a_{m,-n}a_{m,n}.$$

Obviously,

$$\prod_{m=1}^{\infty} a_{-m,0}a_{m,0} = \prod_{m=1}^{\infty}\left|\left(\frac{\zeta - z_k}{6m} + 1\right)\left(\frac{\zeta - z_k}{6m} - 1\right)\right|^2 = \prod_{m=1}^{\infty}\left|\frac{(\zeta - z_k)^2}{36m^2} - 1\right|^2$$

and

$$\prod_{n=1}^{\infty} a_{0,-n}a_{0,n} = \prod_{n=1}^{\infty}\left|\left(\frac{\zeta - z_k}{2i\sqrt{3}n} + 1\right)\left(\frac{\zeta - z_k}{2i\sqrt{3}n} - 1\right)\right|^2 = \prod_{n=1}^{\infty}\left|\frac{(\zeta - z_k)^2}{12n^2} + 1\right|^2$$

are convergent. Moreover,

$$\prod_{m,n=1}^{\infty} a_{-m,-n}a_{m,n} = \prod_{m,n=1}^{\infty}\left|\left(\frac{\zeta - z_k}{6m + 2i\sqrt{3}n} + 1\right)\left(\frac{\zeta - z_k}{6m + 2i\sqrt{3}n} - 1\right)\right|^2$$

$$= \prod_{m,n=1}^{\infty}\left|\frac{(\zeta - z_k)^2}{(6m + 2i\sqrt{3}n)^2} - 1\right|^2$$

and

$$\prod_{m,n=1}^{\infty} a_{-m,n}a_{m,-n} = \prod_{m,n=1}^{\infty} \left| \left(\frac{\zeta - z_k}{6m - 2i\sqrt{3}n} + 1 \right) \left(\frac{\zeta - z_k}{6m - 2i\sqrt{3}n} - 1 \right) \right|^2$$

$$= \prod_{m,n=1}^{\infty} \left| \frac{(\zeta - z_k)^2}{(6m - 2i\sqrt{3}n)^2} - 1 \right|^2$$

converge as

$$\sum_{m,n=1}^{\infty} \left| \frac{(\zeta - z_k)^2}{(6m + 2i\sqrt{3}n)^2} \right|^2$$

converges.

Next the normal derivative on ∂T has to be calculated.

On $\partial_1 T$ the outward normal derivative is

$$\partial_{\nu_z} = \frac{1}{2}(\sqrt{3} + i)\partial_z + \frac{1}{2}(\sqrt{3} - 1)\partial_{\overline{z}}.$$

Thus differentiating $N_1(z, \zeta)$ termwise and taking into account the expressions for $z_1, ..., z_5$ (formulas (1)–(5))

$$\partial_{\nu_z} N_1(z, \zeta) = -\sum_{m,n\in\mathbb{Z}} \sum_{k=0}^{5} \partial_{\nu_z} \left[\log |\zeta - z_k - \Omega_{m,n}|^2 + \log |\zeta - \overline{z_k} - \Omega_{m,n}|^2 \right]$$

$$= i \sum_{m,n\in\mathbb{Z}} \left[\frac{1}{\zeta - z_5 - \Omega_{m,n}} - \frac{1}{\overline{\zeta - z_5 - \Omega_{m,n}}} + \frac{1}{\overline{\zeta} - z_5 - \overline{\Omega_{m,n}}} - \frac{1}{\zeta - \overline{z_5} - \Omega_{m,n}} \right] \quad (7)$$

is seen. This series is absolutely convergent as the two twice double sums

$$\sum_{m,n\in\mathbb{Z}} \left[\frac{1}{\zeta - z_5 - \Omega_{m,n}} - \frac{1}{\zeta - \overline{z_5} - \Omega_{m,n)}} \right],$$

$$\sum_{m,n\in\mathbb{Z}} \left[\frac{1}{\overline{\zeta} - z_5 - \overline{\Omega_{m,n}}} - \frac{1}{\overline{\zeta - z_5 - \Omega_{m,n)}}} \right]$$

are absolutely convergent.

As on $\partial_1 T$ hold $z = z_1$, $z_2 = \overline{z_2}$, $z_3 = z_4$, $\overline{z_5} = z_5 - 2\sqrt{3}i$ this in turn gives that (7) equals to

$$i \sum_{m,n\in\mathbb{Z}} \left[\frac{1}{\zeta - z_5 - \Omega_{m,n}} - \frac{1}{\zeta - z_5 - \Omega_{m,n-1}} + \frac{1}{\overline{\zeta} - z_5 - \Omega_{m,-n}} - \frac{1}{\overline{\zeta} - z_5 - \Omega_{m,-(n+1)}} \right].$$

From

$$\sum_{n=-N}^{N} \left[\frac{1}{\zeta - z_5 - \Omega_{m,n}} - \frac{1}{\zeta - z_5 - \Omega_{m,n-1}} + \frac{1}{\overline{\zeta} - z_5 - \Omega_{m,-n}} - \frac{1}{\overline{\zeta} - z_5 - \Omega_{m,-(n+1)}} \right]$$

$$= \sum_{n=-N}^{N} \left[\frac{1}{\zeta - z_5 - \Omega_{m,n}} - \frac{1}{\zeta - z_5 - \Omega_{m,n-1}} + \frac{1}{\overline{\zeta} - z_5 - \Omega_{m,n}} - \frac{1}{\overline{\zeta} - z_5 - \Omega_{m,n-1}} \right]$$

$$= \frac{1}{\zeta - z_5 - \Omega_{m,N}} - \frac{1}{\zeta - z_5 - \Omega_{m,-(N+1)}} + \frac{1}{\overline{\zeta} - z_5 - \Omega_{m,N}} - \frac{1}{\overline{\zeta} - z_5 - \Omega_{m,-(N+1)}}$$

$$= \frac{2i\sqrt{3}(2N+1)}{(\zeta - z_5 - 6m - 2i\sqrt{3}N)(\zeta - z_5 - 6m + 2i\sqrt{3}(N+1))}$$

$$+ \frac{2i\sqrt{3}(2N+1)}{(\overline{\zeta} - z_5 - 6m - 2i\sqrt{3}N)(\overline{\zeta} - z_5 - 6m + 2i\sqrt{3}(N+1))}$$

finally

$$\partial_{\nu_z} N_1(z, \zeta) = 0$$

is seen for z on $\partial_1 T$, $\zeta \in T$.

On $\partial_2 T$ the outward normal derivative is

$$\partial_{\nu_z} = \frac{1}{2}(\sqrt{3} - i)\partial_z + \frac{1}{2}(\sqrt{3} + i)\partial_{\overline{z}}.$$

On this line $\overline{z_1} = z_1 - 2i\sqrt{3}$, $z_2 = z_3$, $\overline{z_4} = z_4$, $z_5 = z + 6$.

Direct calculations show

$$\partial_{\nu_z} N_1(z, \zeta) =$$

$$\frac{1}{2}(\sqrt{3}-i) \sum_{m,n\in\mathbb{Z}} \left[\frac{1}{\zeta - z - \Omega_{m,n}} - \frac{1}{\zeta - z - \Omega_{m+1,n}} + \frac{1}{\overline{\zeta} - z - \overline{\Omega_{m,n}}} - \frac{1}{\overline{\zeta} - z - \overline{\Omega_{m+1,n}}} \right]$$

$$+\frac{1}{2}(\sqrt{3}+i) \sum_{m,n\in\mathbb{Z}} \left[\frac{1}{\overline{\zeta - z - \Omega_{m,n}}} - \frac{1}{\overline{\zeta - z - \Omega_{m+1,n}}} + \frac{1}{\zeta - \overline{z} - \Omega_{m,n}} - \frac{1}{\zeta - \overline{z} - \Omega_{m+1,n}} \right]$$

$$-i \sum_{m,n\in\mathbb{Z}} \left[\frac{1}{\zeta - z_1 - \Omega_{m,n}} - \frac{1}{\zeta - z_1 - \Omega_{m,n-1}} + \frac{1}{\overline{\zeta} - z_1 - \overline{\Omega_{m,n}}} - \frac{1}{\overline{\zeta} - z_1 - \overline{\Omega_{m,n-1}}} \right].$$

All these sums are of the above type as in step i and hence equal zero. Thus also on $\partial_2 T$

$$\partial_{\nu_z} N_1(z, \zeta) = 0 \text{ for any } \zeta \in T.$$

On $\partial_3 T$ the outward normal derivative is

$$\partial_{\nu_z} = -i\partial_y = -i\partial_z + i\partial_{\overline{z}}.$$

On this line as part of the real axis

$$z = \overline{z_1}, \quad z_1 = z_2, \quad \overline{z_3} = z_3 - 2i\sqrt{3}, \quad z_4 = z_5.$$

Thus

$$\partial_{\nu_z} N_1(z, \zeta)$$

$$= i \sum_{m,n\in\mathbb{Z}} \left[\frac{1}{\zeta - z_3 - \Omega_{m,n}} - \frac{1}{\overline{\zeta - z_3 - \Omega_{m,n}}} + \frac{1}{\overline{\zeta} - z_3 - \overline{\Omega_{m,n}}} - \frac{1}{\zeta - \overline{z_3} - \Omega_{m,n}} \right],$$

from where as before

$$\partial_{\nu_z} N_1(z,\zeta) = 0$$

is seen for z on $\partial_3 T$ and $\zeta \in T$. □

Remark 2. *The function*

$$\widehat{N}_1(z,\zeta) = -\log\left|(\zeta - z)(\zeta - z_1)(\zeta - z_2) \prod_{\substack{m,n\in\mathbb{Z},\\ 0<m^2+n^2}} \prod_{k=0}^{2} \left(\frac{\zeta - z_k}{\omega_{m,n}} - 1\right)\left(\frac{\zeta - \overline{z_k}}{\omega_{m,n}} - 1\right)\right|^2,$$

where $z_0 = z$ *and* $\omega_{m,n} = 3m + i\sqrt{3}n$, *differs from* $N_1(z,\zeta)$ *just by an additive constant. It therefore can serve as Neumann function also having the same boundary behavior.*

As the normal derivative of the Neumann functions vanishes on ∂T outside the corner points, the Neumann problem for the Poisson equation in T is unconditionally solvable.

That $\widehat{N}_1(z,\zeta)$ differs only by an additive constant from $N_1(z,\zeta)$ follows from the facts

$$z_k = \overline{z_{k-3}} + 3 + i\sqrt{3} \text{ for } 3 \le k \le 5.$$

For those k

$$\frac{\zeta - z_k}{\Omega_{m,n}} - 1 = \left[\frac{\zeta - \overline{z_{k-3}}}{\omega_{2m+1,2n+1}} - 1\right]\left[1 + \frac{3 + i\sqrt{3}}{\Omega_{m,n}}\right],$$

$$\frac{\zeta - \overline{z_k}}{\Omega_{m,n}} - 1 = \left[\frac{\zeta - z_{k-3}}{\omega_{2m+1,2n-1}} - 1\right]\left[1 + \frac{3 - i\sqrt{3}}{\Omega_{m,n}}\right]$$

and

$$\prod_{m,n\in\mathbb{Z}} \left|1 + \frac{3 + i\sqrt{3}}{\Omega_{m,n}}\right|^2 = \prod_{m,n=1}^{\infty} \left|\left[1 - \left(\frac{3 + i\sqrt{3}}{\Omega_{m,n}}\right)^2\right]\left[1 - \left(\frac{3 - i\sqrt{3}}{\Omega_{m,n}}\right)^2\right]\right|^2.$$

This product is convergent and the mentioned constant is just the sixth power of this value.

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