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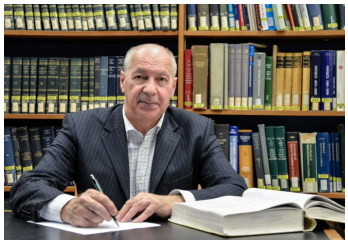
VICTOR IVANOVICH BURENKOV

(to the 80th birthday)



On July 15, 2021 was the 80th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), professor of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (Moscow), chairman of the Dissertation Council at the RUDN University, research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), honorary academician of the National Academy of Sciences of the Republic of Kazakhstan, doctor of physical and mathematical sciences(1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK,2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University. Through 2015-2017 he was head of the Department of Mathematical Analysis and Theory of Functions (RUDN University). He was one of the organisers and the first director of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (2016-2017).

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem on composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 190 scientific papers. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces. In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

V.I. Burenkov is still very active in research. Through 2016-2021 he published 20 papers in leading mathematical journals.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 80th birthday and wishes him good health and new achievements in science and teaching!

Short communications

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ON THE INEQUALITY OF DIFFERENT METRICS FOR MULTIPLE FOURIER-HAAR SERIES

A.N. Bashirova, E.D. Nursultanov

Communicated by N.A. Bokayev

Key words: Fourier series, Haar system, inequality of different metrics, anisotropic Lebesgue and Lorentz spaces.

AMS Mathematics Subject Classification: 42B05, 46E30.

Abstract. Let $1 < p < q < \infty$, $f \in L_p[0,1]$. Then, according to the inequality of different metrics due to S.M. Nikol'skii, for the sequence of norms of partial sums of the Fourier-Haar series $\left\{\|S_{2^k}(f)\|_{L_q}\right\}_{k=0}^{\infty}$ the following relation is true $\|S_{2^k}(f)\|_{L_q} = O\left(2^{k\left(\frac{1}{p}-\frac{1}{q}\right)}\right)$. In this paper, we study the asymptotic behavior of partial sums in the Lorentz spaces. In particular, it is obtained that $\|S_{2^{k_1}2^{k_2}}(f)\|_{L_{\bar{q}}} = o\left(2^{k_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)+k_2\left(\frac{1}{p_2}-\frac{1}{q_2}\right)}\right)$ for $f \in L_{\bar{p},\bar{\tau}}[0,1]^2$.

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1 Introduction

The Haar system is the system of functions $\chi = \{\chi_k^j(x)\}_{k=0,j=1}^{\infty,2^k}$, $x \in [0,1]$, in which $\chi_1(x) \equiv 1$, and the function $\chi_k^j(x)$, where $k = 0, 1, \ldots$, $j = 1, 2, \ldots, 2^k$ is defined as:

$$\chi_k^j(x) = \begin{cases} 2^{\frac{k}{2}}, & \dfrac{2j-2}{2^{k+1}} < x < \dfrac{2j-1}{2^{k+1}}, \\ -2^{\frac{k}{2}}, & \dfrac{2j-1}{2^{k+1}} < x < \dfrac{2j}{2^{k+1}}, \\ 0, & x \notin \left(\dfrac{j-1}{2^k}; \dfrac{j}{2^k}\right). \end{cases}$$

The Fourier-Haar series of a function $f \in L_1[0,1]$ is the series of the form

$$\sum_{k=0}^{\infty}\sum_{j=1}^{2^k} a_k^j(f)\chi_k^j(x),$$

where $a_k^j(f) = (f, \chi_k^j)$ are the Fourier-Haar coefficients of the function f.

The inequality of different metrics due to S.M. Nikol'skii for the Fourier-Haar series in the Lebesgue spaces is well known [13]. It can be written in terms of partial sums as follows

$$\|S_{2^n}(f)\|_{L_q} \le c2^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L_p}, \tag{1.1}$$

where $1 \le p < q \le \infty$ and $c > 0$ is independent of f. Hence,

$$\|S_{2^n}(f)\|_{L_q} = O\left(2^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\right).$$

Here

$$S_{2^n}(f) = \sum_{k=0}^{n} \sum_{j=1}^{2^k} a_k^j(f)\chi_k^j(x)$$

is the partial sum of the Fourier-Haar series.

In [9] the following relations were obtained for the Fourier series over the trigonometric system

$$\|S_n(f)\|_{L_q} = o\left(n^{\frac{1}{p}-\frac{1}{q}}\right), \quad n \to \infty,$$

and

$$\sum_{n=1}^{\infty} \frac{\left(n^{\frac{1}{q}-\frac{1}{p}}\|S_n(f)\|_{L_q}\right)^p}{n} \le c\|f\|_{L_p}^p. \tag{1.2}$$

These inequalities were essentially used in the study of the multiplicators of the Fourier series [10, 11, 12].

Let $\bar{p} = (p_1, p_2)$, $0 < p_i \le \infty$. The Lebesgue space with mixed metric $L_{\bar{p}}[0,1]^2$ is the set of all measurable on $[0,1]^2$ functions $f(x_1, x_2)$ for which

$$\|f\|_{L_{\bar{p}}[0,1]^2} := \left(\int_0^1 \left(\int_0^1 |f(x_1,x_2)|^{p_1}\, dx_1\right)^{\frac{p_2}{p_1}} dx_2\right)^{\frac{1}{p_2}}.$$

Let f be a function Lebesgue measurable on $[0,1]^2$, via $f^{*_1*_2}$ we denote the function obtained by applying the non-increasing rearrangement to the function $f(x_1, x_2)$ sequentially in variables x_1, x_2.

Let $\bar{p} = (p_1, p_2)$, $\bar{\tau} = (\tau_1, \tau_2)$ be such that if $0 < p_i < \infty$, then $0 < \tau_i \le \infty$, if $p_i = \infty$, then $\tau_i = \infty$, $i = 1, 2$.

By the anisotropic Lorentz space $L_{\bar{p},\bar{\tau}}([0,1]^2)$ [8, 14] we mean the set of all functions for which the the following expression is finite

$$\|f\|_{L_{\bar{p},\bar{\tau}}([0,1]^2)} = \left(\int_0^1 \left(\int_0^1 \left(t_2^{\frac{1}{p_2}} t_1^{\frac{1}{p_1}} f^{*_1*_2}(t_1,t_2)\right)^{\tau_1} \frac{dt_1}{t_1}\right)^{\frac{\tau_2}{\tau_1}} \frac{dt_2}{t_2}\right)^{\frac{1}{\tau_2}}.$$

Here and in the sequel, when $\tau = \infty$, the integral $\left(\int_0^1 (\varphi(t))^\tau \frac{dt}{t}\right)^{\frac{1}{\tau}}$ is understood as a $\sup_{t>0} \varphi(t)$.

2 Main result

Theorem 2.1. *Let $1 < p < q < \infty$, $0 < \tau \le \infty$. If $f \in L_{p,\tau}[0,1]$, then*

$$\left(\sum_{k=0}^{\infty} \left(2^{k\left(\frac{1}{q}-\frac{1}{p}\right)} \|S_{2^k}(f)\|_{L_q}\right)^\tau\right)^{\frac{1}{\tau}} \le C\,\|f\|_{L_{p,\tau}}, \tag{2.1}$$

where $C > 0$ is independent of f.

Remark 1. From inequality (2.1) it follows that for the norm of partial sums of a series of fixed function $f \in L_{p,\tau}[0,1]$, the following relations is true

$$\|S_{2^k}(f)\|_{L_q} = o\left(2^{k\left(\frac{1}{p}-\frac{1}{q}\right)}\right).$$

In addition, this relation gives us information about the rate of convergence to zero for the sequence $\left\{2^{k\left(\frac{1}{q}-\frac{1}{p}\right)}\|S_{2^k}(f)\|_{L_q}\right\}$.

Lemma 2.1. *Let $f \in L_{\bar{p}}[0,1]^2$, $S_{2^{k_1}2^{k_2}}(f)$ be the partial sums of its Fourier-Haar series and $1 < \bar{p} < \bar{q} < \infty$, where $\bar{p} = (p_1, p_2)$ and $\bar{q} = (q_1, q_2)$. Then*

$$2^{\left(\frac{1}{q_1}-\frac{1}{p_1}\right)k_1+\left(\frac{1}{q_2}-\frac{1}{p_2}\right)k_2}\|S_{2^{k_1}2^{k_2}}(f)\|_{L_{\bar{q}}} \le c\,\|f\|_{L_{\bar{p}}},$$

where $c > 0$ is independent of f.

Theorem 2.2. *Let $1 < \bar{p} < \bar{q} < \infty$, $0 < \bar{\tau} \le \infty$. If $f \in L_{\bar{p},\bar{\tau}}[0,1]^2$, then the following inequality takes place*

$$\left(\sum_{k_2=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\left(2^{k_1\left(\frac{1}{q_1}-\frac{1}{p_1}\right)+k_2\left(\frac{1}{q_2}-\frac{1}{p_2}\right)}\|S_{2^{k_1}2^{k_2}}(f)\|_{L_{\bar{q}}}\right)^{\tau_1}\right)^{\frac{\tau_2}{\tau_1}}\right)^{\frac{1}{\tau_2}} \le C\,\|f\|_{L_{\bar{p},\bar{\tau}}}, \tag{2.2}$$

where $C > 0$ is independent of f.

The proof of this theorem is based on Lemma 2.1, which is in some sense a "weak" inequality. We also use interpolation methods for anisotropic spaces and interpolation properties of the Lebesgue spaces with a mixed metric. These interpolation methods allow us to interpolate anisotropic spaces (see works [1] - [3], [5, 6], [8]).

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