ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2021, Volume 12, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

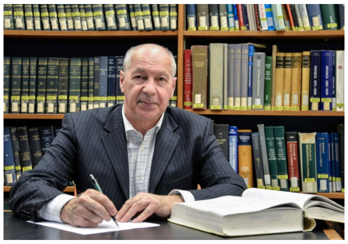
eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 562
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

VICTOR IVANOVICH BURENKOV

(to the 80th birthday)



On July 15, 2021 was the 80th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), professor of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (Moscow), chairman of the Dissertation Council at the RUDN University, research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), honorary academician of the National Academy of Sciences of the Republic of Kazakhstan, doctor of physical and mathematical sciences(1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK,2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University. Through 2015-2017 he was head of the Department of Mathematical Analysis and Theory of Functions (RUDN University). He was one of the organisers and the first director of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (2016-2017).

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem on composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 190 scientific papers. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces. In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

V.I. Burenkov is still very active in research. Through 2016-2021 he published 20 papers in leading mathematical journals.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 80th birthday and wishes him good health and new achievements in science and teaching!

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 12, Number 3 (2021), 78 – 89

STOKES-TYPE INTEGRAL EQUALITIES FOR SCALARLY ESSENTIALLY INTEGRABLE LOCALLY CONVEX VECTOR-VALUED FORMS WHICH ARE FUNCTIONS OF AN UNBOUNDED SPECTRAL OPERATOR

B. Silvestri

Communicated by V.I. Burenkov

Key words: unbounded spectral operators in Banach spaces, functional calculus, integration of locally convex vector-valued forms on manifolds, Stokes equalities.

AMS Mathematics Subject Classification: 46G10, 47B40, 47A60, 58C35.

Abstract. In this work we establish a Stokes-type integral equality for scalarly essentially integrable forms on an orientable smooth manifold with values in the locally convex linear space $\langle B(G), \sigma(B(G), \mathcal{N})\rangle$, where G is a complex Banach space and $\mathcal{N}$ is a suitable linear subspace of the norm dual of $B(G)$. This result widely extends the Newton-Leibnitz-type equality stated in one of our previous articles. To obtain our equality we generalize the main result of those articles, and employ the Stokes theorem for smooth locally convex vector-valued forms established there. Two facts are remarkable. First, the forms integrated involved in the equality are functions of a possibly unbounded scalar-type spectral operator in G. Secondly, these forms need not be smooth nor even continuously differentiable.

DOI: https://doi.org/10.32523/2077-9879-2021-12-3-78-89

1 Introduction

In this work we establish in Theorem 3.3 a Stokes-type integral equality for scalarly essentially integrable $\langle B(G), \sigma(B(G), \mathcal{N})\rangle$-valued forms on an orientable smooth manifold, where G is a complex Banach space. This result widely extends the Newton-Leibnitz-type equality established in [3, Corollary 2.33]. To obtain the equality we employ the Extension Theorem 3.2 a generalization of [3, Theorem 2.25] along with the Stokes theorem for smooth locally convex vector-valued forms [4, Theorem 2.54]. Two facts are remarkable. First, these forms are functions of a possibly unbounded scalar-type spectral operator in G. Secondly these forms need not be smooth nor even continuously differentiable.

2 Notation

In the present work we employ the notation of [3] and of [4], with the following two remarks. First, what in [3] is called "Radon measure" and meant measure in the sense of Bourbaki [1, Chapter III, §1, $n^{\circ}3$, Definition 2], here accordingly will be called simply "measure". Secondly, if Z is a $\mathbb{K}$-locally convex vector space with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then $Z' = \mathcal{L}(Z, \mathbb{K})$ denotes the topological dual of Z.

If G is a $\mathbb{C}$-Banach space, then let $\mathrm{ClO}(G)$ denote the set of all closed operators in G. If X is a locally compact space and μ is a measure on X, then a map $f : X \to \mathbb{C}$ is scalarly essentially μ-integrable or simply essentially μ-integrable if and only if $\mathfrak{R} \circ \imath_{\mathbb{C}}^{\mathbb{C}_{\mathbb{R}}} \circ f$ and $\mathfrak{I} \circ \imath_{\mathbb{C}}^{\mathbb{C}_{\mathbb{R}}} \circ f$ are essentially μ-integrable, where $\mathfrak{R}, \mathfrak{I} \in \mathcal{L}(\mathbb{C}_{\mathbb{R}}, \mathbb{R})$ are the real and imaginary parts, respectively.

We recall from [3, pages 39 - 40] that if $\langle Z,\tau\rangle$ is a Hausdorff locally convex space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, then by definition $f:X\to\langle Z,\tau\rangle$ is scalarly essentially (μ,Z)-integrable, or $f:X\to Z$ is scalarly essentially (μ,Z)-integrable with respect to the topology τ, if and only if $\psi\circ f$ is essentially μ-integrable for every $\psi\in\langle Z,\tau\rangle'$ and the weak integral of f belongs to Z, namely there exists a necessarily unique element $s\in Z$ such that $\psi(s)=\int(\psi\circ f)\,d\mu$ for every $\psi\in\langle Z,\tau\rangle'$. In such a case we shall define $\int f\,d\mu\doteqdot s$.

Let $N\in\mathbb{Z}_+^*$, define $\mathrm{P}^{[N]}:\mathbb{R}^N\to\mathbb{R}^{N-1}$, $x\mapsto x\upharpoonright[1,N-1]\cap\mathbb{Z}$ if $N>1$; $x\mapsto 0$ if $N=1$. Let M be a nonzero dimensional manifold with boundary and let (U,ϕ) be a boundary chart of M, define $\phi^{\partial M}\doteqdot(\mathrm{P}^{[\dim M]}\circ\imath_{\phi(U)}^{\mathbb{R}^{\dim M}}\circ\phi\circ\imath_{U\cap\partial M}^{U})_\natural$, where $f_\natural=f\upharpoonright^{\mathrm{Range}(f)}$ for any map f. Let $\mathcal{U}$ be a collection of charts of M, and let $\mathcal{U}_\partial$ be the subcollection of all those elements in $\mathcal{U}$ that are boundary charts, define $\mathcal{U}^\partial\doteqdot\{(U\cap\partial M,\phi^{\partial M})\,|\,(U,\phi)\in\mathcal{U}_\partial\}$. If $\mathcal{U}$ is an atlas of M, then $\mathcal{U}^\partial$ is an atlas of ∂M, moreover if M is oriented and $\mathcal{U}$ is oriented, then $\mathcal{U}^\partial$ is oriented and $(U\cap\partial M,\phi^{\partial M})$ is γ-oriented if and only if $(U,\phi)\in\mathcal{U}$ is γ-oriented, with $\gamma\in\{1,-1\}$.

We fix the following data. G is a $\mathbb{C}$-Banach space; R is a possibly **unbounded** scalar-type spectral operator on G; let $\sigma(R)$ be its spectrum and let E be its resolution of identity; an $E-$appropriate set $\mathcal{N}$ [3, Definition 2.11]; a scalar-type spectral operator $T\in B(G)$ and let $\sigma(T)$ denote its spectrum; locally compact spaces X,Y and measures μ and ν on X and Y respectively; a finite dimensional smooth manifold M, with or without boundary, such that $N\doteqdot\dim M\neq 0$.

3 Main results

Theorem 3.1. *Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be an $E-$sequence, let maps $X\ni x\mapsto f_x\in\mathrm{Bor}(\sigma(R))$ and $Y\ni y\mapsto u_y\in\mathrm{Bor}(\sigma(R))$ be such that $\widetilde{f_x}\in\mathfrak{L}_E^\infty(\sigma(R))$, $\mu-l.a.e.(X)$ and $\widetilde{u_y}\in\mathfrak{L}_E^\infty(\sigma(R))$, $\nu-l.a.e.(Y)$. Let $X\ni x\mapsto f_x(R)\in\langle B(G),\sigma(B(G),\mathcal{N})\rangle$ and $Y\ni y\mapsto u_y(R)\in\langle B(G),\sigma(B(G),\mathcal{N})\rangle$ be scalarly essentially $(\mu,B(G))-$integrable and $(\nu,B(G))-$integrable respectively, let $g,h\in\mathrm{Bor}(\sigma(R))$. If for all $n\in\mathbb{N}$,*

$$g(R_{\sigma_n}\upharpoonright G_{\sigma_n})\int f_x(R_{\sigma_n}\upharpoonright G_{\sigma_n})\,d\,\mu(x)\subseteq h(R_{\sigma_n}\upharpoonright G_{\sigma_n})\int u_y(R_{\sigma_n}\upharpoonright G_{\sigma_n})\,d\,\nu(y), \tag{3.1}$$

then

$$g(R)\int f_x(R)\,d\,\mu(x)\upharpoonright\Theta=h(R)\int u_y(R)\,d\,\nu(y)\upharpoonright\Theta. \tag{3.2}$$

In (3.1) the weak-integrals are with respect to the measures μ and ν and with respect to the $\sigma(B(G_{\sigma_n}),\mathcal{N}_{\sigma_n})$-topology, while in (3.2)

$$\Theta\doteqdot\mathrm{Dom}\left(g(R)\int f_x(R)\,d\,\mu(x)\right)\cap\mathrm{Dom}\left(h(R)\int u_y(R)\,d\,\nu(y)\right),$$

and the weak-integrals are with respect to the measures μ and ν and with respect to the $\sigma(B(G),\mathcal{N})$-topology.

Proof. (3.1) is meaningful by [3, Theorem 2.22]. By [3, (1.18)], for all $z\in\Theta$

$$g(R)\int f_x(R)\,d\,\mu(x)\,z=\lim_{n\in\mathbb{N}}E(\sigma_n)g(R)\int f_x(R)\,d\,\mu(x)\,z$$

by [2, Theorem 18.2.11(g)] and [3, (2.25)]

$$=\lim_{n\in\mathbb{N}}g(R)\int f_x(R)\,d\,\mu(x)\,E(\sigma_n)z$$

by[3, (2.31)] and [3, Lemma 1.7] applied to $g(R)$

$$= \lim_{n\in\mathbb{N}} g(R_{\sigma_n} \restriction G_{\sigma_n}) \int f_x(R_{\sigma_n} \restriction G_{\sigma_n})\, d\,\mu(x)\, E(\sigma_n)z$$

by hypothesis (3.1)

$$= \lim_{n\in\mathbb{N}} h(R_{\sigma_n} \restriction G_{\sigma_n}) \int u_y(R_{\sigma_n} \restriction G_{\sigma_n})\, d\,\nu(y)\, E(\sigma_n)z$$

by the above, replacing g by h, f by u and μ by ν,

$$= h(R) \int u_y(R)\, d\,\nu(y)\, z. \tag{3.3}$$

□

Theorem 3.2 ($\sigma(B(G),\mathcal{N})-$**Extension Theorem**). *Let $X \ni x \mapsto f_x \in \mathrm{Bor}(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^{\infty}(\sigma(R))$, $\mu - l.a.e.(X)$ and $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ be scalarly essentially $(\mu, B(G))-$integrable. Moreover, let $Y \ni y \mapsto u_y \in \mathrm{Bor}(\sigma(R))$ be such that $\widetilde{u_y} \in \mathfrak{L}_E^{\infty}(\sigma(R))$, $\nu - l.a.e.(Y)$ and $Y \ni y \mapsto u_y(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ be scalarly essentially $(\nu, B(G))-$integrable. Finally, let $g, h \in \mathrm{Bor}(\sigma(R))$ and assume that* [1]

$$h(R) \int u_y(R)\, d\,\nu(y) \in B(G). \tag{3.4}$$

If $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence and for all $n \in \mathbb{N}$

$$g(R_{\sigma_n} \restriction G_{\sigma_n}) \int f_x(R_{\sigma_n} \restriction G_{\sigma_n})\, d\,\mu(x) \subseteq h(R_{\sigma_n} \restriction G_{\sigma_n}) \int u_y(R_{\sigma_n} \restriction G_{\sigma_n})\, d\,\nu(y), \tag{3.5}$$

then

$$g(R) \int f_x(R)\, d\,\mu(x) = h(R) \int u_y(R)\, d\,\nu(y). \tag{3.6}$$

In (3.5) the weak-integral are with respect to the measures μ and ν and with respect to the $\sigma(B(G_{\sigma_n}),\mathcal{N}_{\sigma_n})$-topology, while in (3.6) the weak-integral is with respect to the measures μ and ν and with respect to the $\sigma(B(G),\mathcal{N})$-topology.

Notice that $g(R)$ and $h(R)$ are possibly **unbounded** operators in G.

Proof. (3.4) and (3.2) imply

$$g(R) \int f_x(R)\, d\,\mu(x) \subseteq h(R) \int u_y(R)\, d\,\nu(y). \tag{3.7}$$

Let us set

$$(\forall n \in \mathbb{N})(\delta_n \doteqdot \overset{-1}{|g|}([0,n])). \tag{3.8}$$

We claim that

$$\begin{cases} \bigcup_{n\in\mathbb{N}} \delta_n = \sigma(R) \\ n \geq m \Rightarrow \delta_n \supseteq \delta_m \\ (\forall n \in \mathbb{N})(g(\delta_n) \text{ is bounded.}) \end{cases} \tag{3.9}$$

[1] For instance, but not necessarily, when $\widetilde{h} \in \mathfrak{L}_E^{\infty}(\sigma(R))$, since in such a case [2, Theorem 18.2.11] implies that $h(R) \in B(G)$.

Since $|g| \in \text{Bor}(\sigma(R))$ we have $\delta_n \in \mathcal{B}(\mathbb{C})$ for all $n \in \mathbb{N}$, so $\{\delta_n\}_{n\in\mathbb{N}}$ is an $E-$sequence, hence by [3, (1.18)]

$$\lim_{n\in\mathbb{N}} E(\delta_n) = \mathbf{1}; \tag{3.10}$$

with respect to the strong operator topology on $B(G)$. Indeed, the first equality follows by the equality

$$\bigcup_{n\in\mathbb{N}} \delta_n \doteq \bigcup_{n\in\mathbb{N}} \overset{-1}{|g|}([0,n]) = \overset{-1}{|g|}\left(\bigcup_{n\in\mathbb{N}}[0,n]\right) = \overset{-1}{|g|}(\mathbb{R}^+) = \text{Dom}(g) \doteqdot \sigma(R),$$

the second by the fact that $\overset{-1}{|g|}$ preserves the inclusion, the third by the inclusion $|g|(\delta_n) \subseteq [0,n]$. Hence our claim follows. By the third statement of (3.9), $\delta_n \in \mathcal{B}(\mathbb{C})$ and [3, Lemma 1.7(3)] we obtain

$$(\forall n \in \mathbb{N})(E(\delta_n)G \subseteq \text{Dom}(g(R))). \tag{3.11}$$

By [3, (2.25)] and (3.11) for all $n \in \mathbb{N}$

$$\int f_x(R)\, d\,\mu(x) E(\delta_n)G \subseteq E(\delta_n)G \subseteq \text{Dom}(g(R)).$$

Therefore,

$$(\forall n \in \mathbb{N})(\forall v \in G)\left(E(\delta_n)v \in \text{Dom}\left(g(R)\int f_x(R)\, d\,\mu(x)\right)\right).$$

Hence, by (3.10)

$$\mathbf{D} \doteqdot \text{Dom}\left(g(R)\int f_x(R)\, d\,\mu(x)\right) \text{ is dense in } G. \tag{3.12}$$

Now, $\int f_x(R)\, d\,\mu(x) \in B(G)$ and $g(R)$ is closed by [2, Theorem 18.2.11], so by [3, Lemma 1.15] we find that

$$g(R)\int f_x(R)\, d\,\mu(x) \text{ is closed.} \tag{3.13}$$

Next, (3.4) and (3.7) imply

$$g(R)\int f_x(R)\, d\,\mu(x) \in B(\mathbf{D}, G). \tag{3.14}$$

Now, (3.13), (3.14) and [3, Lemma 1.16] imply that $\mathbf{D}$ is closed in G, therefore by (3.12)

$$\mathbf{D} = G;$$

therefore the statement follows by (3.7). □

Definition 1. Let V be an open neighbourhood of $\sigma(R)$, $l \in \mathbb{R}_+^* \cup \{+\infty\}$ such that $]-l,l[\cdot V \subseteq V$, and $F : V \to \mathbb{C}$ be analytic. Moreover, let W be a set and $g : W \to \mathbb{R}$ such that $g(W) \subseteq]-l,l[$. Let $F_t : V \ni \lambda \mapsto F(t\lambda) \in \mathbb{C}$ with $t \in]-l,l[$. We define the following operator valued map originating by the Borel functional calculus of the operator R

$$\zeta_{F,g}^R : W \ni x \mapsto F_{g(x)}(R) \in \text{ClO}(G).$$

Corollary 3.1. *Let V be an open neighbourhood of $\sigma(T)$, $l \in \mathbb{R}_+^* \cup \{+\infty\}$ such that $]-l,l[\cdot V \subseteq V$, and $F : V \to \mathbb{C}$ be analytic. Moreover, let $n,p \in \mathbb{Z}_+^*$, W be an open set of $\mathbb{R}^n$, and $g \in \mathcal{C}^p(W,\mathbb{R})$ such that $g(W) \subseteq]-l,l[$. Thus, $\zeta_{F,g}^T \in \mathcal{C}^p(W, B(G))$, and for every $i \in [1,n] \cap \mathbb{Z}$ we have*

$$\frac{\partial \zeta_{F,g}^T}{\partial e_i} = \frac{\partial g}{\partial e_i} \cdot T\zeta_{\frac{dF}{d\lambda},g}^T.$$

Proof. $]-l,l[\ni t\mapsto F_t(T)\in B(G)$ is smooth by [3, Theorem 1.21], therefore, the first sentence of the statement follows since a composition of $\mathcal{C}^p$-maps is a $\mathcal{C}^p$-map, while the equality follows by the Chain Rule and by [3, Theorem 1.21]. □

Definition 2. Let $k\in\mathbb{Z}_+$, $\omega\in\mathrm{Alt}_c^k(M)$ and $(U,\phi:U\to W)$ be a chart of M. Define

$$\begin{cases}\omega^\phi : M(k,N,<)\to\mathcal{A}(W),\\ I\mapsto(\imath_U^M)^*(\omega)(\partial_{I_1}^\phi,\dots,\partial_{I_k}^\phi)\circ\phi^{-1}.\end{cases}$$

Moreover, let V be an open neighbourhood of $\sigma(R)$ such that $\mathbb{R}\cdot V\subseteq V$, $F:V\to\mathbb{C}$ be analytic and let $\delta\in\mathcal{B}(\mathbb{C})$ be such that[2]

$$\begin{aligned}&\mathrm{Range}(\zeta_{F,\omega_I^\phi}^{R_\delta\upharpoonright G_\delta})\subseteq B(G_\delta),\\ &\zeta_{F,\omega_I^\phi}^{R_\delta\upharpoonright G_\delta}\in\mathfrak{L}_c^1(W,\langle B(G_\delta),\sigma(B(G_\delta),\mathcal{N}_\delta)\rangle,\lambda).\end{aligned}\tag{3.15}$$

Define

$$\mathrm{f}_{\omega,\phi}^{\delta,F}:M(k,N,<)\ni I\mapsto\mathrm{f}_{\omega,\phi,I}^{\delta,F}\doteqdot\zeta_{F,\omega_I^\phi}^{R_\delta\upharpoonright G_\delta}\circ\phi,$$

and then define $[\omega,\phi,\delta,F]\in\mathrm{Alt}^k(U,M;\langle B(G_\delta),\sigma(B(G_\delta),\mathcal{N}_\delta)\rangle,\lambda)$ such that

$$[\omega,\phi,\delta,F]\doteqdot\sum_{I\in M(k,N,<)}\mathrm{f}_{\omega,\phi,I}^{\delta,F}\otimes\bigwedge_{s=1}^k dx_{I_s}^\phi.$$

Remark 1. Let $k\in\mathbb{Z}_+$, $\omega\in\mathrm{Alt}_c^k(M)$ and $(U,\phi:U\to W)$ be a chart of M. Let $\sigma\in\mathcal{B}(\mathbb{C})$ be *bounded*, thus, $R_\sigma\upharpoonright G_\sigma\in B(G_\sigma)$ by [3, Lemma 1.7]. Moreover, $\zeta_{F,\omega_I^\phi}^{R_\sigma\upharpoonright G_\sigma}\in\mathcal{A}_c(W,\langle B(G_\sigma),\|\cdot\|\rangle)$ by Corollary 3.1, so, $\mathrm{f}_{\omega,\phi,I}^{\sigma,F}\in\mathcal{A}_c(U,\langle B(G_\sigma),\|\cdot\|\rangle)$ and $[\omega,\phi,\sigma,F]$ is smooth with respect to the norm topology, namely $[\omega,\phi,\sigma,F]\in\mathrm{Alt}^k(U,M;\langle B(G_\delta),\|\cdot\|\rangle)$. Finally, as a result, $\zeta_{F,\omega_I^\phi}^{R_\sigma\upharpoonright G_\sigma}$ is norm continuous and compactly supported, therefore, $\zeta_{F,\omega_I^\phi}^{R_\sigma\upharpoonright G_\sigma}$ is Lebesgue integrable with respect to the norm topology and its integral belongs to $B(G_\sigma)$.

Remark 2. Let $\delta\in\mathcal{B}(\mathbb{C})$. The norm topology on $B(G_\delta)$ is stronger than the topology $\sigma(B(G_\delta),\mathcal{N}_\delta)$ since the latter is the weakest topology on $B(G_\delta)$ among those for which $\mathcal{N}_\delta$ is a set of continuous functionals, and since $\mathcal{N}_\delta\subseteq B(G_\delta)'$. Thus, we can and shall identify $\mathcal{A}(U,\langle B(G_\delta),\|\cdot\|\rangle)$ as a $\mathcal{A}(U)$-submodule of $\mathcal{A}(U,\langle B(G_\delta),\sigma(B(G_\delta),\mathcal{N}_\delta)\rangle)$ and $\mathrm{Alt}^k(U,M;\langle B(G_\delta),\|\cdot\|\rangle)$ as a $\mathcal{A}(U)$-submodule of $\mathrm{Alt}^k(U,M;\langle B(G_\delta),\sigma(B(G_\delta),\mathcal{N}_\delta)\rangle)$.

Remark 3. Let $\delta\in\mathcal{B}(\mathbb{C})$, then any map defined on X and with values in $B(G_\delta)$, that is scalarly essentially μ-integrable with respect to the norm topology, it is also scalarly essentially μ-integrable with respect to the $\sigma(B(G_\delta),\mathcal{N}_\delta)$-topology since $\mathcal{N}_\delta\subseteq B(G_\delta)'$.

Definition 3. Let $k\in\mathbb{Z}_+$, $\omega\in\mathrm{Alt}_c^k(M)$ and $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in D}$ be an atlas of M. Let V be an open neighbourhood of $\sigma(R)$ such that $\mathbb{R}\cdot V\subseteq V$, $F:V\to\mathbb{C}$ be analytic and $\delta\in\mathcal{B}(\mathbb{C})$ be such that (3.15) holds for $\phi=\phi_\alpha$ and for every $\alpha\in D$. Define $[\omega,\delta,F]\in\mathrm{Alt}^k(M;\langle B(G_\delta),\sigma(B(G_\delta),\mathcal{N}_\delta)\rangle,\lambda)$ such that for all $\alpha\in D$

$$(\imath_{U_\alpha}^M)^\times([\omega,\delta,F])=[\omega,\phi_\alpha,\delta,F].$$

[2]For instance, when δ is bounded, see Remark 1.

Definition 4. Let $k \in \mathbb{Z}_+^*$, $\omega \in \mathrm{Alt}^{k-1}(M)$ and $i \in [1,k] \cap \mathbb{Z}$. Define $\mathfrak{d}_i(\omega) \in \mathcal{A}(M)$ and $\mathfrak{n}_i(\omega) \in \mathrm{Alt}^k(M)$ such that for any given atlas $\mathcal{U}$ of M we have for every $(U,\phi) \in \mathcal{U}$

$$(\imath_U^M)^*(\mathfrak{d}_i(\omega)) \doteqdot \partial_i^\phi[(\imath_U^M)^*(\omega)(\partial_1^\phi,\ldots,\widehat{\partial_i^\phi},\ldots,\partial_k^\phi)],$$
$$(\imath_U^M)^*(\mathfrak{n}_i(\omega)) \doteqdot (\imath_U^M)^*(\omega)(\partial_1^\phi,\ldots,\widehat{\partial_i^\phi},\ldots,\partial_k^\phi)\bigwedge_{s=1}^{k} dx_s^\phi,$$

where $\widehat{z}$ stands for z missing.

The above two definitions are well-set by the usual gluing lemma for smooth forms, since the extension of the gluing lemma via charts at scalarly essentially integrable locally convex vector-valued maps [4, Remark 1.2], and since the extension of the gluing lemma via charts at smooth locally convex vector-valued maps [4, Notation], where the compatibility in both the definitions is ensured by the following simple fact

$$(\imath_{U_\alpha}^M)^*(\omega)(\partial_{I_1}^{\phi_\alpha},\ldots,\partial_{I_k}^{\phi_\alpha}) \circ \imath_{U_{\alpha,\beta}}^{U_\alpha} = (\imath_{U_{\alpha,\beta}}^M)^*(\omega)(\partial_{I_1}^{\phi_{\alpha,\beta}},\ldots,\partial_{I_k}^{\phi_{\alpha,\beta}}),$$

where $U_{\alpha,\beta} = U_\alpha \cap U_\beta$ and $\phi_{\alpha,\beta} = (\phi_\alpha \circ \imath_{U_{\alpha,\beta}}^{U_\alpha})_\natural$.

Theorem 3.3 (**Stokes equality for $\sigma(B(G),\mathcal{N})$-integrable forms functions of an unbounded operator**). *Let M be oriented with boundary and $\omega \in \mathrm{Alt}_c^{N-1}(M)$. Let V be an open neighbourhood of $\sigma(R)$ such that $\mathbb{R}\cdot V \subseteq V$ and $F : V \to \mathbb{C}$ is analytic. Assume that there exists a finite family $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in D}$ of oriented charts of M such that $\{U_\alpha\}_{\alpha\in D}$ is a covering of* $\mathrm{supp}(\omega)$ *and*

1. *$\widetilde{F}_t \in \mathfrak{L}_E^\infty(\sigma(R))$ for every $t \in \mathbb{R}$, and for all $\alpha \in D$ such that ϕ_α is a boundary chart, the map*

$$\zeta_{F,\omega_N^{\phi_\alpha}}^R \circ \imath_{\phi_\alpha(U_\alpha\cap\partial M)}^{\phi_\alpha(U_\alpha)} : \phi_\alpha(U_\alpha\cap\partial M) \to \langle B(G),\sigma(B(G),\mathcal{N})\rangle,$$

is scalarly essentially $(\lambda_{\phi_\alpha(U_\alpha\cap\partial M)}, B(G))$-integrable,

2. *$(\widetilde{\frac{dF}{d\lambda}})_t \in \mathfrak{L}_E^\infty(\sigma(R))$ for every $t \in \mathbb{R}$, and for all $\alpha \in D$ and $i \in [1,N]\cap\mathbb{Z}$, the map*

$$\zeta_{\frac{dF}{d\lambda},\omega_i^{\phi_\alpha}}^R : \phi_\alpha(U_\alpha) \to \langle B(G),\sigma(B(G),\mathcal{N})\rangle,$$

is scalarly essentially $(\lambda_{\phi_\alpha(U_\alpha)}, B(G))$-integrable, where

$$\omega_i^{\phi_\alpha} \doteqdot (\imath_{U_\alpha}^M)^*(\omega)(\partial_1^{\phi_\alpha},\ldots,\widehat{\partial_i^{\phi_\alpha}},\ldots,\partial_N^{\phi_\alpha}) \circ \phi_\alpha^{-1}.$$

Thus,

$$R\int\sum_{i=1}^{N}(-1)^{i-1}\mathfrak{d}_i(\omega)\cdot[\mathfrak{n}_i(\omega),\sigma(R),\frac{dF}{d\lambda}] = \int(\imath_{\partial M}^M)^\times([\omega,\sigma(R),F]),$$

where the integrals belong to $B(G)$ and are with respect to the $\sigma(B(G),\mathcal{N})$ topology. In the case $\partial M = \emptyset$ the integral in the right-hand side has to be understood equal to $\mathbf{0}$.

Remark 4. Let $\mathcal{U} = \{(U_\alpha,\phi_\alpha)\}_{\alpha\in D}$ and $\mathcal{U}^\partial$ be as in Notation. Thus $\mathcal{U}^\partial$ is a family of oriented charts of ∂M such that $\{Q_\alpha\}_{\alpha\in D}$, with $Q_\alpha = U_\alpha \cap \partial M$ for every $\alpha \in D$, is a collection of open sets of ∂M and a covering of $\mathrm{supp}(\omega)\cap\partial M$ compact set of ∂M. Next, set $D^\dagger = D \cup \{\dagger\}$, $U_\dagger = \complement_M \mathrm{supp}(\omega)$, $Q_\dagger = \complement_{\partial M}(\mathrm{supp}(\omega)\cap\partial M)$ and let $\{\psi_\alpha\}_{\alpha\in D^\dagger}$ be a smooth partition of unity of M subordinate to

$\{U_\alpha\}_{\alpha\in D^\dagger}$ and $\{k_\alpha\}_{\alpha\in D^\dagger}$ be a smooth partition of unity of ∂M subordinate to $\{Q_\alpha\}_{\alpha\in D^\dagger}$. Thus, by (3.31) and (3.30) applied to $\delta=\sigma(R)$ the statement of Theorem 3.3 reads as follows

$$R\int\sum_{\alpha\in D}\gamma_{\phi_\alpha}(\imath_{U_\alpha}^M\circ\phi_\alpha^{-1})^*(\psi_\alpha)\sum_{i=1}^N(-1)^{i-1}\frac{\partial\omega_i^{\phi_\alpha}}{\partial e_i}\zeta^R_{\frac{dF}{d\lambda},\omega_i^{\phi_\alpha}}\,d\lambda_{\phi_\alpha(U_\alpha)}$$
$$=\int\sum_{\alpha\in D}\gamma_{\phi_\alpha}\left(\imath^{\partial M}_{U_\alpha\cap\partial M}\circ(\phi_\alpha^{\partial M})^{-1}\right)^*(k_\alpha)\left(\zeta^R_{F,\omega_N^{\phi_\alpha}}\circ\imath^{\phi_\alpha(U_\alpha)}_{\phi_\alpha(U_\alpha\cap\partial M)}\circ(\mathbf{i}^{\mathbb{R}^N}_{\mathbb{R}^{N-1}}\circ\imath^{\mathbb{R}^{N-1}}_{\phi_\alpha^{\partial M}(U_\alpha\cap\partial M)})_\natural\right)d\lambda_{\phi_\alpha^{\partial M}(U_\alpha\cap\partial M)},$$

where $\mathbf{i}^{\mathbb{R}^N}_{\mathbb{R}^{N-1}}:\mathbb{R}^{N-1}\to\mathbb{R}^N$ is such that if $N>1$, then $\mathrm{Pr}_k^{\mathbb{R}^N}\circ\mathbf{i}^{\mathbb{R}^N}_{\mathbb{R}^{N-1}}=\mathrm{Pr}_k^{\mathbb{R}^{N-1}}$ if $k\in[1,N-1]\cap\mathbb{Z}$, and $\mathrm{Pr}_N^{\mathbb{R}^N}\circ\mathbf{i}^{\mathbb{R}^N}_{\mathbb{R}^{N-1}}=\mathbf{0}_{\mathbb{R}^{N-1}}$ the constant map on $\mathbb{R}^{N-1}$ equal to 0; while $\mathbf{i}^{\mathbb{R}^1}_{\mathbb{R}^0}:0\to 0$. Notice that $(\mathbf{i}^{\mathbb{R}^N}_{\mathbb{R}^{N-1}}\circ\imath^{\mathbb{R}^{N-1}}_{\phi_\alpha^{\partial M}(U_\alpha\cap\partial M)})_\natural$ is a diffeomorphism of $\phi_\alpha^{\partial M}(U_\alpha\cap\partial M)$ onto $\phi_\alpha(U_\alpha\cap\partial M)$ thus the right-hand side of the above equality is well-set by hypothesis (1) and the theorem on change of variable in multiple integrals.

Remark 5. The strategy employed to obtain Theorem 3.3 is as follows: Given an E-sequence of bounded sets $\{\sigma_n\}_{n\in\mathbb{N}}$ we apply for every $n\in\mathbb{N}$ the Stokes theorem for locally convex vector-valued forms [4, Theorem 2.54] to the $\langle B(G_{\sigma_n}),\sigma(B(G_{\sigma_n}),\mathcal{N}_{\sigma_n})\rangle$-valued form $[\omega,\sigma_n,F]$ which is smooth as a result of Remark 1. Then develop the terms of these equalities by employing the families of oriented charts $\mathcal{U}$ and $\mathcal{U}^\partial$, and the families of smooth maps $\{\psi_\alpha\}_{\alpha\in D}$ and $\{k_\alpha\}_{\alpha\in D}$. Finally, we apply the Extension Theorem 3.2 to the sequence of the resulting equalities.

Remark 6. Theorem 3.3 establishes a Stokes-type equality for $\langle B(G),\sigma(B(G),\mathcal{N})\rangle$-valued integrable forms: (1) that arise from the Borelian functional calculus of the possibly **unbounded** operator R; (2) that might be **not** smooth nor even continuously differentiable. To this regard we notice that the rigidity of analytic functions prevents any reasonable attempt to use the strong operator derivability on $\mathrm{Dom}(R)$ in [3, Theorem 1.23(2)] in order to prove regularity of these forms.

Proof of Theorem 3.3. We maintain the data and notation introduced in Remark 4, in addition we let (U,ϕ) be an oriented chart of M and $h\in\mathcal{A}(M)$ and $k\in\mathcal{A}(\partial M)$ be such that

$$\begin{cases}\mathrm{supp}(h)\subseteq U,\\ \mathrm{supp}(k)\subseteq U\cap\partial M.\end{cases}\tag{3.16}$$

Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be an E-sequence of bounded sets and $n\in\mathbb{N}$, let $\delta\in\{\sigma_n,\sigma(R)\}$, let R^δ denote $R_\delta\restriction G_\delta$ and let $\psi\in\mathcal{N}$. By Remark 1 and Remark 2 we have that $[\omega,\sigma_n,F]\in\mathrm{Alt}^{N-1}(M,\langle B(G_{\sigma_n}),\sigma(B(G_{\sigma_n})),\mathcal{N}_{\sigma_n}\rangle)$ so by [4, Theorem 2.42], (3.16), since the unique element of a smooth partition of the unity subordinated to the open covering $\{U\}$ of U equals 1 when evaluated on U, and finally by [4, Proposition 1.45], we have

$$\begin{aligned}\int\psi_\times(hd[\omega,\sigma_n,F])&=\int hd(\psi_\times[\omega,\sigma_n,F])\\&=\gamma_\phi\int(\imath_U^M\circ\phi^{-1})^*(h)(\imath_U^M\circ\phi^{-1})^\times(d\psi_\times[\omega,\sigma_n,F]).\end{aligned}\tag{3.17}$$

Next,

$$\begin{aligned}(\imath_U^M\circ\phi^{-1})^\times d\psi_\times[\omega,\sigma_n,F]&=(\phi^{-1})^\times(\imath_U^M)^\times d\psi_\times[\omega,\sigma_n,F]\\&=\psi_\times d(\phi^{-1})^\times(\imath_U^M)^\times[\omega,\sigma_n,F]\\&=\psi_\times d(\phi^{-1})^\times[\omega,\phi,\sigma_n,F],\end{aligned}\tag{3.18}$$

where the second equality follows by [4, Theorem 2.42], the third one by Definition 3 applied to any atlas containing (U,ϕ). Now, by definition

$$[\omega,\phi,\delta,F]=\sum_{i=1}^{N}\left(\zeta_{F,\omega_i^\phi}^{R^\delta}\circ\phi\right)\otimes(dx_1^\phi\wedge\ldots\widehat{dx_i^\phi}\wedge\ldots dx_N^\phi); \tag{3.19}$$

thus,

$$\begin{aligned} d(\phi^{-1})^\times([\omega,\phi,\sigma_n,F]))&=\sum_{i=1}^{N}(-1)^{i-1}\frac{\partial\zeta_{F,\omega_i^\phi}^{R^{\sigma_n}}}{\partial e_i}\otimes(dx_1^{\mathrm{Id}_{\phi(U)}}\wedge\ldots dx_i^{\mathrm{Id}_{\phi(U)}}\wedge\ldots dx_N^{\mathrm{Id}_{\phi(U)}})\\ &=\sum_{i=1}^{N}(-1)^{i-1}R^{\sigma_n}\frac{\partial\omega_i^\phi}{\partial e_i}\zeta_{\frac{dF}{d\lambda},\omega_i^\phi}^{R^{\sigma_n}}\otimes(dx_1^{\mathrm{Id}_{\phi(U)}}\wedge\ldots dx_i^{\mathrm{Id}_{\phi(U)}}\wedge\ldots dx_N^{\mathrm{Id}_{\phi(U)}}), \end{aligned} \tag{3.20}$$

where the second equality follows by Corollary 3.1. Next, R^{σ_n} is norm continuous, thus by the end of Remark 1 we have that

$$\int(\imath_U^M\circ\phi^{-1})^*(h)R^{\sigma_n}\frac{\partial\omega_i^\phi}{\partial e_i}\zeta_{\frac{dF}{d\lambda},\omega_i^\phi}^{R^{\sigma_n}}d\lambda_{\phi(U)}=R^{\sigma_n}\int(\imath_U^M\circ\phi^{-1})^*(h)\frac{\partial\omega_i^\phi}{\partial e_i}\zeta_{\frac{dF}{d\lambda},\omega_i^\phi}^{R^{\sigma_n}}d\lambda_{\phi(U)}\in B(G_{\sigma_n}), \tag{3.21}$$

the integrals being with respect to the norm topology on $B(G_{\sigma_n})$, hence, also with respect to the $\langle B(G_{\sigma_n}),\sigma(B(G_{\sigma_n}),\mathcal{N}_{\sigma_n})\rangle$ topology by Remark 3. Now, $(3.17),(3.18),(3.20)$ and (3.21) yield

$$\begin{aligned} \int hd[\omega,\sigma_n,F]&=R^{\sigma_n}\gamma_\phi\int(\imath_U^M\circ\phi^{-1})^*(h)\sum_{i=1}^{N}(-1)^{i-1}\frac{\partial\omega_i^\phi}{\partial e_i}\zeta_{\frac{dF}{d\lambda},\omega_i^\phi}^{R^{\sigma_n}}d\lambda_{\phi(U)}\\ &=R^{\sigma_n}\int h\sum_{i=1}^{N}(-1)^{i-1}\mathfrak{d}_i(\omega)\cdot[\mathfrak{n}_i(\omega),\sigma_n,\frac{dF}{d\lambda}], \end{aligned} \tag{3.22}$$

the integrals are with respect to the $\langle B(G_{\sigma_n}),\sigma(B(G_{\sigma_n}),\mathcal{N}_{\sigma_n})\rangle$ topology, where the second equality follows by the following equality obtained by direct calculation

$$\int h\sum_{i=1}^{N}(-1)^{i-1}\mathfrak{d}_i(\omega)\cdot[\mathfrak{n}_i(\omega),\delta,\frac{dF}{d\lambda}]=\gamma_\phi\int(\imath_U^M\circ\phi^{-1})^*(h)\sum_{i=1}^{N}(-1)^{i-1}\frac{\partial\omega_i^\phi}{\partial e_i}\zeta_{\frac{dF}{d\lambda},\omega_i^\phi}^{R^\delta}d\lambda_{\phi(U)}. \tag{3.23}$$

Now, by (3.22) applied to $(U,\phi)=(U_\alpha,\phi_\alpha)$ and $h=\psi_\alpha$ for every $\alpha\in D$ and since [4, Corollary2.53] we obtain

$$\begin{aligned} \int d[\omega,\sigma_n,F]&=R^{\sigma_n}\int\sum_{i=1}^{N}(-1)^{i-1}\mathfrak{d}_i(\omega)\cdot[\mathfrak{n}_i(\omega),\sigma_n,\frac{dF}{d\lambda}]\\ &=R^{\sigma_n}\int\sum_{\alpha\in D}\gamma_{\phi_\alpha}(\imath_{U_\alpha}^M\circ\phi_\alpha^{-1})^*(\psi_\alpha)\sum_{i=1}^{N}(-1)^{i-1}\frac{\partial\omega_i^{\phi_\alpha}}{\partial e_i}\zeta_{\frac{dF}{d\lambda},\omega_i^{\phi_\alpha}}^{R^{\sigma_n}}d\lambda_{\phi_\alpha(U_\alpha)}, \end{aligned} \tag{3.24}$$

where all the three integrals are with respect tothe $\langle B(G_{\sigma_n}),\sigma(B(G_{\sigma_n}),\mathcal{N}_{\sigma_n})\rangle$ topology. Now, if $\partial M=\emptyset$ the statement follows by the above equality, [4, Theorem 2.54] and by our Extension Theorem 3.2. Thus, in what follows assume in addition that $\partial M\neq\emptyset$ and that (U,ϕ) is a boundary chart, therefore $(U,\phi^{\partial M})$ is a chart of ∂M such that $\gamma_{\phi^{\partial M}}=\gamma_\phi$. Next, since the unique element of a smooth partition of the unity subordinated to the open covering $\{U\cap\partial M\}$, with respect to the topological space ∂M, of $U\cap\partial M$ equals 1 when evaluated on $U\cap\partial M$, we have by [4, Theorem 2.42], (3.16), [4, Theorem 1.45] and $\gamma_{\phi^{\partial M}}=\gamma_\phi$

$$\begin{aligned}\int \psi_\times\left(k(\imath_{\partial M}^{M})^\times[\omega,\delta,F]\right) &= \int k\psi_\times(\imath_{\partial M}^{M})^\times[\omega,\delta,F]\\ &= \gamma_\phi\int\left((\imath_{U\cap\partial M}^{\partial M}\circ(\phi^{\partial M})^{-1})^\ast k\right)(\imath_{U\cap\partial M}^{\partial M}\circ(\phi^{\partial M})^{-1})^\times\psi_\times(\imath_{\partial M}^{M})^\times[\omega,\delta,F]\\ &= \gamma_\phi\int\left((\imath_{U\cap\partial M}^{\partial M}\circ(\phi^{\partial M})^{-1})^\ast k\right)\psi_\times((\phi^{\partial M})^{-1})^\times(\imath_{U\cap\partial M}^{\partial M})^\times(\imath_{\partial M}^{M})^\times[\omega,\delta,F]\\ &= \gamma_\phi\int\left((\imath_{U\cap\partial M}^{\partial M}\circ(\phi^{\partial M})^{-1})^\ast k\right)\psi_\times((\phi^{\partial M})^{-1})^\times(\imath_{U\cap\partial M}^{M})^\times[\omega,\delta,F]\\ &= \gamma_\phi\int\left((\imath_{U\cap\partial M}^{\partial M}\circ(\phi^{\partial M})^{-1})^\ast k\right)\psi_\times((\phi^{\partial M})^{-1})^\times(\imath_{U\cap\partial M}^{U})^\times(\imath_{U}^{M})^\times[\omega,\delta,F].\end{aligned}\tag{3.25}$$

Next, by (3.19) and since $(\imath_{U\cap\partial M}^{U})^\ast(dx_N^\phi)=\mathbf{0}$, we obtain

$$\begin{aligned}(\imath_{U\cap\partial M}^{U})^\times(\imath_{U}^{M})^\times[\omega,\delta,F] &= (\imath_{U\cap\partial M}^{U})^\times[\omega,\phi,\delta,F]\\ &= (\zeta_{F,\omega_N^\phi}^{R^\delta}\circ\phi\circ\imath_{U\cap\partial M}^{U})\otimes\bigwedge_{s=1}^{N-1}(\imath_{U\cap\partial M}^{U})^\ast(dx_s^\phi)\\ &= (\zeta_{F,\omega_N^\phi}^{R^\delta}\circ\phi\circ\imath_{U\cap\partial M}^{U})\otimes\bigwedge_{s=1}^{N-1}dx_s^{\phi^{\partial M}};\end{aligned}$$

and by letting $Z\doteqdot\phi^{\partial M}(U\cap\partial M)$, we get

$$((\phi^{\partial M})^{-1})^\times(\imath_{U\cap\partial M}^{U})^\times(\imath_{U}^{M})^\times[\omega,\delta,F]=(\zeta_{F,\omega_N^\phi}^{R^\delta}\circ\phi\circ\imath_{U\cap\partial M}^{U}\circ(\phi^{\partial M})^{-1})\otimes\bigwedge_{s=1}^{N-1}dx_s^{\mathrm{Id}_Z}.\tag{3.26}$$

Next,

$$\mathfrak{i}_{\mathbb{R}^{N-1}}^{\mathbb{R}^N}(Z)=\phi(U\cap\partial M)\subseteq\partial\mathbb{H}^N$$

by the definition of boundary chart of M. Define $\mathrm{P}_{[N]}\doteqdot\mathfrak{i}_{\mathbb{R}^{N-1}}^{\mathbb{R}^N}\circ\mathrm{P}^{[N]}$, thus by letting $\Pr(\mathbb{R}^N)$ be the set of projectors of $\mathbb{R}^N$, we have

$$\begin{cases}\mathrm{P}_{[N]}\in\Pr(\mathbb{R}^N),\\ \partial\mathbb{H}^N=\mathrm{P}_{[N]}(\mathbb{R}^N);\end{cases}\tag{3.27}$$

moreover, by the definition of $\phi^{\partial M}$ we have

$$(\forall x\in Z)\left(\mathrm{P}_{[N]}\left((\phi\circ\imath_{U\cap\partial M}^{U}\circ(\phi^{\partial M})^{-1})(x)\right)=\mathfrak{i}_{\mathbb{R}^{N-1}}^{\mathbb{R}^N}(x)\right).\tag{3.28}$$

Now, $(\phi\circ\imath_{U\cap\partial M}^{U}\circ(\phi^{\partial M})^{-1})(x)\in\partial\mathbb{H}^N$ since ϕ is a boundary chart of M, therefore, by (3.28) and (3.27) we obtain

$$\phi\circ\imath_{U\cap\partial M}^{U}\circ(\phi^{\partial M})^{-1}=\imath_{\phi(U\cap\partial M)}^{\phi(U)}\circ(\mathfrak{i}_{\mathbb{R}^{N-1}}^{\mathbb{R}^N}\circ\imath_{Z}^{\mathbb{R}^{N-1}})_\natural.$$

Therefore, by (3.26)

$$((\phi^{\partial M})^{-1})^\times(\imath_{U\cap\partial M}^{U})^\times(\imath_{U}^{M})^\times[\omega,\delta,F]=\left(\zeta_{F,\omega_N^\phi}^{R^\delta}\circ\imath_{\phi(U\cap\partial M)}^{\phi(U)}\circ(\mathfrak{i}_{\mathbb{R}^{N-1}}^{\mathbb{R}^N}\circ\imath_{Z}^{\mathbb{R}^{N-1}})_\natural\right)\otimes\bigwedge_{s=1}^{N-1}dx_s^{\mathrm{Id}_Z};$$

hence, by (3.25) we obtain

$$\int k(\imath_{\partial M}^{M})^{\times}[\omega,\delta,F]=\gamma_{\phi}\int((\imath_{U\cap\partial M}^{\partial M}\circ(\phi^{\partial M})^{-1})^{*}k)\left(\zeta_{F,\omega_{N}^{\phi}}^{R^{\delta}}\circ\imath_{\phi(U\cap\partial M)}^{\phi(U)}\circ(\mathfrak{i}_{\mathbb{R}^{N-1}}^{\mathbb{R}^{N}}\circ\imath_{Z}^{\mathbb{R}^{N-1}})_{\natural}\right)\,d\lambda_{Z},\quad(3.29)$$

where the integrals are with respect to the $\langle B(G_{\delta}),\sigma(B(G_{\delta}),\mathcal{N}_{\delta})\rangle$ topology. Now, by (3.29) applied to $(U,\phi)=(U_{\alpha},\phi_{\alpha})$ and $k=k_{\alpha}$ for every $\alpha\in D$ and since [4, Corollary 2.53] we obtain by letting $Z_{\alpha}\doteqdot\phi_{\alpha}^{\partial M}(U_{\alpha}\cap\partial M)$

$$\int(\imath_{\partial M}^{M})^{\times}[\omega,\delta,F]=\int\sum_{\alpha\in D}\gamma_{\phi_{\alpha}}((\imath_{U_{\alpha}\cap\partial M}^{\partial M}\circ(\phi_{\alpha}^{\partial M})^{-1})^{*}k_{\alpha})\left(\zeta_{F,\omega_{N}^{\phi_{\alpha}}}^{R^{\delta}}\circ\imath_{\phi_{\alpha}(U_{\alpha}\cap\partial M)}^{\phi_{\alpha}(U_{\alpha})}\circ(\mathfrak{i}_{\mathbb{R}^{N-1}}^{\mathbb{R}^{N}}\circ\imath_{Z_{\alpha}}^{\mathbb{R}^{N-1}})_{\natural}\right)\,d\lambda_{Z_{\alpha}}.\quad(3.30)$$

Next, by (3.23) applied to $(U,\phi)=(U_{\alpha},\phi_{\alpha})$ and $h=\psi_{\alpha}$ for every $\alpha\in D$ and since [4, Corollary 2.53] we obtain

$$\int\sum_{i=1}^{N}(-1)^{i-1}\mathfrak{d}_{i}(\omega)\cdot[\mathfrak{n}_{i}(\omega),\delta,\frac{dF}{d\lambda}]=\int\sum_{\alpha\in D}\gamma_{\phi_{\alpha}}(\imath_{U_{\alpha}}^{M}\circ\phi_{\alpha}^{-1})^{*}(\psi_{\alpha})\sum_{i=1}^{N}(-1)^{i-1}\frac{\partial\omega_{i}^{\phi_{\alpha}}}{\partial e_{i}}\zeta_{\frac{dF}{d\lambda},\omega_{i}^{\phi_{\alpha}}}^{R^{\delta}}\,d\lambda_{\phi_{\alpha}(U_{\alpha})}.\quad(3.31)$$

Now, by [4, Theorem 2.54] applied to the form $[\omega,\sigma_{n},F]$, by (3.24) and by (3.30) applied to $\delta=\sigma_{n}$ we obtain

$$\begin{aligned}R^{\sigma_{n}}\int\sum_{\alpha\in D}\gamma_{\phi_{\alpha}}(\imath_{U_{\alpha}}^{M}\circ\phi_{\alpha}^{-1})^{*}(\psi_{\alpha})\sum_{i=1}^{N}(-1)^{i-1}\frac{\partial\omega_{i}^{\phi_{\alpha}}}{\partial e_{i}}\zeta_{\frac{dF}{d\lambda},\omega_{i}^{\phi_{\alpha}}}^{R^{\sigma_{n}}}\,d\lambda_{\phi_{\alpha}(U_{\alpha})}\\=\int\sum_{\alpha\in D}\gamma_{\phi_{\alpha}}((\imath_{U_{\alpha}\cap\partial M}^{\partial M}\circ(\phi_{\alpha}^{\partial M})^{-1})^{*}k_{\alpha})\left(\zeta_{F,\omega_{N}^{\phi_{\alpha}}}^{R^{\sigma_{n}}}\circ\imath_{\phi_{\alpha}(U_{\alpha}\cap\partial M)}^{\phi_{\alpha}(U_{\alpha})}\circ(\mathfrak{i}_{\mathbb{R}^{N-1}}^{\mathbb{R}^{N}}\circ\imath_{Z_{\alpha}}^{\mathbb{R}^{N-1}})_{\natural}\right)\,d\lambda_{Z_{\alpha}}.\end{aligned}$$

Now, we can employ our Extension Theorem 3.2 to the above sequence of equalities to obtain

$$\begin{aligned}&R\int\sum_{\alpha\in D}\gamma_{\phi_{\alpha}}(\imath_{U_{\alpha}}^{M}\circ\phi_{\alpha}^{-1})^{*}(\psi_{\alpha})\sum_{i=1}^{N}(-1)^{i-1}\frac{\partial\omega_{i}^{\phi_{\alpha}}}{\partial e_{i}}\zeta_{\frac{dF}{d\lambda},\omega_{i}^{\phi_{\alpha}}}^{R}\,d\lambda_{\phi_{\alpha}(U_{\alpha})}\\&=\int\sum_{\alpha\in D}\gamma_{\phi_{\alpha}}\left(\imath_{U_{\alpha}\cap\partial M}^{\partial M}\circ(\phi_{\alpha}^{\partial M})^{-1}\right)^{*}(k_{\alpha})\left(\zeta_{F,\omega_{N}^{\phi_{\alpha}}}^{R}\circ\imath_{\phi_{\alpha}(U_{\alpha}\cap\partial M)}^{\phi_{\alpha}(U_{\alpha})}\circ(\mathfrak{i}_{\mathbb{R}^{N-1}}^{\mathbb{R}^{N}}\circ\imath_{\phi_{\alpha}^{\partial M}(U_{\alpha}\cap\partial M)}^{\mathbb{R}^{N-1}})_{\natural}\right)\,d\lambda_{\phi_{\alpha}^{\partial M}(U_{\alpha}\cap\partial M)},\end{aligned}$$

and the statement follows by (3.31) and (3.30) applied to $\delta=\sigma(R)$. □

Corollary 3.2. *Let M be oriented with boundary and $\omega\in\mathrm{Alt}_{c}^{N-1}(M)$. Let V be an open neighbourhood of $\sigma(R)$ such that $\mathbb{R}\cdot V\subseteq V$ and $F:V\to\mathbb{C}$ be analytic. Assume that there exists a finite collection $\mathcal{U}=\{(U_{\alpha},\phi_{\alpha})\}_{\alpha\in D}$ of oriented charts of M such that $\{U_{\alpha}\}_{\alpha\in D}$ is a covering of the support of ω and*

1. *$\widetilde{F}_{t}\in\mathfrak{L}_{E}^{\infty}(\sigma(R))$ for every $t\in\mathbb{R}$, and for all $\alpha\in D$ such that ϕ_{α} is a boundary chart, the map*

$$\psi\circ\zeta_{F,\omega_{N}^{\phi_{\alpha}}}^{R}\circ\imath_{\phi_{\alpha}(U_{\alpha}\cap\partial M)}^{\phi_{\alpha}(U_{\alpha})}$$

is $\lambda_{\phi_{\alpha}(U_{\alpha}\cap\partial M)}$-measurable for every $\psi\in\mathcal{N}$; and

$$\int^{*}\|\cdot\|_{\infty}^{E}\circ\widetilde{F}_{\omega_{N}^{\phi_{\alpha}}\circ\imath_{\phi_{\alpha}(U_{\alpha}\cap\partial M)}^{\phi_{\alpha}(U_{\alpha})}}\,d\lambda_{\phi_{\alpha}(U_{\alpha}\cap\partial M)}<\infty;$$

2. $(\widetilde{\frac{dF}{d\lambda}})_t \in \mathfrak{L}_E^{\infty}(\sigma(R))$ *for every* $t \in \mathbb{R}$, *and for all* $\alpha \in D$ *and* $i \in [1, N] \cap \mathbb{Z}$, *the map*

$$\psi \circ \zeta^R_{\frac{dF}{d\lambda}, \omega_i^{\phi_\alpha}}$$

is $\lambda_{\phi_\alpha(U_\alpha)}$*-measurable for every* $\psi \in \mathcal{N}$; *and*

$$\int^* \| \cdot \|_\infty^E \circ \left(\widetilde{\frac{dF}{d\lambda}}\right)_{\omega_i^{\phi_\alpha}} d\lambda_{\phi_\alpha(U_\alpha)} < \infty.$$

Then the statement of Theorem 3.3 *holds true. Moreover, if in addition* $\mathcal{N}$ *is an* E*-appropriate set with the isometric duality property and* $C \doteqdot \sup_{\sigma \in \mathcal{B}(\mathbb{C})} \|E(\sigma)\|$, *then we obtain the following estimates*

$$\left\| \int \zeta^R_{F, \omega_N^{\phi_\alpha}} \circ \imath^{\phi_\alpha(U_\alpha)}_{\phi_\alpha(U_\alpha \cap \partial M)} \, d\lambda_{\phi_\alpha(U_\alpha \cap \partial M)} \right\| \leq 4C \int^* \| \cdot \|_\infty^E \circ \widetilde{F}_{\omega_N^{\phi_\alpha} \circ \imath^{\phi_\alpha(U_\alpha)}_{\phi_\alpha(U_\alpha \cap \partial M)}} \, d\lambda_{\phi_\alpha(U_\alpha \cap \partial M)},$$

and

$$\left\| \int \zeta^R_{\frac{dF}{d\lambda}, \omega_i^{\phi_\alpha}} \, d\lambda_{\phi_\alpha(U_\alpha)} \right\| \leq 4C \int^* \| \cdot \|_\infty^E \circ \left(\widetilde{\frac{dF}{d\lambda}}\right)_{\omega_i^{\phi_\alpha}} d\lambda_{\phi_\alpha(U_\alpha)}.$$

Remark 7. By letting $\mathrm{Bor}(\mathbb{C})$ be the set of complex valued Borelian maps on $\mathbb{C}$, we have

$$\begin{cases} \widetilde{F}_{\omega_N^{\phi_\alpha} \circ \imath^{\phi_\alpha(U_\alpha)}_{\phi_\alpha(U_\alpha \cap \partial M)}} : \phi_\alpha(U_\alpha \cap \partial M) \to \mathrm{Bor}(\mathbb{C}) \\ x \mapsto \widetilde{F}_{(\omega_N^{\phi_\alpha} \circ \imath^{\phi_\alpha(U_\alpha)}_{\phi_\alpha(U_\alpha \cap \partial M)})(x)}, \end{cases}$$

and

$$\begin{cases} \left(\widetilde{\frac{dF}{d\lambda}}\right)_{\omega_i^{\phi_\alpha}} : \phi_\alpha(U_\alpha) \to \mathrm{Bor}(\mathbb{C}) \\ x \mapsto \left(\widetilde{\frac{dF}{d\lambda}}\right)_{\omega_i^{\phi_\alpha}(x)}; \end{cases}$$

where we recall that $L_t : \lambda \mapsto L(t\lambda)$ for any $L \in \mathrm{Bor}(\mathbb{C})$ and any $t \in \mathbb{R}$. Therefore, the integrals in hypotheses (1) and (2) are well-set.

Proof. By [3, (1.42)], hypotheses, and [2, Theorem 18.2.11(c)] we obtain that

$$\| \cdot \| \circ \zeta^R_{F, \omega_N^{\phi_\alpha}} \circ \imath^{\phi_\alpha(U_\alpha)}_{\phi_\alpha(U_\alpha \cap \partial M)} \in \mathfrak{F}_1(\phi_\alpha(U_\alpha \cap \partial M), \lambda_{\phi_\alpha(U_\alpha \cap \partial M)}),$$

and

$$\| \cdot \| \circ \zeta^R_{\frac{dF}{d\lambda}, \omega_i^{\phi_\alpha}} \in \mathfrak{F}_1(\phi_\alpha(U_\alpha), \lambda_{\phi_\alpha(U_\alpha)}).$$

Then the hypotheses of Theorem 3.3 are satisfied by [3, footnote 1, page 39] and by [3, Theorem 2.2] and the first part of the statement follows. The estimates in the statement follow by the estimate in [3, Theorem 2.2] and by [2, Theorem 18.2.11(c)]. □

Corollary 3.3. *The statement of Corollary* 3.2 *holds if* G *is a complex Hilbert space and* $\mathcal{N}$ *is replaced by* $\mathcal{N}_{pd}(G)$.

Proof. Follows by the end of [3, Remark 2.12] and by Corollary 3.2. □

Corollary 3.4. *The statement of Corollary* 3.2 *holds if* G *is reflexive and* $\mathcal{N}$ *is replaced by* $\mathcal{N}_{st}(G)$.

Proof. By employing [3, Corollary 2.6] instead of [3, Theorem 2.2] the proof runs exactly as the one in Corollary 3.2. □

References

[1] N. Bourbaki, *Integration 1,2*, Springer, 2003.

[2] N. Dunford, J.T. Schwartz, *Linear operators, Part I, II, III* Interscience Publ.

[3] B. Silvestri, *Integral equalities for functions of unbounded spectral operators in Banach spaces* Dissertationes Math., 464 (2009), 60 pp.

[4] B. Silvestri, *Scalarly essentially integrable locally convex vector-valued tensor fields. Stokes theorem*, preprint in `https://arxiv.org/abs/2010.02327`

Benedetto Silvestri
Dipartimento di Matematica Pura ed Applicata
Universita' degli Studi di Padova
Via Trieste, 63
35121 Padova, Italy
E-mail: 6260rstlvs@gmail.com

Received: 09.01.2021