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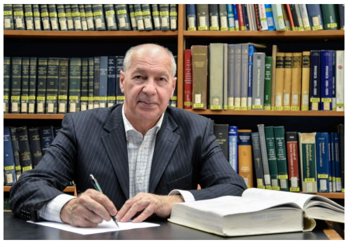
VICTOR IVANOVICH BURENKOV

(to the 80th birthday)



On July 15, 2021 was the 80th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), professor of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (Moscow), chairman of the Dissertation Council at the RUDN University, research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), honorary academician of the National Academy of Sciences of the Republic of Kazakhstan, doctor of physical and mathematical sciences(1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK,2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University. Through 2015-2017 he was head of the Department of Mathematical Analysis and Theory of Functions (RUDN University). He was one of the organisers and the first director of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (2016-2017).

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem on composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 190 scientific papers. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces. In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

V.I. Burenkov is still very active in research. Through 2016-2021 he published 20 papers in leading mathematical journals.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 80th birthday and wishes him good health and new achievements in science and teaching!

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η-INVARIANT AND INDEX FOR OPERATORS ON THE REAL LINE PERIODIC AT INFINITY

A.Yu. Savin, K.N. Zhuikov

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Key words: elliptic operator, operator with periodic coefficients, η-invariant, index.

AMS Mathematics Subject Classification: 58J20, 58J28, 58J40.

Abstract. We define η-invariants for periodic pseudodifferential operators on the real line and establish their main properties. In particular, it is proved that the η-invariant satisfies logarithmic property and a formula for the derivative of the η-invariant of an operator family with respect to the parameter is obtained. Furthermore, we establish an index formula for elliptic pseudodifferential operators on the real line periodic at infinity. The contribution of infinity to the index formula is given by the constructed η-invariant. Finally, we compute η-invariants of differential operators in terms of the spectrum of their monodromy matrices.

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1 Introduction

The present paper is devoted to the index problem for elliptic pseudodifferential operators on the real line with coefficients periodic at infinity (for details, see Section 2).

Elliptic equations with periodic coefficients on noncompact manifolds arise in many problems. In particular, they play an important role in quantum mechanics and solid state physics (see, for example, the survey [13] and the references cited therein), as well as in geometry and topology (see, for example, [3, 19, 23]). In those problems, as a rule, the kernel (and/or cokernel) of the operator is either trivial or infinite-dimensional. Therefore, the Fredholm index for such operators is not interesting and its modifications are studied (e.g., L^2-index in the von Neumann sense or the index with values in the K-theory of C^*-algebras, etc.).

At the same time, a number of geometrical problems (for example, the problem on smooth structures in $\mathbb{R}^4$ [25], the problem of studying the moduli spaces of Yamabe metrics [16], etc.) and analysis (see [5, 9, 11]) lead to the study of operators with coefficients, which are periodic at infinity (in contrast to the situation considered above, where the periodicity condition is satisfied everywhere). In the literature, this theory is referred to as elliptic theory on manifolds with periodic ends. Fredholm solvability of such operators is investigated and the index formulas are established.

It should be noted that manifolds with cylindrical ends are an important special case of manifolds with periodic ends. In this case, a noncompact manifold is obtained from a compact manifold with boundary by gluing an infinite cylinder $\Omega \times [0, \infty)$, where the base Ω of the cylinder is the boundary of the original manifold. Studying the Fredholm property in this particular case goes back to Kondrat'ev [12]. In the cited paper, the ellipticity condition that ensures the Fredholm property of the problem in Sobolev spaces was stated in terms of invertibility of the following two objects: the symbol as a function on the cotangent bundle of the manifold and a certain family of operators on Ω. The index theorem for Dirac-type operators on manifolds with cylindrical ends was established by Atiyah, Patodi and Singer in [2]. The formula they found contains a contribution of infinity described by the so-called *η-invariant* of Atiyah–Patodi–Singer. The latter is a ζ-type regularized signature of a quadratic form associated with a certain self-adjoint operator on the base of the cylinder. For general elliptic operators on manifolds with cylindrical ends, some index formulas were obtained in [22, 10].

In elliptic theory on manifolds with periodic ends, the Fredholmness criteria for operators in Sobolev spaces were obtained (see [21, 25]). In this case, the operator is Fredholm if its interior symbol is invertible and a certain family of operators on the product of the base of the cylinder by the circle is invertible. The index problem was also studied. Namely, the index formula for Dirac-type operators on manifolds with periodic ends was recently obtained in [20]. The authors found a generalization of the Atiyah–Patodi–Singer η-invariant, in which terms the index formula was given.

It should be noted that the index problem for elliptic operators of general form on manifolds with periodic ends remains open even in the one-dimensional case. The problem was studied in the one-dimensional case, i.e. for pseudodifferential operators on the real line. In particular, the K-group of the C^*-algebra of symbols of pseudodifferential operators was calculated in [17], the index formulas for some examples were given in [9, 7, 8, 11]. However, an index formula for general operators was not presented. The purpose of this work is to fill in this gap and present an index formula for operators on the real line. Our approach uses the η-invariants in the considered situation. We express the index in terms of the η-invariants and contributions of Atiyah–Singer type. To define the η-invariant, we use Melrose's approach [18]. According to this approach, the η-invariant is defined as a special regularization of the winding number for families of parameter-dependent pseudodifferential operators. Equivalently (after the Fourier transform), Melrose's approach gives the η-invariant for invertible pseudodifferential operators on a cylinder that are invariant with respect to shifts of the cylinder along its generatrix. Melrose investigated the basic properties of the η-invariant, in particular, he established the logarithmic property $\eta(D_1D_2) = \eta(D_1)+\eta(D_2)$ and showed that the Atiyah–Patodi–Singer η-invariant coincides with the η-invariant of a certain parameter-dependent family (see the cited paper and also [15, 14]).

The work has the following structure. In Section 2, we recall the definition of periodic pseudodifferential operators. Then in Section 3 we define the regularized traces in t- and p-spaces and study their properties. Next, we define the η-invariant and establish its main properties in Section 4. As an example, we calculate the η-invariant of the first-order scalar differential operators in Section 5. The main result of the work is given in Section 6. Namely, the index formula is established. Our index formula includes three terms: η-invariants of the limit operators at plus and minus infinity and a regularization of the standard Atiyah–Singer integral of the interior symbol of the operator. Finally, in Section 7, the invertibility of differential operators with periodic coefficients is expressed in terms of the spectrum of their monodromy matrices and an index formula is obtained. Moreover, as a corollary of the previous results, a formula for the η-invariant of differential operators with periodic coefficients is given in terms of the spectrum of the corresponding monodromy matrix.

Note that Melrose's definition of the η-invariant easily extends to the case of operators with periodic coefficients on manifolds of arbitrary dimensions. However, this approach gives η-invariant, which does not have the logarithmic property if the dimension of the manifold is ≥ 2. This is due to the fact that the regularized integral included in Melrose's definition is not translation invariant. For this reason, in this paper we have limited ourselves to the one-dimensional case.

2 Periodic pseudodifferential operators

Throughout what follows, most structures are defined on the real line $\mathbb{R}$, so we will mostly omit the symbol $\mathbb{R}$ in their notations.

Recall that *the space $S^m_{\rm cl}$ of classical symbols of order $\leq m$, $m \in \mathbb{Z}$, on* $\mathbb{R}$ is the set of smooth functions $A(p) \in C^\infty(\mathbb{R})$ satisfying the estimates

$$|\partial_p^\alpha A(p)| \le C_\alpha (1+|p|)^{(m-\alpha)} \quad \forall p \in \mathbb{R},\ \alpha \ge 0,$$

where $\partial_p = \partial/\partial p$ and C_α is some constant. In addition, it is assumed that there is an asymptotic expansion

$$A(p) \sim \sum_{j\le m} A_j(p), \quad \text{where} \quad A_j(p) \in C^\infty(\mathbb{R}) \text{ and } A_j(\lambda p) = \lambda^j A_j(p)\ \forall \lambda \ge 1,\ |p| \ge 1.$$

The linear space $S_{\mathrm{cl}} = \bigcup_m S^m_{\mathrm{cl}}$ is called *the space of classical symbols on the real line.* The space S_{cl} is an algebra with respect to multiplication.

The space Ψ^m *of pseudodifferential operators* (ψ*DO*) *of order* $\le m$ *with constant coefficients* is the space of operators

$$A(-i\partial_t) \stackrel{\text{def}}{=} \mathcal{F}^{-1} A(p) \mathcal{F} \colon \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \qquad A(p) \in S^m_{\mathrm{cl}},$$

where $\mathcal{F}$ is the Fourier transform, $\mathcal{S}(\mathbb{R})$ is the Schwartz space on $\mathbb{R}$. The space $\Psi = \bigcup_m \Psi^m$ is called *the algebra of pseudodifferential operators with constant coefficients.* The spaces S^m_{cl} and Ψ^m are Fréchet spaces (see [24, Section 27]).

Definition 1. *The space of periodic* ψ*DOs of order* $\le m$ is the space of operators

$$\Psi^m_{\mathrm{per}} = \left\{ D = \sum_{k\in\mathbb{Z}} D_k(-i\partial_t) e^{ikt} \colon \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) \right\}. \tag{2.1}$$

It is assumed here that the elements $D_k(-i\partial_t) \in \Psi^m$ rapidly tend to 0 in the following sense: given $k \in \mathbb{Z}$ and $N \ge 1$, we have the estimate

$$\|D_k(-i\partial_t)\|_\ell \le C_{\ell N}(1+|k|)^{-N},$$

where $\|\cdot\|_\ell$ is an arbitrary semi-norm on the Fréchet space Ψ^m. The space $\Psi_{\mathrm{per}} = \bigcup_m \Psi^m_{\mathrm{per}}$ is called *the algebra of periodic* ψ*DOs.*

Lemma 2.1. *The representation of operator* $D \in \Psi^m_{\mathrm{per}}$ *as a series in* (2.1) *is unique.*

Proof. We will give the proof in the case $m = 0$ (the general case is reduced to this one by the order reduction). After applying the Fourier transform $\mathcal{F}$, the operator D in (2.1) will take the form of an operator with shifts

$$\widetilde{D} = \mathcal{F} D \mathcal{F}^{-1} = \sum_{k\in\mathbb{Z}} D_k(p) T^k, \quad \text{where} \quad (Tu)(p) = u(p-1). \tag{2.2}$$

The closure of the set of operators of form (2.2) with respect to the operator norm in $L^2(\mathbb{R})$ corresponds to the C^*-dynamical system $(A, \mathbb{Z}, \tau)$. Here $A \subset C_b(\mathbb{R})$ is the C^*-algebra of continuous functions on $\mathbb{R}$ having limits as $p \to +\infty$ and $p \to -\infty$ (this algebra is isomorphic to the C^*-algebra $C[0,1]$ and is the closure of S^0_{cl}), and $\tau \colon \mathbb{Z} \to \mathrm{Aut}(A)$ is the group action by automorphisms

$$\big(\tau(k) f\big)(p) = f(p-k).$$

Since the action of τ on the spectrum $\widehat{A} \simeq [-\infty, +\infty] \simeq [0,1]$ is topologically free (see [1, Definition 12.13]), by the isomorphism theorem [1, Corollary 12.17], representation (2.2) of operator $\widetilde{D}$ as a sum is unique. Hence, the inverse Fourier transform gives us the desired unique decomposition of D in (2.1). $\square$

Definition 2. *The principal symbol* of periodic ψDOs is the mapping

$$\begin{array}{rcl} \sigma=(\sigma_+,\sigma_-)\colon \Psi^m_{\text{per}} & \longrightarrow & C^\infty(\mathbb{S}^1)\oplus C^\infty(\mathbb{S}^1),\\ D=\sum\limits_{k\in\mathbb{Z}} D_k(-i\partial_t)e^{ikt} & \longmapsto & \sigma_\pm(D)(\varphi)=\sum\limits_{k\in\mathbb{Z}}\sigma_\pm(D_k(-i\partial_t))e^{ik\varphi}, \qquad \varphi\in[0,2\pi], \end{array}$$

where $\sigma_\pm\big(D_k(-i\partial_t)\big)=\lim\limits_{p\to\pm\infty}|p|^{-m}D_k(p)$.

It is easy to check that for all $A,B\in\Psi_{\text{per}}$ the *composition formula*

$$\sigma_\pm(AB)=\sigma_\pm(A)\sigma_\pm(B) \tag{2.3}$$

holds.

3 Regularized traces

Define the averaging operation

$$\begin{array}{rcl} \operatorname{Av}\colon\quad \Psi_{\text{per}} & \longrightarrow & \Psi,\\ D & \longmapsto & \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} T_\varphi D T_{-\varphi}d\varphi, \end{array} \qquad \text{where} \quad T_\varphi u(t)=u(t-\varphi).$$

If $K_D(t,t')$ is the Schwartz kernel of D, then $\operatorname{Av} D$ has the Schwartz kernel

$$K_{\operatorname{Av}D}(t,t')=\frac{1}{2\pi}\int_0^{2\pi}K_D(t+\varphi,t'+\varphi)d\varphi.$$

Also, we obviously have

$$\operatorname{Av}\left(\sum_{k\in\mathbb{Z}}D_k(-i\partial_t)e^{ikt}\right)=D_0(-i\partial_t). \tag{3.1}$$

Recall (see [18, Definition 2]) that *the regularized integral* of a function $f(p)\in S_{\text{cl}}$ is the value of the constant term in the asymptotic expansion of its integral over the segment $[-P,P]$ as $P\to+\infty$:

$$⨍_{\mathbb{R}} f(p)=a_0,\quad \text{where}\quad \int_{-P}^{P}f(p)dp\sim\sum_{k\le N}a_kP^k+b_0\ln P, \tag{3.2}$$

$N>0$ and $a_k,b_k\in\mathbb{C}$.

Lemma 3.1. *For the functional*

$$\begin{array}{rcl} \alpha\colon\quad \Psi^{-1} & \longrightarrow & \mathbb{C},\\ D(-i\partial_t) & \longmapsto & ⨍_{\mathbb{R}} D(p)dp, \end{array} \quad \text{where} \quad D(p)=\mathcal{F}D(-i\partial_t)\mathcal{F}^{-1},$$

the following equality holds:

$$\alpha\big(D(-i\partial_t)\big)=\sqrt{2\pi}\lim_{t\to0}\left[\frac{K_D(t,0)+K_D(-t,0)}{2}-c_1(\ln|t|+\gamma)\right]. \tag{3.3}$$

Here, $K_D(t,t')$ is the Schwartz kernel of operator $D(-i\partial_t)$, $c_1=\lim\limits_{t\to0}(K_D(t,0)/\ln|t|)$ and γ is the Euler constant.

Proof. For convenience, we denote $\widetilde{f}(p) = D(p)$. The inverse Fourier transform of this function equals $f(t) = \mathcal{F}^{-1}(\widetilde{f}(p)) = \sqrt{2\pi}K_D(t,0)$. Since $D(-i\partial_t) \in \Psi^{-1}$, there is an expansion

$$\widetilde{f}(p) = \widetilde{f}_0(p) + C_1\chi_+(p)p^{-1} + C_2\chi_-(p)p^{-1}, \tag{3.4}$$

where $\widetilde{f}_0(p) = O\big((1+|p|)^{-2}\big)$ and

$$\chi_+(p) = \begin{cases} 0, & p < 1, \\ 1, & p \geq 1, \end{cases} \qquad \chi_-(p) = \begin{cases} 1, & p \leq -1, \\ 0, & p > -1. \end{cases}$$

A straightforward computation gives the following expression for the left-hand side in (3.3):

$$⨍_{\mathbb{R}} \widetilde{f}(p)dp = \int_{\mathbb{R}} \widetilde{f}_0(p)dp + C_1 \cdot 0 + C_2 \cdot 0 = \sqrt{2\pi} f_0(0). \tag{3.5}$$

Let us now calculate the right-hand side in (3.3) for the function

$$f(t) = f_0(t) + \mathcal{F}^{-1}\left(C_1\chi_+(p)p^{-1} + C_2\chi_-(p)p^{-1}\right). \tag{3.6}$$

By linearity, we substitute the first and the second terms in (3.6) to (3.3) separately. For the first term, we have

$$\lim_{t\to 0}\left[\frac{f_0(t)+f_0(-t)}{2} - c_1(\ln|t|+\gamma)\right] = \lim_{t\to 0}\frac{f_0(t)+f_0(-t)}{2} = f_0(0) \tag{3.7}$$

since $f_0(t)$ is continuous at $t=0$ and therefore $c_1 = \lim_{t\to 0}(f_0(t)/\ln|t|) = 0$. We write the second term in (3.6) as

$$\begin{aligned}
g(t) =& \mathcal{F}^{-1}\left(C_1\chi_+(p)p^{-1} + C_2\chi_-(p)p^{-1}\right) = \frac{1}{\sqrt{2\pi}}\left[C_1\int_1^\infty e^{ipt}\frac{dp}{p} + C_2\int_{-\infty}^{-1} e^{ipt}\frac{dp}{p}\right]\\
=& \frac{1}{\sqrt{2\pi}}\left[C_1'\int_1^\infty \cos pt\frac{dp}{p} + C_2'\int_1^\infty \sin pt\frac{dp}{p}\right]\\
=& \frac{1}{\sqrt{2\pi}}\left[C_1'\int_{|t|}^\infty \cos p\frac{dp}{p} + C_2'\operatorname{sgn} t\int_{|t|}^\infty \sin p\frac{dp}{p}\right] = \frac{1}{\sqrt{2\pi}}\left[-C_1'\operatorname{Ci}(|t|) - C_2'\operatorname{sgn} t\operatorname{si}(|t|)\right]\\
=& \frac{1}{\sqrt{2\pi}}\left[-C_1'(\gamma+\ln|t| - \operatorname{Cin}(|t|)) - C_2'\operatorname{sgn} t\left(\operatorname{Si}(|t|) - \frac{\pi}{2}\right)\right]\\
=& \frac{1}{\sqrt{2\pi}}\left[-C_1'\ln|t| + \left(-C_1'\gamma + C_2'\operatorname{sgn} t\frac{\pi}{2}\right) + O(t)\right],
\end{aligned}$$

where $C_1' = C_1 - C_2$ and $C_2' = i(C_1 + C_2)$. Here we use the special functions (see [4]): the integral cosine

$$\operatorname{Ci}(t) = -\int_x^\infty \frac{\cos p}{p}dp, \quad \operatorname{Cin}(t) = \int_0^x \frac{1-\cos p}{p}dp, \quad \text{wherein } \operatorname{Ci}(t) + \operatorname{Cin}(t) = \gamma + \ln t,$$

and the integral sine

$$\operatorname{si}(t) = -\int_x^\infty \frac{\sin p}{p}dp, \quad \operatorname{Si}(t) = \int_0^x \frac{\sin p}{p}dp, \quad \text{wherein } \operatorname{si}(t) = \operatorname{Si}(t) - \pi/2.$$

Hence, for $g(t)$ we obtain

$$c_1 = \lim_{t\to 0}\frac{g(t)}{\ln|t|} = -\frac{C_1'}{\sqrt{2\pi}}, \tag{3.8}$$

$$\frac{g(t)+g(-t)}{2} - c_1(\ln|t|+\gamma) = -C_1'\ln|t| - C_1'\gamma + C_1'(\ln|t|+\gamma) + O(t) = O(t). \tag{3.9}$$

Thus, it follows from (3.7) and (3.9) that the right-hand side in (3.3) for function (3.4) equals $\sqrt{2\pi}f_0(0)$ and coincides with the left-hand side (see (3.5)). This completes the proof of Lemma 3.1. □

Proposition 3.1. *The functional* $\operatorname{Tr} \overset{\text{def}}{=} \alpha \circ \operatorname{Av}\colon \Psi^{-1}_{\text{per}} \to \mathbb{C}$ *is a trace. More precisely, for all* $A, B \in \Psi_{\text{per}}$ *such that* $\operatorname{ord} A + \operatorname{ord} B \leq -1$*, the following equality holds:*

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA).$$

Proof. By (3.1), it suffices to prove the desired equality for operators $A_k(-i\partial_t)e^{ikt}$ and $B_{-k}(-i\partial_t)e^{-ikt}$, $k \in \mathbb{Z}$. A straightforward computation gives

$$e^{ikt}B_{-k}(-i\partial_t) = B_{-k}(-i\partial_t - k)e^{ikt}.$$

Hence, we have

$$\operatorname{Tr}\left(A_k(-i\partial_t)e^{ikt}B_{-k}(-i\partial_t)e^{-ikt}\right) = \operatorname{Tr}(A_k(-i\partial_t)B_{-k}(-i\partial_t - k)) = ⨍_{\mathbb{R}} A_k(p)B_{-k}(p-k)dp. \tag{3.10}$$

Similarly we obtain

$$\operatorname{Tr}\left(B_{-k}(-i\partial_t)e^{-ikt}A_k(-i\partial_t)e^{ikt}\right) = ⨍_{\mathbb{R}} B_{-k}(p)A_k(p+k)dp = ⨍_{\mathbb{R}} A_k(p+k)B_{-k}(p)dp. \tag{3.11}$$

The integrands in (3.10) and (3.11) are elements of S^{-1}_{cl} and differ from each other by a shift by k.

Lemma 3.2. *Regularized integral* (3.2) *is translation invariant, i.e. given* $\widetilde{f} \in S^{-1}_{\text{cl}}$*, we have* $⨍_{\mathbb{R}} \widetilde{f}(p+k)dp = ⨍_{\mathbb{R}} \widetilde{f}(p)dp$ *for all* $k \in \mathbb{R}$*.*

Proof. The regularized integral of function (3.4) shifted by k equals

$$\begin{aligned}
⨍_{\mathbb{R}} \widetilde{f}(p+k)dp &= ⨍_{\mathbb{R}} \left(\widetilde{f}_0(p+k) + C_1\frac{1}{p+k}\chi_+(p+k) + C_2\frac{1}{p+k}\chi_-(p+k)\right)dp \\
&= \int_{\mathbb{R}} \widetilde{f}_0(p+k)dp + ⨍_{\mathbb{R}} \left(C_1\frac{1}{p}\chi_+(p) + C_1\left(\frac{1}{p+k}\chi_+(p+k) - \frac{1}{p}\chi_+(p)\right)\right. \\
&\quad \left. + C_2\frac{1}{p}\chi_-(p) + C_2\left(\frac{1}{p+k}\chi_-(p+k) - \frac{1}{p}\chi_-(p)\right)\right)dp \\
&= \int_{\mathbb{R}} \widetilde{f}_0(p)dp + C_1\int_{\mathbb{R}}\left[\frac{1}{p+k}\chi_+(p+k) - \frac{1}{p}\chi_+(p)\right]dp \\
&\quad + C_2\int_{\mathbb{R}}\left[\frac{1}{p+k}\chi_-(p+k) - \frac{1}{p}\chi_-(p)\right]dp.
\end{aligned}$$

Hence

$$\begin{aligned}
&⨍_{\mathbb{R}} \widetilde{f}(p+k)dp - ⨍_{\mathbb{R}} \widetilde{f}(p)dp \\
=&C_1\int_{\mathbb{R}}\left[\frac{1}{p+k}\chi_+(p+k) - \frac{1}{p}\chi_+(p)\right]dp + C_2\int_{\mathbb{R}}\left[\frac{1}{p+k}\chi_-(p+k) - \frac{1}{p}\chi_-(p)\right]dp.
\end{aligned} \tag{3.12}$$

Evaluating the first integral in (3.12) for $k > 0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\left[\frac{1}{p+k}\chi_+(p+k) - \frac{1}{p}\chi_+(p)\right]dp = \int_{-k+1}^{1}\frac{dp}{p+k} + \int_1^\infty\left(\frac{1}{p+k} - \frac{1}{p}\right)dp \\
&= \ln(p+k)\Big|_{-k+1}^{1} + \left(\ln(p+k) - \ln(p)\right)\Big|_1^\infty = \ln(1+k) - \ln(1+k) = 0.
\end{aligned}$$

For $k<0$, (3.12) gives us

$$\int_{\mathbb{R}}\left[\frac{1}{p+k}\chi_+(p+k)-\frac{1}{p}\chi_+(p)\right]dp=-\int_1^{-k+1}\frac{dp}{p}+\int_{-k+1}^{\infty}\left(\frac{1}{p+k}-\frac{1}{p}\right)dp$$
$$=-\ln(p)\Big|_1^{-k+1}+\big(\ln(p+k)-\ln(p)\big)\Big|_{-k+1}^{\infty}=-\ln(-k+1)+\ln(-k+1)=0.$$

Similar calculations for the second integral in (3.12) also give 0. □

Now the desired equality of the expressions in (3.10) and (3.11) follows by Lemma 3.2. □

Definition 3. *The formal trace* on the algebra Ψ^0_{per} is the functional

$$\begin{aligned}\widetilde{\operatorname{Tr}}\colon\quad \Psi^0_{\text{per}} &\longrightarrow \mathbb{C},\\ D &\longmapsto -i\operatorname{Tr}[t,D],\end{aligned}$$

where $[t,D]=tD-Dt$ is the commutator and $\operatorname{Tr}\overset{\text{def}}{=}\alpha\circ\operatorname{Av}$.

Lemma 3.3. *The formal trace is a trace on* Ψ^0_{per}*, i.e. we have* $\widetilde{\operatorname{Tr}}(AB)=\widetilde{\operatorname{Tr}}(BA)$ *for all* $A,B\in\Psi^0_{\text{per}}$*. It can be computed as follows:*

$$\widetilde{\operatorname{Tr}}\,D=\frac{1}{2\pi}\int_0^{2\pi}\big(\sigma_+(D)(\varphi)-\sigma_-(D)(\varphi)\big)d\varphi,\qquad D\in\Psi^0_{\text{per}}. \tag{3.13}$$

Proof. The cyclic property is checked directly. Let us prove equality (3.13). Let $D\in\Psi^0_{\text{per}}$. On the one hand, we have

$$\widetilde{\operatorname{Tr}}\,D=-i\operatorname{Tr}[t,D]=⨍_{\mathbb{R}}[\partial_p,\widetilde{D}]_0dp=⨍_{\mathbb{R}}\partial_p(\widetilde{D}_0)dp=D_0(+\infty)-D_0(-\infty),\ \widetilde{D}=\mathcal{F}D\mathcal{F}^{-1}, \tag{3.14}$$

where D_0 is a term from the representation $D=\sum D_kT^k$. The limits in (3.14) exist since $D_k(p)\in S^0_{\text{cl}}$. On the other hand, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\big(\sigma_+(D)(\varphi)-\sigma_-(D)(\varphi)\big)d\varphi$$
$$=\sigma_+\big(D_0(-i\partial_t)\big)-\sigma_-\big(D_0(-i\partial_t)\big)=D_0(+\infty)-D_0(-\infty). \tag{3.15}$$

From (3.14) and (3.15), we obtain the desired equality (3.13). □

Table 1 summarizes most useful operations in t- and p-spaces. The last row of the table is defined in the next section.

Remark 1. All the results in this section can be generalized to operators in the algebra $\Psi_{\text{per}}\otimes\operatorname{Mat}_N$ of matrix $N\times N$ ψDOs. In particular, the regularized trace $\operatorname{Tr}\colon\Psi^{-1}_{\text{per}}\otimes\operatorname{Mat}_N\to\mathbb{C}$ is defined as $\operatorname{Tr}=\operatorname{tr}\circ\alpha\circ\operatorname{Av}$, where tr is the trace for matrices.

4 η-invariant

Definition 4. Let $D\in\Psi^m_{\text{per}}\otimes\operatorname{Mat}_N$ be an invertible matrix operator. We assume that $D^{-1}\in\Psi^{-m}_{\text{per}}\otimes\operatorname{Mat}_N$. Then the number

$$\eta(D)\overset{\text{def}}{=}-\frac{1}{2\pi}\operatorname{Tr}(D^{-1}[t,D]) \tag{4.1}$$

is called *the η-invariant* of D.

	t-**space**	p-**space**
Operator	$D=\sum_{k\in\mathbb{Z}} D_k\left(-i\partial_t\right)e^{ikt}$	$\widetilde{D}=\sum_{k\in\mathbb{Z}} D_k(p)T^k,$ $Tu(p)=u(p-1)$
Differentiation	$D\mapsto -i[t,D]$	$\widetilde{D}\mapsto[\partial_p,\widetilde{D}]$
Averaging	$D\mapsto \operatorname{Av} D$	$\widetilde{D}\mapsto D_0(p)$
Regularized integral	$f(t)\mapsto\sqrt{2\pi}\lim_{t\to 0}\left[\frac{f(t)+f(-t)}{2}-\right.$ $\left.c_1(\ln\lvert t\rvert+\gamma)\right]$, where $c_1=\lim_{t\to 0}(f(t)/\ln\lvert t\rvert)$	$⨍_{\mathbb{R}} f(p)dp=c_0,$ where $\int_{-P}^{P} f(p)dp\sim\sum_{j\le 0}c_jP^j+$ $d_0\ln P$ as $P\to+\infty$
Formal trace	$\widetilde{\operatorname{Tr}}\, D=-i\alpha\left(\operatorname{Av}[t,D]\right)=$ $\frac{1}{2\pi}\int_0^{2\pi}(\sigma_+(D)-\sigma_-(D))d\varphi$	$\widetilde{\operatorname{Tr}}\, D(p)=⨍_{\mathbb{R}}\partial_p D_0(p)dp=$ $D_0(+\infty)-D_0(-\infty)$
η-invariant	$\eta(D)=-\frac{1}{2\pi}\alpha\left(\operatorname{Av}(D^{-1}[t,D])\right)$	$\eta(D)=\frac{1}{2\pi i}⨍_{\mathbb{R}}(D^{-1}[\partial_p,D])_0dp$

Table 1: Transition between t- and p-spaces

Proposition 4.1. *The η-invariant satisfies the logarithmic property*

$$\eta(AB)=\eta(A)+\eta(B)$$

for all invertible $A,B\in\Psi_{\mathrm{per}}\otimes\mathrm{Mat}_N$.

Proof. We have

$$\begin{aligned}-2\pi\eta(AB)&=\operatorname{Tr}\big((AB)^{-1}[t,AB]\big)=\operatorname{Tr}\big(B^{-1}A^{-1}([t,A]B+A[t,B])\big)\\&=\operatorname{Tr}\big(BB^{-1}A^{-1}[t,A]\big)+\operatorname{Tr}\big(B^{-1}A^{-1}A[t,B]\big)\\&=\operatorname{Tr}\big(A^{-1}[t,A]\big)+\operatorname{Tr}\big(B^{-1}[t,B]\big)=-2\pi\big(\eta(A)+\eta(B)\big).\end{aligned}$$

Here the third equality follows by linearity and the cyclic property of Tr (see Proposition 3.1). Note that the conditions in Proposition 3.1 are satisfied since $\operatorname{ord}\big(B^{-1}A^{-1}[t,A]B\big)\le -1$. □

Proposition 4.2. *Let $D_\varepsilon\in\Psi^m_{\mathrm{per}}\otimes\mathrm{Mat}_N$, $\varepsilon\in[0,1]$ be a smooth homotopy of invertible operators. Then the derivative of the η-invariant of D_ε with respect to ε is equal to*

$$\partial_\varepsilon\eta(D_\varepsilon)=\frac{1}{4\pi^2 i}\int_0^{2\pi}\operatorname{tr}\left[\sigma_+^{-1}(D_\varepsilon)\partial_\varepsilon\sigma_+(D_\varepsilon)-\sigma_-^{-1}(D_\varepsilon)\partial_\varepsilon\sigma_-(D_\varepsilon)\right]d\varphi. \tag{4.2}$$

Proof. We have

$$\begin{aligned}-2\pi\partial_\varepsilon\eta(D_\varepsilon)&=\operatorname{Tr}\left(\partial_\varepsilon(D_\varepsilon^{-1}[t,D_\varepsilon])\right)=\operatorname{Tr}\left(-D_\varepsilon^{-1}(\partial_\varepsilon D_\varepsilon)D_\varepsilon^{-1}[t,D_\varepsilon]+D_\varepsilon^{-1}[t,\partial_\varepsilon D_\varepsilon]\right)\\&=\operatorname{Tr}\left(-D_\varepsilon^{-1}[t,D_\varepsilon]D_\varepsilon^{-1}(\partial_\varepsilon D_\varepsilon)+D_\varepsilon^{-1}[t,\partial_\varepsilon D_\varepsilon]\right)\\&=\operatorname{Tr}\left([t,D_\varepsilon^{-1}]\partial_\varepsilon D_\varepsilon+D_\varepsilon^{-1}[t,\partial_\varepsilon D_\varepsilon]\right)=\operatorname{Tr}\left([t,D_\varepsilon^{-1}\partial_\varepsilon D_\varepsilon]\right)=i\,\widetilde{\operatorname{Tr}}(D_\varepsilon^{-1}\partial_\varepsilon D_\varepsilon)\\&=\frac{i}{2\pi}\int_0^{2\pi}\operatorname{tr}\left(\sigma_+^{-1}(D_\varepsilon)\partial_\varepsilon\sigma_+(D_\varepsilon)-\sigma_-^{-1}(D_\varepsilon)\partial_\varepsilon\sigma_-(D_\varepsilon)\right)d\varphi.\end{aligned}$$

Here, the second equality follows by the Leibniz rule, the third by the cyclic property of Tr (see Proposition 3.1), the fourth by the Leibniz rule for the commutator, the fifth by the commutator definition, the sixth by the definition of $\widetilde{\mathrm{Tr}}$ and the last one follows by Lemma 3.3. □

5 Example. η-invariant of first-order operators

We consider the operator

$$D = -i\partial_t + a(t)\colon H^s(\mathbb{R}) \longrightarrow H^{s-1}(\mathbb{R}), \tag{5.1}$$

where $a(t)$ is a smooth periodic complex-valued function with period 2π and $H^s(\mathbb{R})$ is the Sobolev space with the smoothness exponent s. Denote $\mathbb{S}^1_z = \{z \in \mathbb{C}\colon |z| = 1\}$.

Proposition 5.1.

1. *Operator* (5.1) *is invertible if and only if*

$$\operatorname{Im} \int_0^{2\pi} a(t)dt \neq 0.$$

2. *The η-invariant of invertible operator* (5.1) *equals*

$$\eta(D) = -\frac{1}{2} \operatorname{sgn} \operatorname{Im} \int_0^{2\pi} a(t)dt.$$

Proof. 1. Let us obtain invertibility conditions for operator (5.1). By $\mathcal{F}_z\colon L^2(\mathbb{R}) \to L^2(\mathbb{S}^1_t \times \mathbb{S}^1_z)$ we denote *the Fourier-Laplace transform*(see [25, 20])

$$(\mathcal{F}_z u)(t, z) = z^{t/2\pi} \sum_{n\in\mathbb{Z}} z^n u(t + 2\pi n) \tag{5.2}$$

of a compactly supported function u for a fixed branch of the complex logarithm $\ln z$. Note that many variants of this transform exist in the literature (for historical notes see [13, p. 359]). The inverse Fourier-Laplace transform is given by the formula

$$u(t) = \frac{1}{2\pi i} \oint_{|z|=1} z^{-t/2\pi} (\mathcal{F}_z u)(t, z) \frac{dz}{z}.$$

By [21, Theorem 2] (see also [25, Lemma 4.3]) it follows that the operator $D\colon H^s(\mathbb{R}) \to H^{s-1}(\mathbb{R})$ is invertible if and only if the operator

$$D_z = \mathcal{F}_z D \mathcal{F}_z^{-1}\colon H^s(\mathbb{S}^1_t) \to H^{s-1}(\mathbb{S}^1_t)$$

is invertible for all $z = e^{i\theta}$, $\theta \in [0, 2\pi]$. We have

$$\begin{aligned} D\mathcal{F}_z u(t) &= (-i\partial_t + a(t))\Big(z^{t/2\pi} \sum_n z^n u(t + 2\pi n)\Big) = -\frac{i}{2\pi} z^{t/2\pi} \ln z \sum_n z^n u(t + 2\pi n) \\ &- iz^{t/2\pi} \sum_n z^n u'(t + 2\pi n) + a(t)\Big(z^{t/2\pi} \sum_n z^n u(t + 2\pi n)\Big) \\ = &-\frac{i}{2\pi} \ln z \mathcal{F}_z u(t) + z^{t/2\pi} \sum_n z^n \big(-i\partial_t + a(t)\big) u(t + 2\pi n) = -\frac{i}{2\pi} \ln z \mathcal{F}_z u(t) + \mathcal{F}_z D u(t). \end{aligned}$$

Hence, $\mathcal{F}_z D = \big(D + \frac{i}{2\pi}\ln z\big)\mathcal{F}_z$, and therefore

$$D_z = D + \frac{i}{2\pi}\ln z = -i\partial_t + a(t) - \frac{\theta}{2\pi}. \tag{5.3}$$

The latter operator has index zero, thence, its invertibility is equivalent to triviality of its kernel. Let us calculate the kernel of D_z. The equation $D_z u = 0$ has the solution

$$u(t) = C\exp\left(-i\int_0^t \left(a(t') - \frac{\theta}{2\pi}\right)dt'\right),$$

where C is some constant. The solution is required to satisfy the periodicity conditions

$$u(0) = C, \quad u(2\pi) = C\exp\left(-i\int_0^{2\pi}\left(a(t') - \frac{\theta}{2\pi}\right)dt'\right) = C.$$

Hence, the kernel of D_z is trivial if and only if

$$\int_0^{2\pi}\left(a(t') - \frac{\theta}{2\pi}\right)dt' \neq 2\pi k \iff \int_0^{2\pi} a(t')dt' \neq 2\pi k + \theta, \qquad \forall k \in \mathbb{Z}.$$

Therefore, all operators D_z, $z = e^{i\theta}$, are invertible if and only if

$$\operatorname{Im}\int_0^{2\pi} a(t)dt \neq 0.$$

2. Let us calculate the η-invariant of invertible operator (5.1). First, we describe operator D^{-1}. A straightforward computation gives us

Lemma 5.1. *Let f be a compactly supported function. The equation $Du = f$ has the solution*

$$u(t) = \begin{cases} i\displaystyle\int_{-\infty}^{t}\exp\left(-i\int_{t'}^{t} a(t'')dt''\right)f(t')dt' & \text{if} \quad \operatorname{Im}\displaystyle\int_0^{2\pi} a(t)dt < 0. \\ -i\displaystyle\int_{t}^{+\infty}\exp\left(i\int_{t}^{t'} a(t'')dt''\right)f(t')dt' & \text{if} \quad \operatorname{Im}\displaystyle\int_0^{2\pi} a(t)dt > 0. \end{cases}$$

By Lemma 5.1, for $\operatorname{Im}\int_0^\infty a(t)dt < 0$, the Schwartz kernel of D^{-1} is equal to

$$K_{D^{-1}}(t,t') = i\exp\left(-i\int_{t'}^{t} a(t'')dt''\right)\chi(t-t'), \quad \text{where} \quad \chi(t-t') = \begin{cases} 1 & \text{if} \quad t-t' \geq 0, \\ 0 & \text{if} \quad t-t' < 0. \end{cases}$$

Let $B = D^{-1}[t,D]$. Since $[t,D] = i\,\mathrm{Id}$, we have $K_B = iK_{D^{-1}}$. The Schwartz kernel of the averaged operator $\operatorname{Av} B$ equals

$$K_{\operatorname{Av} B}(t,t') = \frac{1}{2\pi}\int_0^{2\pi}\exp\left(-i\int_{t'+\delta}^{t+\delta} a(t'')dt''\right)\chi(t-t')d\delta. \tag{5.4}$$

In what follows, to calculate the η-invariant, we use formula (3.3). Since the Schwartz kernel $K_{\operatorname{Av} B}$ (see (5.4)) is bounded, the constant c_1 in (3.3) equals 0 and we obtain

$$\eta(D) = \frac{1}{2\pi}\lim_{h\to 0+}\int_0^{2\pi}\frac{1}{2}\left(\exp\left(-i\int_\delta^{h+\delta} a(t'')dt''\right)\cdot 1 + \exp\left(-i\int_\delta^{-h+\delta} a(t'')dt''\right)\cdot 0\right)d\delta = \frac{1}{2}.$$

For $\operatorname{Im}\int_0^{2\pi} a(t)dt > 0$, we similarly obtain

$$\eta(D) = -\frac{1}{2}.$$

□

6 Index of pseudodifferential operators

ψDOs on the real line periodic at infinity. We assume that the following objects are given:

- operators with periodic coefficients $D_+, D_- \in \Psi^n_{\mathrm{per}} \otimes \mathrm{Mat}_N$;
- a classical symbol $D(t,p) \in C^\infty(\mathbb{R}, S^n_{\mathrm{cl}} \otimes \mathrm{Mat}_N)$;
- the following compatibility condition for the principal symbols of operators $D_\pm$ and the principal symbol $D_n(t,p)$: there exists a number $T > 0$ large enough such that

$$D_n(t,\pm 1) = \sigma_\pm(D_+)(t) \text{ as } t \geq T \quad \text{and} \quad D_n(t,\pm 1) = \sigma_\pm(D_-)(t) \text{ as } t \leq -T; \tag{6.1}$$

- a partition of unity $1 = \chi_-^2 + \chi_0^2 + \chi_+^2$ subordinate to the cover

$$\mathbb{R} = (-\infty, -T+\varepsilon) \cup (-T-\varepsilon, T+\varepsilon) \cup (T-\varepsilon, +\infty).$$

With this data, we define the operator

$$D = \chi_- D_- \chi_- + \chi_0 D_0 \chi_0 + \chi_+ D_+ \chi_+ \colon H^s(\mathbb{R}, \mathbb{C}^N) \longrightarrow H^{s-n}(\mathbb{R}, \mathbb{C}^N), \tag{6.2}$$

where D_0 is a ψDO on the real line with the symbol $D(t,p)$:

$$(D_0 u)(t) = \frac{1}{\sqrt{2\pi}} \int e^{itp} D(t,p) \widetilde{u}(p) dp, \quad \widetilde{u} = \mathcal{F}u.$$

Operator (6.2) is called *a ψDO periodic at infinity.* Note that operator (6.2) does not depend on a partition of unity up to summands of order -1 with compactly supported Schwarz kernels.

Theorem 6.1. *Operator* (6.2) *is Fredholm if the following conditions hold:*

1) *the principal symbol $D_n(t,p)$ is invertible for all $t \in \mathbb{R}$ and $p \neq 0$;*

2) *operators $D_\pm \colon H^s(\mathbb{R}, \mathbb{C}^N) \to H^{s-n}(\mathbb{R}, \mathbb{C}^N)$ are invertible.*

Proof. Let us construct an almost inverse operator for D:

$$R = \chi_- R_- \chi_- + \chi_0 R_0 \chi_0 + \chi_+ R_+ \chi_+ \colon H^{s-n}(\mathbb{R}, \mathbb{C}^N) \longrightarrow H^s(\mathbb{R}, \mathbb{C}^N). \tag{6.3}$$

Here $R_\pm = D_\pm^{-1} \in \Psi^{-n}_{\mathrm{per}} \otimes \mathrm{Mat}_N$ and R_0 is a ψDO on the real line with the principal symbol $D_n(t,p)^{-1}$. Direct calculation shows that the differences

$$DR - \mathrm{Id} \qquad \text{and} \qquad RD - \mathrm{Id}$$

are operators of order -1 with compactly supported Schwartz kernels. Such operators are compact in $H^s(\mathbb{R}, \mathbb{C}^N)$. Hence, D is a Fredholm operator by Atkinson's theorem. $\square$

A straightforward computation gives us

Lemma 6.1. *Let $f(t)$ be a continuous function on the real line, which is periodic with period 2π for $|t| > T_0$. Then for $T > T_0$ we have*

$$\int_{-T}^{T} f(t) dt = kT + \varphi(T), \tag{6.4}$$

where $k \in \mathbb{C}$ and a periodic function $\varphi(T)$ are defined uniquely.

Definition 5. Given a function f satisfying the conditions of Lemma 6.1, *the regularized integral* of f is equal to

$$\fint_{\mathbb{R}} f(t)dt = \frac{1}{2\pi}\int_0^{2\pi} \varphi(T)dT.$$

Theorem 6.2. *Let the conditions of Theorem* 6.1 *hold. Then the following index formula holds:*

$$\operatorname{ind} D = -\frac{1}{2\pi i}\fint_{\mathbb{R}} \operatorname{tr}\left(D_n^{-1}\partial_t D_n\Big|_{p=-1}^{p=1}\right)dt + \eta(D_+) - \eta(D_-). \tag{6.5}$$

Proof. It suffices to prove the theorem for operators of order 0. Let us prove auxiliary results.

Lemma 6.2. *The left- and right-hand sides in* (6.5) *do not change under smooth homotopies of operators* (6.2)*, where operators* $D_\pm(\varepsilon) \in C^\infty\big([0,1], \Psi^n_{\mathrm{per}} \otimes \mathrm{Mat}_N\big)$ *and symbols* $D(\varepsilon,t,p) \in C^\infty\big([0,1]\times \mathbb{R}, S^n_{\mathrm{cl}} \otimes \mathrm{Mat}_N\big)$ *satisfy the conditions of Theorem* 6.1 *for all* $\varepsilon \in [0,1]$.

Proof. For the left-hand side (i.e., for the Fredholm index) the desired statement is a reformulation of the known homotopy invariance of the index. Let us prove that the right-hand side in formula (6.5) does not change under homotopies. On one hand, from (4.2), we obtain

$$\partial_\varepsilon \eta(D_\pm) = \frac{1}{4\pi^2 i}\int_0^{2\pi} \operatorname{tr}\left[\sigma_+^{-1}(D_\pm)\partial_\varepsilon\sigma_+(D_\pm) - \sigma_-^{-1}(D_\pm)\partial_\varepsilon\sigma_-(D_\pm)\right]d\varphi. \tag{6.6}$$

On the other hand, the integrand in (6.5) satisfies the conditions of Lemma 6.1 and we have

$$\begin{aligned}
&\partial_\varepsilon \fint_{\mathbb{R}} \operatorname{tr}\left(\left(D_n^{-1}\partial_t D_n\right)\Big|_{p=-1}^{p=1}\right)dt\\
=&\fint_{\mathbb{R}} \operatorname{tr}\left(-D_n^{-1}(\partial_\varepsilon D_n)D_n^{-1}\partial_t D_n + D_n^{-1}\partial^2_{\varepsilon t} D_n\Big|_{p=-1}^{p=1}\right)dt\\
=&\fint_{\mathbb{R}} \operatorname{tr}\left(\partial_t\big(D_n^{-1}\partial_\varepsilon D_n\big)\Big|_{p=-1}^{p=1}\right)dt = \text{reg-}\lim_{T\to+\infty} \operatorname{tr}\left(\big(D_n^{-1}\partial_\varepsilon D_n\big)\Big|_{p=-1}^{p=1}\right)\Bigg|_{-T}^{T}\\
=&\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\left[\big(D_n^{-1}\partial_\varepsilon D_n\big)(\varepsilon,T,p)\Big|_{p=-1}^{p=1} - \big(D_n^{-1}\partial_\varepsilon D_n\big)(\varepsilon,-T,p)\Big|_{p=-1}^{p=1}\right]dT\\
=&\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\Big[\sigma_+^{-1}(D_+)\partial_\varepsilon\sigma_+(D_+) - \sigma_-^{-1}(D_+)\partial_\varepsilon\sigma_-(D_+)\\
&- \sigma_+^{-1}(D_-)\partial_\varepsilon\sigma_+(D_-) + \sigma_-^{-1}(D_-)\partial_\varepsilon\sigma_-(D_-)\Big]dT.
\end{aligned} \tag{6.7}$$

The second equality in (6.7) follows from the cyclic property of tr. Then, the regularized limit reg-lim in (6.7) of function f such that $f(T) = kT + \varphi(T)$ for large T, where φ is a periodic function with period 2π, is defined as

$$\text{reg-}\lim_{T\to+\infty} f(T) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(T)dT.$$

The last equality in (6.7) follows from the compatibility conditions (6.1). Finally, the right-hand side in (6.5) is computed using (6.6) and (6.7):

$$-\frac{1}{2\pi i}\partial_\varepsilon \fint_{\mathbb{R}} \operatorname{tr}\left(D_n^{-1}\partial_t D_n\Big|_{p=-1}^{p=1}\right)dt + \partial_\varepsilon\eta(D_+) - \partial_\varepsilon\eta(D_-) = 0.$$

Thus, the right-hand side in (6.5) does not change under homotopies. □

Lemma 6.3. *Let an elliptic operator D of order 0 have form (6.2). Then there exist numbers $N' \in \mathbb{N}$, $k_\pm \in \mathbb{Z}$ and a smooth homotopy of elliptic operators*

$$D_\varepsilon \colon H^s(\mathbb{R}, \mathbb{C}^N \oplus \mathbb{C}^{N'}) \longrightarrow H^s(\mathbb{R}, \mathbb{C}^N \oplus \mathbb{C}^{N'}), \quad \varepsilon \in [0,1], \tag{6.8}$$

such that $D_0 = D \oplus \mathrm{Id}_{N'}$ and the limit operators $D_{1\pm}$ of operator D_1 at infinity are products by exponentials: $D_{1\pm} = e^{ik_\pm t} \oplus \mathrm{Id}_{N+N'-1}$.

Proof. This result follows from [17]. For completeness, we present an independent proof.

Consider the limit operators $D_\pm \in \Psi^0_{\mathrm{per}} \otimes \mathrm{Mat}_N$ of operator D. These operators are invertible and, therefore, define classes in the K-theory

$$[D_\pm] = K_1(\Psi^0_{\mathrm{per}})$$

of algebra Ψ^0_{per}. The closure of this algebra with respect to the norm is isomorphic to the crossed product

$$\overline{\Psi^0_{\mathrm{per}}} \simeq C(I) \rtimes \mathbb{Z}. \tag{6.9}$$

Here, $I = [-\infty, +\infty]$ and $\mathbb{Z}$ acts by shifts $f(p) \mapsto f(p+k)$. Isomorphism (6.9) is defined by the Fourier transform. We obtain isomorphisms of K-groups:

$$K_1(\Psi^0_{\mathrm{per}}) \simeq K_1(\overline{\Psi^0_{\mathrm{per}}}) \simeq K_1(C(I) \rtimes \mathbb{Z}) \simeq K_1(\mathbb{C} \rtimes \mathbb{Z}) = K_1(C(\mathbb{Z})) = K_1(C(\mathbb{S}^1)) = \mathbb{Z}. \tag{6.10}$$

Here, the first isomorphism follows by the spectral invariance of subalgebra $\Psi^0_{\mathrm{per}} \subset \overline{\Psi^0_{\mathrm{per}}}$, the second one is induced by isomorphism (6.9), the third one follows by the properties of the crossed products by $\mathbb{Z}$ (see [6]), the fourth one is the isomorphism between $\mathbb{C} \rtimes \mathbb{Z}$ and the group C^*-algebra of $\mathbb{Z}$. It follows from isomorphisms (6.10) that after stabilization (i.e. passing to the direct sums $D_\pm \oplus \mathrm{Id}_{N'}$, where N' is sufficiently large) there exists a homotopy between the operators

$$D_\pm \oplus \mathrm{Id}_{N'} \quad \text{and} \quad e^{ik_\pm t} \oplus \mathrm{Id}_{N+N'-1}, \qquad \text{where } k_\pm \in \mathbb{Z}. \tag{6.11}$$

Homotopy (6.11) of invertible operators can be lifted to the desired homotopy of elliptic operators (6.8). □

For elliptic operator (6.2), we consider the homotopy D_ε from Lemma 6.3. Then, for the Fredholm index, we have

$$\mathrm{ind}\, D = \mathrm{ind}\, D_0 = \mathrm{ind}\, D_1. \tag{6.12}$$

Here in the second equality we used the homotopy invariance of the index. Now we denote the right-hand side in (6.5) by $\mathrm{ind}_t\, D$. So we have by Lemma 6.2

$$\mathrm{ind}_t\, D = \mathrm{ind}_t\, D_0 = \mathrm{ind}_t\, D_1. \tag{6.13}$$

Since the limit operators of D_1 are multiplication operators, the η-invariants of the limit operators of D_1 at $\pm\infty$ are equal to zero. Thus, we have

$$\mathrm{ind}_t\, D_1 = -\frac{1}{2\pi i} \fint_{\mathbb{R}} \mathrm{tr}\left(D_1^{-1} \partial_t D_1\right)\Big|_{p=-1}^{p=1} dt = -\frac{1}{2\pi i} \int_{-T}^{T} \mathrm{tr}\left(D_1^{-1} \partial_t D_1\right)\Big|_{p=-1}^{p=1} dt \tag{6.14}$$

since $D_1 = e^{ik\pm t} \oplus \mathrm{Id}_{N+N'-1}$ as $|t| > T$ and, consequently,

$$D^{-1} \partial_t D_1 \Big|_{p=-1}^{p=1} = 0 \quad \text{whenever} \quad |t| > T.$$

Further, the analytical index of D_1 is calculated by the Atiyah–Singer formula. It equals

$$\mathrm{ind}\, D_1 = -\frac{1}{2\pi i} \int_{-T}^{T} \mathrm{tr}\left(D_1^{-1} \partial_t D_1\right)\Big|_{p=-1}^{p=1} dt. \tag{6.15}$$

Now the desired index formula (6.5) follows from (6.12), (6.13), (6.14) and (6.15). □

7 η-invariant and index of differential operators

In this section, we obtain an index formula in the case of differential operators in terms of the monodromy matrices of the limit operators at infinity. We also express the η-invariant of differential operators in terms of the spectrum of the monodromy matrix by comparing the resulting index formula with the index formula in Section 5.

Invertibility of operators of arbitrary order. Consider a linear differential operator with periodic coefficients (the period is 2π)

$$D=\sum_{0\le k\le n} d_k(t)(-i\partial_t)^k\colon H^s(\mathbb{R},\mathbb{C}^N)\longrightarrow H^{s-n}(\mathbb{R},\mathbb{C}^N). \tag{7.1}$$

Recall that the monodromy matrix of operator (7.1) is a matrix $M\in \mathrm{Mat}_{nN}$ such that the following equality holds:

$$M(v_0,\dots,v_{n-1})=(u(t),u'(t),\dots,u^{(n-1)}(t))\Big|_{t=T+2\pi}.$$

Here $u(t)$ is the solution of the homogeneous equation $Du=0$ with the Cauchy data at $t=T$

$$u(T)=v_0,\ \dots,\ u^{(n-1)}(T)=v_{n-1}.$$

Proposition 7.1. *Differential operator* (7.1) *is invertible if and only if one has* $\operatorname{Spec} M\cap \mathbb{S}^1_\lambda=\emptyset$. *Here* $\operatorname{Spec} M\subset\mathbb{C}$ *is the spectrum of the monodromy matrix M, while* $\mathbb{S}^1_\lambda=\{\lambda\in\mathbb{C}\colon|\lambda|=1\}$.

Proof. The conjugation of D with the Fourier-Laplace transform (5.2) gives the operator family (see (5.3))

$$D_z=\sum_{k\in\mathbb{Z}} d_k(t)\left(-i\partial_t+\frac{i}{2\pi}\ln z\right)^k\colon H^s(\mathbb{S}^1_t,\mathbb{C}^N)\longrightarrow H^{s-n}(\mathbb{S}^1_t,\mathbb{C}^N),\quad z\in\mathbb{S}^1_z.$$

Similarly to the proof of Proposition 5.1, D is invertible if and only if D_z is invertible on the circle $\mathbb{S}^1=\mathbb{R}_t/2\pi\mathbb{Z}$ for all $z\in\mathbb{S}^1_z$. Since the index of D_z is equal to zero, D_z is invertible if and only if its kernel is trivial for all $z\in\mathbb{S}^1_z$. Thus, it is necessary to find the conditions under which the problem

$$\begin{cases} D_z u &= 0,\\ u(0) &= u(2\pi),\\ &\dots\\ u^{(n-1)}(0) &= u^{(n-1)}(2\pi)\end{cases} \tag{7.2}$$

has only trivial solution. It is easy to see that $D_z=z^{t/2\pi}Dz^{-t/2\pi}$. Therefore, (7.2) is equivalent to the problem

$$\begin{cases} D(z^{-t/2\pi}u) &= 0,\\ u(0) &= u(2\pi),\\ &\dots\\ u^{(n-1)}(0) &= u^{(n-1)}(2\pi).\end{cases} \tag{7.3}$$

We claim that (7.3) is equivalent to the problem

$$\begin{cases} Dw &= 0,\\ z^{-1}w(0) &= w(2\pi),\\ &\dots\\ z^{-1}w^{(n-1)}(0) &= w^{(n-1)}(2\pi)\end{cases} \tag{7.4}$$

for $w(t) = z^{-t/2\pi}u(t)$. Indeed, we obtain

$$w^{(k)}(t) = z^{-t/2\pi} \sum_{0\le j\le k} C_k^j \left(-\frac{\ln z}{2\pi}\right)^k u^{(k-j)}(t),$$
$$w^{(k)}(0) = \sum_{0\le j\le k} C_k^j \left(-\frac{\ln z}{2\pi}\right)^k u^{(k-j)}(0),$$
$$w^{(k)}(2\pi) = z^{-1} \sum_{0\le j\le k} C_k^j \left(-\frac{\ln z}{2\pi}\right)^k u^{(k-j)}(2\pi).$$

Hence, the equivalence of problems (7.3) and (7.4) follows directly.

We introduce a vector function $W(t) = \big(w(t), w'(t), \dots, w^{(n-1)}(t)\big)$ to solve problem (7.4). By the definition of the monodromy matrix, we have $W(2\pi) = MW(0)$. Therefore, problem (7.4) has a nontrivial solution if and only if there is a nontrivial solution to the equation

$$MW(0) = z^{-1}W(0).$$

The latter condition is equivalent to the condition $z \in \operatorname{Spec} M$. Hence, the invertibility condition of the family D_z is equivalent to the condition $\operatorname{Spec} M \cap \mathbb{S}^1_z = \emptyset$. □

The index formula. Let the coefficients of a differential operator of order n

$$D = \sum_{0\le k\le n} d_k(t)(-i\partial_t)^k \colon H^s(\mathbb{R}, \mathbb{C}^N) \longrightarrow H^{s-n}(\mathbb{R}, \mathbb{C}^N) \tag{7.5}$$

be smooth functions periodic with period 2π as $|t| > T$. We denote

$$d_k^{\pm}(t) = \lim_{j\to+\infty} d_k(t \pm 2\pi j), \qquad D_{\pm} = \sum_k d_k^{\pm}(t)(-i\partial_t)^k.$$

Theorem 7.1. *Let the principal symbol of operator (7.5) be invertible and the operators $D_+, D_- \colon H^s(\mathbb{R}, \mathbb{C}^N) \to H^{s-n}(\mathbb{R}, \mathbb{C}^N)$ be invertible. Then operator (7.5) is Fredholm and its index is equal to*

$$\operatorname{ind} D = \frac{1}{2}\big(\operatorname{sign} M_- - \operatorname{sign} M_+\big). \tag{7.6}$$

Here $M_\pm$ are the monodromy matrices of the operators $D_\pm$ and

$$\operatorname{sign} M = \#\{|\lambda_M| > 1\} - \#\{|\lambda_M| < 1\}$$

is the signature of M.

Proof. 1. The Fredholm property follows by Theorem 6.1. We define the partition of $\mathbb{R} = (-\infty, -T] \cup [-T, T] \cup [T, +\infty)$, where $T > 0$ is chosen from the conditions $d_k(t) = d_k^+(t)$ as $t \ge T$ and $d_k(t) = d_k^-(t)$ as $t \le -T$. Since the index does not depend on s, we assume that $s = n$. Furthermore, the spaces $L^2(\mathbb{R}, \mathbb{C}^N)$ and $H^n(\mathbb{R}, \mathbb{C}^N)$ are isomorphic to the spaces

$$L^2_T(\mathbb{R}, \mathbb{C}^N) = L^2\big((-\infty, -T], \mathbb{C}^N\big) \oplus L^2\big([-T, T], \mathbb{C}^N\big) \oplus L^2\big([T, +\infty), \mathbb{C}^N\big),$$
$$H^n_T(\mathbb{R}, \mathbb{C}^N) \cap H_B = \Big(H^n\big((-\infty, -T], \mathbb{C}^N\big) \oplus H^n\big([-T, T], \mathbb{C}^N\big) \oplus H^n\big([T, +\infty), \mathbb{C}^N\big)\Big) \cap H_B$$

respectively. Here $H_B = \big\{(u_-, u_0, u_+) \in H^n_T(\mathbb{R}, \mathbb{C}^N)\big\}$, where u_-, u_0, u_+ satisfy the compatibility conditions

$$u_-^{(j)}(-T) = u_0^{(j)}(-T) \text{ and } u_0^{(j)}(T) = u_+^{(j)}(T)\ \forall j = 0, 1, \dots, n-1.$$

Consider the operator $D'\colon H^n_T(\mathbb{R},\mathbb{C}^N)\cap H_B\to L^2_T(\mathbb{R},\mathbb{C}^N)$ isomorphic to D. We have

$$D' = D_-\oplus D_0\oplus D_+\Big|_{H^n_T(\mathbb{R},\mathbb{C}^N)\cap H_B},$$

where

$$D_- = D\Big|_{(-\infty,-T]}\colon H^n\big((-\infty,-T],\mathbb{C}^N\big)\longrightarrow L^2\big((-\infty,-T],\mathbb{C}^N\big),$$
$$D_0 = D\Big|_{[-T,+T]}\colon H^n\big([-T,T],\mathbb{C}^N\big)\longrightarrow L^2\big([-T,T],\mathbb{C}^N\big),$$
$$D_+ = D\Big|_{[T,+\infty)}\colon H^n\big([T,+\infty),\mathbb{C}^N\big)\longrightarrow L^2\big([T,+\infty),\mathbb{C}^N\big).$$

Hence

$$\operatorname{ind} D = \operatorname{ind} D' = \operatorname{ind} D_- + \operatorname{ind} D_0 + \operatorname{ind} D_+ - 2nN, \tag{7.7}$$

where the last term corresponds to the compatibility conditions included in the definition of H_B.

2. Let us calculate the indices of operators in (7.7). Since the equation $D_0u_0 = 0$ has a solution on the segment $[-T,T]$ for any Cauchy data (the solution is unique) and the cokernel of operator D_0 is trivial, we have

$$\operatorname{ind} D_0 = nN. \tag{7.8}$$

To calculate the index of $D_+\colon H^n\big([T,+\infty),\mathbb{C}^N\big)\to L^2\big([T,+\infty),\mathbb{C}^N\big)$, we consider the vector-function $U(t) = \big(u(t),u'(t),\ldots,u^{(n-1)}(t)\big)$ corresponding to the solution u of the equation $D_+u = 0$. We have

$$U(T+2\pi k) = M_+^kU(T)\quad\text{as}\quad k\ge 0,$$

where M_+ is the monodromy matrix of D_+.

Lemma 7.1. *The condition $U\in L^2\big([T,+\infty),\mathbb{C}^N\big)$ holds if and only if $U(T)\in L^{\pm}_{M_+}$, where $L^+_{M_+}$ and $L^-_{M_+}$ are the direct sums of the eigenspaces and the root subspaces of the monodromy matrix M_+ corresponding to the eigenvalues $|\lambda|<1$ for $L^+_{M_+}$ and $|\lambda|>1$ for $L^-_{M_+}$.*

Proof. Given $U(T)\in L^+_{M_+}$, the following estimate holds:

$$\|U(T+2\pi k)\|\le q^k\|U(T)\|,\quad\text{where}\quad q = \max_{\substack{\lambda\in\operatorname{Spec}M_+\\|\lambda|<1}}|\lambda|+\varepsilon$$

and $\varepsilon>0$ is small enough. Since D_+ has periodic coefficients, we obtain

$$\|U(t)\|\le Ce^{-\gamma(t-T)}\|U(T)\|\quad\text{for all } t\ge T\text{, where}\quad \gamma = -\frac{\ln q}{2\pi}.$$

This implies that $U\in L^2\big([T,+\infty),\mathbb{C}^N\big)$. The converse is also true. More precisely, if $U(T)\notin L^+_{M_+}$, then

$$\|U(T+2\pi k)\|\ge Cq^k\|U(T)\|,\quad\text{where}\quad 1<q = \min_{\substack{\lambda\in\operatorname{Spec}M_+\\|\lambda|>1}}|\lambda|-\varepsilon.$$

Since D_+ has periodic coefficients, there exists $\varepsilon>0$ such that for all t satisfying $|t-(T+2\pi k)|<\varepsilon$, we obtain

$$\|U(t)\|\ge C'e^{\gamma(t-T)}\|U(T)\|,\quad\text{where}\quad\gamma = \frac{\ln q}{2\pi}.$$

Consequently, $U\notin L^2\big([T,+\infty),\mathbb{C}^N\big)$. □

It follows by Lemma 7.1 that
$$\dim\ker D_+ = \dim L^+_{M_+}. \tag{7.9}$$
Moreover, we have
$$\begin{cases} \dim L^+_{M_+} + \dim L^-_{M_+} &= \quad nN, \\ \dim L^+_{M_+} - \dim L^-_{M_+} &= -\operatorname{sign} M_+. \end{cases} \tag{7.10}$$
Further, since $\operatorname{coker} D_+ \simeq \ker D_+^*$ and the domain of D_+^* has the form
$$\mathcal{D}(D_+^*) = \left\{u_+(t)\colon D_+^* u_+ \in L^2\big([T,+\infty),\mathbb{C}^N\big) \text{ and } u_+^{(j)}(0) = 0\ \forall j = 0,1,\dots,n-1\right\},$$
the cokernel is trivial: $\operatorname{coker} D_+ = 0$. Consequently, (7.9) and (7.10) imply
$$\operatorname{ind} D_+ = \dim\ker D_+ - \dim\operatorname{coker} D_+ = \frac{1}{2}(nN - \operatorname{sign} M_+). \tag{7.11}$$
For D_-, we similarly obtain
$$\operatorname{ind} D_- = \dim\ker D_- - \dim\operatorname{coker} D_- = \frac{1}{2}(nN + \operatorname{sign} M_-). \tag{7.12}$$

3. Finally, substituting (7.8), (7.11) and (7.12) into (7.7), we obtain the desired formula (7.6):
$$\operatorname{ind} D = \frac{1}{2}(nN - \operatorname{sign} M_+ + nN + \operatorname{sign} M_-) + nN - 2nN = \frac{1}{2}(\operatorname{sign} M_- - \operatorname{sign} M_+).$$
□

Example 1. Let us compute the index of differential operator $D_+ = -i\partial_t + a(t)$, where a is a periodic function with period 2π, on the half-line $[T,\infty)$. The solution of the equation $D_+ u = 0$ is
$$u(t) = C\exp\left(-i\int_T^t a(t)dt\right) = C\exp\left(-i\operatorname{Re}\int_T^t a(t)dt\right)\exp\left(\operatorname{Im}\int_T^t a(t)dt\right).$$
Obviously, the kernel of the operator depends on the value
$$\alpha \stackrel{\text{def}}{=} \operatorname{sgn}\operatorname{Im}\int_0^{2\pi} a(t)dt.$$
More precisely, for the existence of a solution for $\alpha < 0$, the constant C can be chosen arbitrarily, and for $\alpha > 0$, the solution decreases at infinity only when $C = 0$. Thus,
$$\ker D_+ = \begin{cases} \left\{C\exp\left(-i\int_T^t a(t)dt\right)\right\}, & \alpha < 0, \\ 0, & \alpha > 0. \end{cases}$$
Considering $\operatorname{coker} D_+ = \ker D_+^*$ and condition $u(T) = 0$ for equation $D_+^* u = 0$, we obtain
$$\operatorname{coker} D_+ = 0.$$
Consequently,
$$\operatorname{ind} D_+ = \begin{cases} 1, & \alpha < 0, \\ 0, & \alpha > 0. \end{cases} \tag{7.13}$$

Let us express this result in terms of the signature. Since the solution of equation $D_+u = 0$ has the form $u(t) = C\exp\left(-i\int_T^t a(t)dt\right)$ as $t > T$, the corresponding monodromy matrix described by $u(2\pi) = M_+u(0)$ is equal to

$$M_+ = \exp\left(-i\int_0^{2\pi} a(t)dt\right). \tag{7.14}$$

Hence

$$|M_+| = \exp\left(\operatorname{Im}\int_0^{2\pi} a(t)dt\right).$$

From (7.9) and (7.10), we obtain

$$\operatorname{sign} M_+ = \operatorname{sgn}\operatorname{Im}\int_0^{2\pi} a(t)dt, \quad \operatorname{ind} D_+ = \frac{1}{2}\left(1 - \operatorname{sgn}\operatorname{Im}\int_0^{2\pi} a(t)dt\right).$$

The latter coincides with (7.13).

Example 2. Using Example 1, let us calculate the index of the operator $D\colon H^1(\mathbb{R}) \to L^2(\mathbb{R})$ on the real line $\mathbb{R} = (-\infty, -T] \cup [-T, T] \cup [T, +\infty)$. In accordance with such a partition of the real line, the Sobolev space $H^1(\mathbb{R})$ is isomorphic to the space

$$H^1(\mathbb{R}) \simeq \left\{(u_-, u_0, u_+) \in H^1(-\infty, -T] \oplus H^1[-T, T] \oplus H^1[T, +\infty)\right\},$$

where u_-, u_0, u_+ satisfy the compatibility conditions, namely, $u_-(-T) = u_0(-T)$ and $u_0(T) = u_+(T)$. Therefore, the index formula gives us

$$\operatorname{ind} D = \operatorname{ind}_+ D + \operatorname{ind}_- D + \operatorname{ind}_0 D - 2,$$

where $\operatorname{ind}_\pm D = \operatorname{ind} D_\pm$, $\operatorname{ind}_0 D$ is the index of D on the segment $[-T, T]$, and the last term corresponds to the compatibility conditions at $\pm T$. Obviously, $\operatorname{ind}_0 D = 1$, $\operatorname{ind} D_+$ was evaluated in Example 1. Thus, $\operatorname{ind} D_-$ is equal to

$$\operatorname{ind} D_- = \begin{cases} 0, & \alpha < 0, \\ 1, & \alpha > 0. \end{cases}$$

We have

$$\operatorname{ind} D = \frac{1}{2}\left(1 - \operatorname{sgn}\operatorname{Im}\int_0^{2\pi} a(t)dt + 1 + \operatorname{sgn}\operatorname{Im}\int_0^{2\pi} a(t)dt\right) - 1 = 0.$$

The latter coincides with the result in Theorem 7.1 since $\operatorname{sign} M_+ = \operatorname{sign} M_-$ (see (7.14)).

Let us apply theorems 6.2 and 7.1 to compute η-invariants of differential operators.

Corollary 7.1. *The η-invariant of an invertible differential operator $D_+ \in \Psi^n_{\mathrm{per}} \otimes \mathrm{Mat}_N$ equals*

$$\eta(D_+) = -\frac{\operatorname{sign} M_+}{2}. \tag{7.15}$$

Here $\operatorname{sign} M_+ = \#\{|\lambda_{M_+}| > 1\} - \#\{|\lambda_{M_+}| < 1\}$ *is the signature of the monodromy matrix M_+ of D_+ and $\#A$ is the cardinality of a set A.*

Proof. Given an invertible operator $D_+ \in \Psi^n_{\mathrm{per}} \otimes \mathrm{Mat}_N$, we write it as

$$D_+ = \sum_{0\le k\le n} d_k(t)\,(-i\partial_t)^k, \quad \text{where} \quad d_n(t) \neq 0.$$

We define the operator with periodic coefficients

$$D_- = d_n(t)\,(-i\partial_t + i)^n$$

and the operator with coefficients periodic at infinity

$$D = \chi(t)D_+ + (1-\chi(t))D_- \colon H^s(\mathbb{R}, \mathbb{C}^N) \longrightarrow H^{s-n}(\mathbb{R}, \mathbb{C}^N). \tag{7.16}$$

Here $\chi \in C^\infty(\mathbb{R})$, $\chi(t) \geq 0$ and

$$\chi(t) = \begin{cases} 0 & \text{if} \quad t \leq -T, \\ 1 & \text{if} \quad t \geq T. \end{cases}$$

Operator (7.16) satisfies the conditions in Theorem 6.1 and, therefore, has Fredholm property. On one hand, we apply Theorem 6.2 to operator (7.16) and obtain

$$\operatorname{ind} D = \eta(D_+) - \eta(D_-) = \eta(D_+) + \frac{nN}{2}. \tag{7.17}$$

Indeed, by Propositions 4.1 and 5.1, we have

$$\eta(D_-) = \eta(d_n) + nN\eta(-i\partial_t + i) = -\frac{nN}{2},$$

where $\eta(d_n) = 0$ (see (4.1)). On the other hand, Theorem 7.1 implies

$$\operatorname{ind} D = \frac{1}{2}(\operatorname{sign} M_- - \operatorname{sign} M_+) = \frac{1}{2}(nN - \operatorname{sign} M_+). \tag{7.18}$$

From relations (7.17) and (7.18), we obtain the desired equality (7.15). □

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