ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2021, Volume 12, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

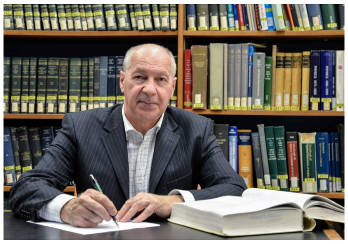
eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 562
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

VICTOR IVANOVICH BURENKOV

(to the 80th birthday)



On July 15, 2021 was the 80th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), professor of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (Moscow), chairman of the Dissertation Council at the RUDN University, research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), honorary academician of the National Academy of Sciences of the Republic of Kazakhstan, doctor of physical and mathematical sciences(1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK,2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University. Through 2015-2017 he was head of the Department of Mathematical Analysis and Theory of Functions (RUDN University). He was one of the organisers and the first director of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (2016-2017).

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem on composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 190 scientific papers. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces. In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

V.I. Burenkov is still very active in research. Through 2016-2021 he published 20 papers in leading mathematical journals.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 80th birthday and wishes him good health and new achievements in science and teaching!

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 12, Number 3 (2021), 46 – 56

SHARP CONFORMALLY INVARIANT HARDY-TYPE INEQUALITIES WITH REMAINDERS

R.G. Nasibullin

Communicated by V.S. Guliyev

Key words: Hardy inequality, half space, remainder terms, hyperbolic domain, the Poincaré metric, hyperbolic radius, distance function.

AMS Mathematics Subject Classification: 26D10.

Abstract. In the present paper we establish new Hardy-Maz'ya-type inequalities with remainders for all continuously differentiable functions with compact support in the half space $\mathbb{R}^n_+$. The weight functions depend on the distance to the boundary or on the distance to the origin. Also new sharp Avkhadiev-Hardy-type inequalities involving the distance to the boundary or the hyperbolic radius are proved. We consider Avkhadiev-Hardy-type inequalities in simply and doubly connected plain domains and in tube-domains.

DOI: https://doi.org/10.32523/2077-9879-2021-12-3-46-56

1 Introduction

The present paper is devoted to Hardy-type inequalities with remainders. There are many inequalities that can be called inequalities of Hardy type. Hardy inequalities are of fundamental importance in many branches of mathematical analysis and mathematical physics. We refer to [1] –[27] and the literature therein for more information about those inequalities.

Hardy inequalities with remainders were first obtained by V.G. Maz'ya [27] in the half-space

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ : x_n > 0\}.$$

For instance, in [27], V.G. Maz'ya for all infinitely differentiable functions with compact support in $\mathbb{R}^n_+$ proved the following improved Hardy-type inequality

$$\int\limits_{\mathbb{R}^n_+} |\nabla f| dx \geq \frac{1}{4} \int\limits_{\mathbb{R}^n_+} \frac{|f|^2}{x_n^2} dx + \frac{1}{16} \int\limits_{\mathbb{R}^n_+} \frac{|f|^2}{x_n (x_{n-1}^2 + x_n^2)^{1/2}} dx. \tag{1.1}$$

Note that the papers [1], [30], [37], [38] are also devoted to inequalities in the half-space.

Many papers devoted to Hardy-type inequalities with additional non-negative terms were published after the release of the work "Hardy's inequality revisited" by H. Brezis and M. Marcus. In [17], H. Brezis and M. Marcus for bounded convex domains $\Omega \subset \mathbb{R}^n$ and functions $f \in C_0^1(\Omega)$ obtained the following inequality

$$\int\limits_{\Omega} |\nabla f|^2 dx \geq \frac{1}{4} \int\limits_{\Omega} \frac{|f|^2}{\delta(x)^2} dx + \frac{1}{4\mathrm{diam}^2(\Omega)} \int\limits_{\Omega} |f|^2 dx, \tag{1.2}$$

where the constant 1/4 is sharp, $\mathrm{diam}(\Omega)$ is the diameter of Ω and $\delta(x)$ is the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$ of Ω, i.e.

$$\delta(x) = \mathrm{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|.$$

In [4], F.G. Avkhadiev established a sharp conformally invariant analogue of (1.2) for a simply connected domains $\Omega \subset \mathbb{R}^2, \Omega \neq \mathbb{R}^2$, when the weight function depends on the hyperbolic radius $R(z, \Omega)$. In fact, he proved that

$$\iint_{\Omega} |\nabla f|^2 dxdy \geq \iint_{\Omega} \frac{|f|^2}{R(z,\Omega)^2} dxdy + \frac{1}{4} \iint_{\Omega} |f|^2 \left| \frac{g'(z)}{g(z)} \right|^2 dxdy, \tag{1.3}$$

where $z = x + iy$ and g is any univalent conformal mapping of Ω onto the upper half-plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.

Notice, that the sharp inequality

$$\iint_{\Omega} |\nabla f|^2 dxdy \geq \iint_{\Omega} \frac{|f|^2}{R(z,\Omega)^2} dxdy$$

is well known in spectral theory of the Laplace-Beltrami operator for simply and doubly connected domains $\Omega \subset \mathbb{R}^n$ (see [35] for more information).

In the present paper we shall prove new Hardy-Maz'ya-type inequalities in the half-space $\mathbb{R}^n_+$. Namely, for all $f \in C^1_0(\mathbb{R}^n_+)$ we have the following inequality

$$\int_{\mathbb{R}^n_+} \frac{|\nabla f|^p}{x_n^{s-p}} dx \geq \frac{1}{p^{p-1}} \left(s - 2 + \frac{1}{p} \right) \int_{\mathbb{R}^n_+} \frac{|f|^p}{x_n^s} dx + \frac{(5-2s)}{2p^{p-1}} \int_{\mathbb{R}^n_+} \frac{|f|^p}{x_n^{s-2} |x_{n-1}^2 + x_n^2|} dx,$$

where $n \geq 2, p \in [2, \infty)$ and $s \in [2 - 1/p, 2.5]$. Notice, that earlier there were no Hardy-type inequalities in the case $s < n$ (see [5], [7], [8] and Section 2 bellow for more information).

We prove an L_p-version of inequality (1.3) in simply and doubly connected domains for $p \geq 2$. For instance, in [28], [29], [36], L_p-inequalities for $p > 1$ are proved. It is well known that for the L_p-spaces with $0 < p < 1$ Hardy inequalities are not valid for arbitrary non-negative measurable functions. In spite of this Hardy-type inequalities for $0 < p < 1$ are also known for non-negative non-increasing functions (see [18] - [20], [33], [34]).

Suppose that Ω is a hyperbolic domain in the complex plane $\mathbb{C}$, i.e. Ω has more than three boundary points in the extended complex plane $\overline{\mathbb{C}}$ and the hyperbolic radius $R(z, \Omega) = 1/\lambda(\Omega)$ is defined at all points $z = x + iy \in \Omega$, where by $\lambda(\Omega)$ we denote the density of the Poincaré metric with the Gaussian curvature $k = -4$.

The main result is the following assertion.

Theorem 1.1. 1) *Suppose that Ω is a simply connected hyperbolic domain in $\mathbb{C}$, g is any univalent conformal mapping of Ω onto the upper half-plane*

$$H_+ = \{\zeta = \xi + i\eta \in \mathbb{C} : \eta > 0\}.$$

Then for all real valued function $f \in C^1_0(\Omega)$ and $p \geq 2$ the following Hardy-Avkhadiev-type inequality

$$\iint_{\Omega} \frac{|\nabla f|^p}{R(z,\Omega)^{2-p}} dxdy \geq \frac{2^p}{p^p} \iint_{\Omega} \frac{|f|^p}{R(z,\Omega)^2} dxdy + \frac{2^{p-3}}{p^{p-1}} \iint_{\Omega} |f|^p \left| \frac{g'(z)}{g(z)} \right|^2 dxdy$$

is valid with the sharp constant $2^p/p^p$, *where* $z = x + iy$.

2) *Let* $\Omega \subset \mathbb{C}$ *be a doubly connected domain and let* g *be any univalent conformal mapping of* Ω *onto the annuli* $A_q = \{\eta \in \mathbb{C} : q < |\eta| < 1\}$, $q = \exp(-2\pi M(\Omega))$. *Then for any real valued function* $f \in C_0^1(\Omega)$ *the following Avkhadiev-Hardy-type inequality*

$$\iint\limits_{\Omega} \frac{|\nabla f|^p}{R(z,\Omega)^{2-p}} dxdy \geq \frac{2^p}{p^p} \iint\limits_{\Omega} \frac{|f|^p}{R(z,\Omega)^2} dxdy + \frac{1}{16M^2(\Omega)} \frac{2^{p-1}}{p^{p-1}} \iint\limits_{\Omega} |f|^p \left| \frac{g'(z)}{g(z)} \right|^2 dxdy$$

holds with the sharp constant $2^p/p^p$, *where* $z = x+iy$ *and* $M(\Omega)$ *is the geometrical parameter defined as the supremum of the moduli of doubly connected domains lying in the domain* Ω *and separating its boundary* $\partial\Omega$.

It is a well known fact (see, for example, [14]) that if Ω is a simply connected domain in $\mathbb{C}$, then

$$R(z,\Omega) \leq 4\delta(z), \quad z \in \Omega,$$

and if Ω is a convex domain, then

$$R(z,\Omega) \leq 2\delta(z), \quad z \in \Omega.$$

Using these estimates, we get

Corollary 1.1. *Suppose that* Ω *is a convex domain in* $\mathbb{C}$, g *is any univalent conformal mapping of* Ω *onto the upper half-plane* $H_+ = \{\zeta = \xi + i\eta \in \mathbb{C} : \eta > 0\}$. *Then for all real-valued function* $f \in C_0^1(\Omega)$ *and* $p \geq 2$ *the following Avkhadiev-Hardy-type inequality*

$$\iint\limits_{\Omega} \frac{|\nabla f|^p}{\delta(z)^{2-p}} dxdy \geq \frac{1}{p^p} \iint\limits_{\Omega} \frac{|f|^p}{\delta(z)^2} dxdy + \frac{1}{2p^{p-1}} \iint\limits_{\Omega} |f|^p \left| \frac{g'(z)}{g(z)} \right|^2 dxdy$$

is valid, where $z = x + iy$. *The constant* $1/p^p$ *is sharp.*

Corollary 1.2. *Suppose that* Ω *is a simply connected hyperbolic domain in* $\mathbb{C}$, g *is any univalent conformal mapping of* Ω *onto the upper half-plane* $H_+ = \{\zeta = \xi + i\eta \in \mathbb{C} : \eta > 0\}$. *Then for all real-valued function* $f \in C_0^1(\Omega)$ *and* $p \geq 2$ *the following Avkhadiev-Hardy-type inequality*

$$\iint\limits_{\Omega} \frac{|\nabla f|^p}{\delta(z)^{2-p}} dxdy \geq \frac{1}{(2p)^p} \iint\limits_{\Omega} \frac{|f|^p}{\delta(z)^2} dxdy + \frac{1}{(2p)^{p-1}} \iint\limits_{\Omega} |f|^p \left| \frac{g'(z)}{g(z)} \right|^2 dxdy$$

is valid, where $z = x + iy$.

Note that our main statements are based on the one-dimensional inequalities. These one-dimensional inequalities are proved in Section 3. Using the one-dimensional inequalities, we will obtain their multidimensional analogues for continuously differentiable functions with compact support.

In the second chapter, we present an overview of results related to inequalities of the form (1.1)-(1.3).

In the last part of the paper we prove conformally invariant inequalities in the half-space and extend them to simply and doubly connected hyperbolic domains. Inequalities in tube-domains are also considered (see for more information [9]).

We prove inequalities with the weight functions dependent on the distance to the boundary, on the distance to the origin or on the hyperbolic radius. Therefore, these types of inequalities also can be called geometric Hardy type inequalities (see [31], [32]).

2 Related results

Let Ω be a domain in the Euclidean space $\mathbb{R}^n$ and let $C_0^1(\Omega)$ denote the space of all continuously differentiable functions $f : \Omega \to \mathbb{R}$, which vanish on the boundary $\partial\Omega$ of the domain. Our paper deals with inequalities of the form

$$\int\limits_{\Omega} \frac{|\nabla f|^p}{\delta(x)^{s-p}} dx \geq c_p(s,\Omega) \int\limits_{\Omega} \frac{|f|^p}{\delta(x)^s} dx, \quad \forall f \in C_0^1(\Omega), \tag{2.1}$$

where $p \in [1,\infty)$, $s \in (-\infty,\infty)$, by $c_p(s,\Omega)$ is denoted the optimal constant in this inequality and $\delta(x)$ is the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$ of Ω, i.e.

$$\delta(x) = \mathrm{dist}(x,\partial\Omega) = \inf_{y\in\partial\Omega} |x-y|.$$

Note that the constant $c_p(s,\Omega)$ depends only on s, n, p and Ω. It is known (see [5], [7]) that if $s > n \geq 2$, then for arbitrary open sets Ω and any $p \in [1,\infty)$ the sharp inequality

$$c_p(s,\Omega) \geq \left(\frac{s-n}{p}\right)^p$$

is valid. If $s = n = 2$, then inequality (2.1) holds in domains with the uniformly perfect boundary $\partial\Omega$ (see, for instance, [5]). In the case $s = n$, there are analogues of (2.1) with logarithmic weights (see [7], [10]). In [8], analogous results with other weight functions have been obtained also in the case of $s < n$. Notice, that earlier there were no Hardy inequalities akin to (2.1) when $s < n$.

Let Ω be the half-space $\mathbb{R}^n_+ = \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n_+ : x_n > 0\}$. In [37], J. Tidblom obtained an estimate for a remainder term for a special case suggested by the result of V.G. Maz'ya. More precisely, he proved that if $\Omega = \mathbb{R}^n_+$ and $f \in C_0^\infty(\Omega)$, then

$$\int\limits_{\mathbb{R}^n_+} |\nabla f|^p dx \geq \left(\frac{p-1}{p}\right)^p \int\limits_{\mathbb{R}^n_+} \frac{|f|^p}{x_n^p} dx + D(p) \left(\frac{p-1}{p}\right)^{p-1} \int\limits_{\mathbb{R}^n_+} \frac{|f|^p}{x_n^{p-1}(x_{n-1}^2 + x_n^2)^{1/2}} dx,$$

where $C_0^\infty(\Omega)$ is the space of smooth real-valued functions on Ω with compact support and

$$D(p) = \begin{cases} \frac{2}{2+3p}, & \text{if } 1 < p < 2, \\ \frac{1}{4(p-1)}, & \text{if } p \geq 2. \end{cases}$$

Many papers are devoted to Hardy-type inequalities in the half-space (see [1], [30], [37] and [38]).

Now suppose that Ω is a hyperbolic domain in the complex plane $\mathbb{C}$, i.e. Ω has more than three boundary points in the extended complex plane $\overline{\mathbb{C}}$ and the hyperbolic radius $R(z,\Omega) = 1/\lambda(\Omega)$ is defined at all points $z = x + iy \in \Omega$, where by $\lambda(\Omega)$ we denote the density of the Poincaré metric with the Gaussian curvature $k = -4$.

In [4], interesting inequalities for compactly supported functions in arbitrary plane domains of hyperbolic type are given and also special cases for domains with uniformly perfect boundaries are studied. The following is a special case of those results: *if Ω is a simply or doubly connected hyperbolic domain in $\mathbb{C}$, $f \in C_0^1(\Omega)$ and $p \in [1,\infty)$, then*

$$\iint\limits_{\Omega} \frac{|\nabla f|^p}{R(z,\Omega)^{2-p}} dxdy \geq \frac{2^p}{p^p} \iint\limits_{\Omega} \frac{|f|^p}{R(z,\Omega)^2} dxdy, \tag{2.2}$$

where the constant $2^p/p^p$ is sharp and $z = x + iy$.

Note that the smoothness of a function f at the point $z = \infty$ is understood as the smoothness of the function $f(1/z)$ at the point $z = 0$.

3 One-dimensional Hardy-type inequalities

In the sequel we will need the following lemma.

Lemma 3.1. *Suppose that $v \in C_0^1[0,\pi]$. Then for all $p \geq 2$, $s \in [2-\frac{1}{m}, 2.5]$ and $m \in [2,p]$ the following inequality*

$$\int_0^\pi \frac{|v(\theta)|^{p-m}|v'(\theta)|^m}{\sin^{s-m}\theta}d\theta \geq \frac{m}{p^m}\left(s-2+\frac{1}{m}\right)\int_0^\pi \frac{|v(\theta)|^p}{\sin^s\theta}d\theta + \frac{m(5-2s)}{2p^m}\int_0^\pi \frac{|v(\theta)|^p}{\sin^{s-2}\theta}d\theta$$

is valid.

Proof. Let $v_0(\theta) := \sqrt{\sin\theta}$. Consider the quantity $I(v)$ defined as

$$I(v) := \int_0^\pi \frac{|v(\theta)|^{p-2}}{v_0(\theta)^{2(s-2)}}\left(v'(\theta) - \frac{2}{p}\frac{v(\theta)v_0'(\theta)}{v_0(\theta)}\right)^2 d\theta.$$

Obviously,

$$0 \leq I(v) = \int_0^\pi \Big(\frac{|v(\theta)|^{p-2}v'(\theta)^2}{v_0(\theta)^{2(s-2)}} + \frac{4}{p^2}\frac{|v(\theta)|^p v_0'(\theta)^2}{v_0(\theta)^{2s-2}} - \frac{4}{p}\frac{|v(\theta)|^{p-1}\mathrm{sign}v(\theta)v'(\theta)v_0'(\theta)}{v_0(\theta)^{2s-3}}\Big)d\theta$$

$$= \int_0^\pi \frac{|v(\theta)|^{p-2}v'(\theta)^2}{v_0(\theta)^{2(s-2)}} + \frac{4}{p^2}\frac{|v(\theta)|^p v_0'(\theta)^2}{v_0(\theta)^{2s-2}}d\theta - \frac{4}{p^2}\int_0^\pi \frac{v_0'(\theta)}{v_0(\theta)^{2s-3}}d|v(\theta)|^p.$$

Integrating by part the second integral, we obtain

$$I(v) = \int_0^\pi \frac{|v(\theta)|^{p-2}v'(\theta)^2}{v_0(\theta)^{2(s-2)}}d\theta + \frac{4}{p^2}\int_0^\pi |v(\theta)|^p\left(\frac{v_0''(\theta)}{v_0(\theta)^{2s-3}} + (4-2s)\frac{v_0'(\theta)^2}{v_0(\theta)^{2s-2}}\right)d\theta$$

$$-\frac{4}{p^2}\left(\lim_{\theta\to\pi^-}|v(\theta)|^p\frac{v_0'(\theta)}{v_0(\theta)^{2s-3}} - \lim_{\theta\to 0^+}|v(\theta)|^p\frac{v_0'(\theta)}{v_0(\theta)^{2s-3}}\right).$$

If $\theta \in (0,\varepsilon_1)$, then by using Hölder's inequality, we have

$$|v(\theta)|^p \leq \left(\int_0^\theta |v'(t)|dt\right)^p = \left(\int_0^\theta \left(\sin^{\frac{s-p}{p-1}}t\right)^{\frac{p-1}{p}}\left(\frac{|v'(t)|^p}{\sin^{s-p}t}\right)^{1/p}dt\right)^p$$

$$\leq \left(\frac{p-1}{s-1}\right)^{p-1}\frac{\sin^{s-1}\theta}{\cos^{p-1}\theta}\int_0^\theta \frac{|v'(t)|^p}{\sin^{s-p}t}dt$$

and if $\theta \in (\pi-\varepsilon_2,\pi)$, then we get

$$v(\theta)^p \leq \left(\int_\theta^\pi |v'(t)|dt\right)^p \leq \left(\int_\theta^\pi \sin^{\frac{s-p}{p-1}}t\right)^{p-1}\int_\theta^\pi \frac{|v'(t)|^p}{\sin^{s-p}t}dt \leq \left(\frac{p-1}{s-1}\right)^{p-1}\frac{\sin^{s-1}\theta}{\cos^{p-1}\theta}\int_\theta^\pi \frac{|v'(t)|^p}{\sin^{s-p}t}dt,$$

where ε_1 and ε_2 are arbitrarily small numbers.

Consequently,

$$\lim_{\theta\to 0^+}|v(\theta)|^p\frac{v_0'(\theta)}{v_0(\theta)^{2s-3}}=\lim_{\theta\to 0^+}|v(\theta)|^p\frac{\cos\theta}{\sin^{s-1}\theta}=0$$

and

$$\lim_{\theta\to \pi^-}|v(\theta)|^p\frac{v_0'(\theta)}{v_0(\theta)^{2s-3}}=\lim_{\theta\to \pi^-}|v(\theta)|^p\frac{\cos\theta}{\sin^{s-1}\theta}=0.$$

Therefore,

$$p^2\int_0^\pi\frac{|v(\theta)|^{p-2}v'(\theta)^2}{v_0(\theta)^{2(s-2)}}d\theta\geq\int_0^\pi|v(\theta)|^p\left(\frac{2s-3}{\sin^s\theta}+\frac{5-2s}{\sin^{s-2}\theta}\right)d\theta. \tag{3.1}$$

Applying the inequality

$$a^{p_1}b^{p_2}\leq\left(\frac{p_1a+p_2b}{p_1+p_2}\right)^{p_1+p_2}$$

to the quantities

$$a=\frac{|v(\theta)|^p}{\sin^s\theta},b=p^m\frac{|v(\theta)|^{p-m}|v'(\theta)|^m}{\sin^{s-m}},p_1=1-\frac{2}{m}\text{ and }p_2=\frac{2}{m},$$

we have

$$\int_0^\pi\frac{|v(\theta)|^{p-2}v'(\theta)^2}{v_0(\theta)^{2(s-2)}}d\theta\leq\left(1-\frac{2}{m}\right)\int_0^\pi\frac{|v(\theta)|^p}{\sin^s\theta}d\theta+\frac{2p^m}{m}\int_0^\pi\frac{|v(\theta)|^{p-m}|v'(\theta)|^m}{\sin^{s-m}}. \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$\frac{2p^m}{m}\int_0^\pi\frac{|v(\theta)|^{p-m}|v'(\theta)|^m}{\sin^{s-m}\theta}\geq\left(2s-4+\frac{2}{m}\right)\int_0^\pi\frac{|v(\theta)|^p}{\sin^s\theta}d\theta+(5-2s)\int_0^\pi\frac{|v(\theta)|^p}{\sin^{s-2}\theta}d\theta.$$

□

Corollary 3.1. *Let $p\geq 2$ and $2-\frac{1}{p}\leq s\leq 2.5$. Then for all $u\in C_0^1[0,\pi]$ the following inequality*

$$\int_0^\pi\frac{|u'(\theta)|^p}{\sin^{s-p}\theta}\geq\frac{1}{p^{p-1}}\left(s-2+\frac{1}{p}\right)\int_0^\pi\frac{|u(\theta)|^p}{\sin^s\theta}d\theta+\frac{(5-2s)}{2}\int_0^\pi\frac{|u(\theta)|^p}{\sin^{s-2}\theta}d\theta$$

holds.

The substitution $\theta=\frac{\pi}{\ln 1/q}\ln r$, where $0<q\leq r<1$, yields

Corollary 3.2. *Let $q\in(0,1),0<q\leq r\leq 1,p\geq 2$ and $s\in[2-\frac{1}{p},2.5]$. For any absolutely continuous function u such that $u(q)=u(1),u\not\equiv 0$ the following inequality*

$$\int_q^1\frac{|u'(r)|^p}{\rho(r)^{s-p}}r^{s-1}dr\geq\frac{2^p}{p^{p-1}}\left(s-2+\frac{1}{p}\right)\int_q^1\frac{|u(r)|^p}{\rho(r)^s}r^{s-1}dr+\frac{2^{p-3}\pi^2(5-2s)}{p^{p-1}\ln^2 q}\int_q^1\frac{|u(r)|^p}{\rho(r)^{s-2}}r^{s-3}dr$$

holds, where $\rho(r)=\frac{2r\ln q}{\pi}\sin\frac{\pi\ln r}{\ln q}$.

4 Inequalities in domains

In this part, using the above one-dimensional inequalities, we obtain inequalities in plane domains. To proof the main results we need

Proposition 4.1. *Let $H_+ = \{\zeta = \xi + i\eta \in \mathbb{C} : \eta > 0\}$, $p \geq 2$ and $2 - \frac{1}{p} \leq s \leq 2.5$. Then for all $u \in C_0^1(H_+)$ the following inequality*

$$\iint\limits_{H+} \frac{|\nabla u|^p}{R(\zeta, Q)^{s-p}} d\xi d\eta \geq \frac{2^p}{p^{p-1}} \left(s - 2 + \frac{1}{p}\right) \iint\limits_{H+} \frac{|u(\zeta)|^p}{R(\zeta, Q)^s} d\xi d\eta + \frac{2^{p-3}(5-2s)}{p^{p-1}} \iint\limits_{H+} \frac{|u(\zeta)|^p}{|\zeta|^2 R(\zeta, Q)^{s-2}} d\xi d\eta$$

is valid.

Proof. Suppose that $\eta = \rho e^{i\varphi}$, $\rho = |\eta| > 0$, $\varphi \in (0, \pi)$, $u \in C_0^1(H_+)$ is a real valued function. By Corollary 3.1, we have

$$\int_0^\pi \left(\frac{1}{\rho^2}\left(\frac{\partial u}{\partial \varphi}\right)^2\right)^{p/2} \frac{\rho d\theta}{(2\rho \sin\theta)^{s-p}} \geq \frac{2^p}{p^{p-1}} \left(s - 2 + \frac{1}{p}\right) \int_0^\pi \frac{|u(\theta)|^p}{(2\rho\sin\theta)^s} \rho d\theta + \frac{2^{p-3}(5-2s)}{p^{p-1}} \int_0^\pi \frac{|u(\theta)|^p}{\rho^2 (2\rho\sin\theta)^{s-2}} \rho d\theta.$$

Integrating the last inequality with respect to ρ, we get

$$\iint\limits_{H+} \left(\frac{1}{\rho^2}\left(\frac{\partial u}{\partial \varphi}\right)^2\right)^{p/2} \frac{d\xi d\eta}{(2\rho \sin\theta)^{s-p}} \geq \frac{2^p}{p^{p-1}} \left(s - 2 + \frac{1}{p}\right) \iint\limits_{H+} \frac{|u(\theta)|^p}{(2\rho\sin\theta)^s} d\xi d\eta + \frac{2^{p-3}(5-2s)}{p^{p-1}} \iint\limits_{H+} \frac{|u(\theta)|^p}{\rho^2 (2\rho\sin\theta)^{s-2}} d\xi d\eta.$$

Using that $R(\rho e^{i\varphi}, H_+) = 2\eta = 2\rho\sin\varphi$ and

$$\left(\frac{\partial u}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial u}{\partial \varphi}\right)^2 = |\nabla u|^2 \geq \frac{1}{\rho^2}\left(\frac{\partial u}{\partial \varphi}\right)^2,$$

we obtain

$$\iint_{H+} \frac{|\nabla u|^p}{R(\zeta, Q)^{s-p}} d\xi d\eta \geq \frac{2^p}{p^{p-1}} \left(s - 2 + \frac{1}{p}\right) \iint_{H+} \frac{|u(\zeta)|^p}{R(\zeta, Q)^s} d\xi d\eta + \frac{2^{p-3}(5-2s)}{p^{p-1}} \iint_{H+} \frac{|u(\zeta)|^p}{|\zeta|^2 R(\zeta, Q)^{s-2}} d\xi d\eta.$$

□

Theorem 4.1. *Let $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, $p \geq 2$ and $2 - \frac{1}{p} \leq s \leq 2.5$. Then for all $f \in C_0^1(\mathbb{R}^n_+)$ the following inequality*

$$\int\limits_{\mathbb{R}^n_+} \frac{|\nabla f|^p}{x_n^{s-p}} dx \geq \frac{1}{p^{p-1}} \left(s - 2 + \frac{1}{p}\right) \int\limits_{\mathbb{R}^n_+} \frac{|f|^p}{x_n^s} dx + \frac{(5-2s)}{2p^{p-1}} \int\limits_{\mathbb{R}^n_+} \frac{|f|^p}{x_n^{s-2}|x_{n-1}^2 + x_n^2|} dx$$

is valid.

Proof. To obtain Theorem 4.1, we apply Proposition 4.1 to the function u defined by $u(x_{n-1}, x_n) := f(x_1, \dots, x_n)$ and integrate with respect to $(x_1, \dots, x_{n-2}) \in \mathbb{R}^{n-2}$. □

Proposition 4.2. *Let $A_q = \{\zeta = \mathbb{C} : q < |\zeta| < 1\}$, $p \geq 2$ and $2 - \frac{1}{p} \leq s \leq 2.5$. Then for all $u \in C_0^1(A_q)$ the following inequality*

$$\iint\limits_{A_q} \frac{|\nabla u|^p}{R(\zeta, A_q)^{2-p}} d\xi d\eta \geq \frac{2^p}{p^p} \iint\limits_{A_q} \frac{|u|^p}{R(\zeta, A_q)^2} d\xi d\eta + \left(\frac{1}{4M(A_q)}\right)^2 \frac{2^{p-1}}{p^{p-1}} \iint\limits_{A_q} \frac{|u|^p}{|\zeta|^2} d\xi d\eta$$

is valid, where $\eta = \xi + i\eta$.

Proof. Suppose that $\eta = \xi + i\eta = re^{i\theta}$ and $u \in C_0^1(A_q)$, where $A_q = \{\zeta = \mathbb{C} : q < |\zeta| < 1\}$. By Corollary 3.2 we get the following estimate

$$\int\limits_q^1 \frac{|u'(r)|^p}{\rho(r)^{2-p}} r dr \geq \frac{2^p}{p^p} \int\limits_q^1 \frac{|u(r)|^p}{\rho(r)^2} r dr + \left(\frac{\pi}{2\ln q}\right)^2 \frac{2^{p-1}}{p^{p-1}} \int\limits_q^1 \frac{|u(r)|^p}{r^2} r dr.$$

Integrating the last inequality with respect to $\theta \in [0, 2\pi]$, we obtain

$$\iint\limits_{A_q} \left|\frac{\partial u}{\partial r}\right|^p \frac{1}{\rho(r)^{2-p}} r dr d\theta \geq \frac{2^p}{p^p} \iint\limits_{A_q} \frac{|u|^p}{\rho(r)^2} r dr d\theta + \left(\frac{\pi}{\ln q}\right)^2 \frac{2^{p-3}}{p^{p-1}} \iint\limits_{A_q} \frac{|u|^p}{r^2} r dr d\theta.$$

Since $R(\zeta, A_q) = -\rho(|\zeta)|$, $M(A_q) = (2\pi)^{-1} \ln(1/q)$ and $|\partial u/\partial r| \leq |\nabla u|$, we immediately obtain

$$\iint\limits_{A_q} \frac{|\nabla u|^p}{R(\zeta, A_q)^{2-p}} d\xi d\eta \geq \frac{2^p}{p^p} \iint\limits_{A_q} \frac{|u|^p}{R(\zeta, A_q)^2} d\xi d\eta + \left(\frac{1}{4M(A_q)}\right)^2 \frac{2^{p-1}}{p^{p-1}} \iint\limits_{A_q} \frac{|u|^p}{|\zeta|^2} d\xi d\eta,$$

where $\zeta = \xi + i\eta$. □

Proof of Theorem 1.1. Let $F : \Omega \to \Omega_\zeta \subset \overline{\mathbb{C}}$ be a conformal mapping of a domain Ω in the z-plane onto a domain Ω_ζ in the ζ-plane. It is known (see, for example, [14]) that

$$\lambda_\Omega(z)|dz| \equiv \lambda_{\Omega_\zeta}(z)|d\zeta|, \quad \lambda_\Omega^2(z) dxdy = \lambda_{\Omega_\zeta}^2(z) d\xi d\eta,$$

where $z = x + iy \in \Omega$ and $\zeta = F(z) = \xi + i\eta \in \Omega_\zeta$.

Using the equalities $|F'(z)|^2 dxdy = d\xi d\eta$ and

$$\nabla U = 2\frac{\partial U\zeta}{\partial \overline{\zeta}} = 2\frac{\partial u(z)}{\partial \overline{z}} \overline{F'(z)} = (\nabla u)\overline{F'(z)},$$

it is easy to prove that

$$\iint\limits_{\Omega} \frac{|\nabla u|^p}{R(z, \Omega)^{2-p}} dxdy = \iint\limits_{\Omega_\zeta} \frac{|\nabla U|^p}{R(\zeta, \Omega)^{2-p}} d\xi d\eta$$

and

$$\iint\limits_{\Omega} \frac{|u|^p}{R(z, \Omega)^2} dxdy = \iint\limits_{\Omega_\zeta} \frac{|U|^p}{R(\zeta, \Omega_\zeta)^2} d\xi d\eta,$$

where $u \in C_0^1(\Omega)$ and $U := u \circ F^{-1} \in C_0^1(\Omega_\zeta)$.

Consequently, the two integrals

$$\iint_\Omega \frac{|\nabla u|^p}{R(z,\Omega)^{2-p}}dxdy \text{ and } \iint_\Omega \frac{|u|^p}{R(z,\Omega)^2}dxdy$$

are conformal invariant. That is to prove Theorem 1.1 in the case of simply connected domains we can assume that Ω is the half-space and in the case of doubly connected domains we can assume that Ω is an annuli. The applications of Proposition 4.1 and 4.2 yield Theorem 1.1. Note that the sharpness of the constant $2^p/p^p$ follows from [4]. $\square$

Now we apply Theorem 1.1 to the function u defined by $u(x_{n-1}, x_n) := f(x_1, \dots, x_n)$ and integrate with respect to $(x_1, \dots, x_{n-2}) \in \mathbb{R}^{n-2}$. We obtain

Theorem 4.2. 1) *Suppose that Ω_1 is a simply connected hyperbolic domain in $\mathbb{C}$, g is any univalent conformal mapping of Ω_1 onto the upper half-plane $H_+ = \{\zeta = \xi + i\eta \in \mathbb{C} : \eta > 0\}$ and $\Omega = \mathbb{R}^{n-2} \times \Omega_1$. Then for all real-valued function $f \in C_0^1(\Omega)$ and $p \geq 2$ the following Avkhadiev-Hardy-type inequality*

$$\int_\Omega \frac{|\nabla f|^p}{R(z,\Omega)^{2-p}}dx \geq \frac{2^p}{p^p}\int_\Omega \frac{|f|^p}{R(z,\Omega)^2}dx + \frac{2^{p-3}}{p^{p-1}}\int_\Omega |f|^p \left|\frac{g'(z)}{g(z)}\right|^2 dx$$

is valid, where $z = x_{n-1} + ix_n$.

2) *Let $\Omega_2 \subset \mathbb{C}$ be a doubly connected domain and let g be any univalent conformal mapping of Ω_2 onto the annuli $A_q = \{\eta \in \mathbb{C} : q < |\eta| < 1\}$, $q = \exp(-2\pi M(\Omega_2))$ and $\Omega = \mathbb{R}^{n-2} \times \Omega_2$. Then for any real-valued function $f \in C_0^1(\Omega)$ the following Avkhadiev-Hardy-type inequality*

$$\frac{p^p}{2^p}\int_\Omega \frac{|\nabla f|^p}{R(z,\Omega)^{2-p}}dx \geq \int_\Omega \frac{|f|^p}{R(z,\Omega)^2}dx + \frac{1}{16M(\Omega_2)^2}\frac{p}{2}\iint_\Omega |f|^p \left|\frac{g'(z)}{g(z)}\right|^2 dx$$

holds, where $z = x_{n-1} + ix_n$ and $M(\Omega_2)$ is the geometrical parameter defined as the supremum of the moduli of doubly connected domains lying in the domain Ω_2 and separating its boundary $\partial\Omega_2$.

Acknowledgments

The author thanks Professor F.G. Avkhadiev for constant attention to this work. This work is supported by the Russian Science Foundation under grant 18-11-00115.

References

[1] A. Alvino, R. Volpicelli, A. Ferone, *Sharp Hardy inequalities in the half space with trace remainder term.* Nonlinear Analysis: Theory, Methods and Applications, 75 (2012), no. 14, 5466–5472.

[2] F.G. Avkhadiev, *A geometric description of domains whose Hardy constant is equal to 1/4.* Izvestiya: Mathematics, 78 (2014), no. 5, 855–876.

[3] F.G. Avkhadiev, *Brezis-Marcus problem and its generalizations.* Complex analysis, Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. Pril. Temat. Obz., VINITI, 153 (2018), 3–12 (in Russian).

[4] F.G. Avkhadiev, *Integral inequalities in domains of hyperbolic type and their applications.* Sbornik: Mathematics, 206 (2015), no. 12, 1657–1681.

[5] F.G. Avkhadiev, *Hardy type inequalities in higher dimensions with explicit estimate of constants.* Lobachevskii J. Math., 21 (2006), 3–31.

[6] F.G. Avkhadiev, *Sharp Hardy constants for annuli.* Journal of Mathematical Analysis and Applications, 466 (2018), no. 1 , 936–951.

[7] F.G. Avkhadiev, *Hardy-type inequalities on planar and spatial open sets.* Proceedings of the Steklov Institute of Mathematics, 255 (2006), no. 1 , 2–12.

[8] F.G. Avkhadiev, R.G. Nasibullin, *Hardy-type inequalities in arbitrary domains with finite inner radius.* Siberian Mathematical Journal, 55 (2014), no. 2, 191–200.

[9] F.G. Avkhadiev, I.K. Shafigullin, *Estimates of Hardy's constants for tubular extensions of sets and domains with finite boundary moments.* Sib. Adv. Math. 24 (2014), no. 3, 153–158.

[10] F.G. Avkhadiev, R.G. Nasibullin, I.K. Shafigullin, *Hardy-type inequalities with power and logarithmic weights in domains of the Euclidean space.* Russian Mathematics, 55 (2011), no. 9, 76–79.

[11] F.G. Avkhadiev, R.G. Nasibullin, I.K. Shafigullin, *L_p-versions of one conformally invariant inequality.* Russian Mathematics, 62 (2018), no. 8, 76–79.

[12] F.G. Avkhadiev, K.-J. Wirths, *Unified Poincaré and Hardy inequalities with sharp constants for convex domains.* Z. Angew. Math. Mech., 87 (2007), 632–642.

[13] F.G. Avkhadiev, K.-J. Wirths, *Sharp Hardy-type inequalities with Lamb's constants.* Bull. Belg. Math. Soc. Simon Stevin, 18 (2011), 723–736.

[14] F.G. Avkhadiev, K.-J. Wirths, *Schwarz-Pick type inequalities.* Birkhäuser, Boston-Berlin-Bern, 2009.

[15] A.A. Balinsky, W.D. Evans, R.T. Lewis, *The analysis and geometry of Hardy's inequality.* Universitext, Springer, Heidelberg - New York - Dordrecht - London, 2015.

[16] G. Barbatis, S. Filippas, A. Tertikas, *Series expansion for L^p Hardy inequalities.* Indiana Univ. Math. J., 52 (2003), no. 1 , 171–190.

[17] H. Brezis, M. Marcus, *Hardy's inequality revisited.* Ann. Scuola Norm.Sup. Pisa Cl. Sci., (4). 25 (1997), no. 1-2, 217–237.

[18] V.I. Burenkov, A. Senouci, T.V. Tararykova, *Hardy-type inequality for $0 < p < 1$ and hypodecreasing functions.* Eurasian Mathematical Journal, 1 (2010), no. 3, 27 – 42.

[19] V.I. Burenkov, *Function spaces. Main integral inequalities related to L_p-spaces.* Peoples' Friendship University, Moscow, 1989 (in Russian).

[20] V.I. Burenkov, *On the exact constant in the Hardy inequality with $0 < p < 1$ for monotone functions.* Proc. Steklov Inst. Math., 194 (1993), no. 4 , 59 – 63.

[21] E.B. Davies, *The Hardy constant.* Quart. J. Math. Oxford (2), 46 (1995), no. 4, 417–431.

[22] E.B. Davies, *Spectral theory and differential operators.* Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, Cambridge, 1995.

[23] W.D. Evans, R.T. Lewis, *Hardy and Rellich inequalities with remainders.* J. Math. Inequal., 1 (2007), no. 4 , 473–490.

[24] J.L. Fernández, J.M. Rodríguez, *The exponent of convergence of Riemann surfaces, bass Riemann surfaces.* Ann. Acad. Sci. Fenn. Series A. I. Mathematica, 15 (1990), 165–182.

[25] S. Filippas, V.G. Maz'ya, A. Tertikas, *On a question of Brezis and Marcus.* Calc. Var. Partial Differential Equations, 25 (2006), no. 4, 491–501.

[26] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev, *A geometrical version of Hardy's inequality.* J. Funct. Anal., 189 (2002), no. 2 , 539–548.

[27] V.G. Maz'ya, *Sobolev spaces.* Springer, 1985.

[28] R.G. Nasibullin, *A geometrical version of Hardy-Rellich type inequalities.* Mathematica Slovaca, 69 (2019), no. 4, 785–800.

[29] R.G. Nasibullin, *Multidimensional Hardy type inequalities with remainders.* Lobachevskii Journal of Mathematics, 40 (2019), no. 9, 1383-1396.

[30] V.H. Nguyen, *Some trace Hardy type inequalities and trace Hardy-Sobolev-Maz'ya type inequalities.* Journal of Functional Analysis, 270 (2016), no. 11 , 4117–4151.

[31] M. Ruzhansky, D. Suragan, *Hardy inequalities on homogeneous groups.* Progress in Mathematics, Vol. 327, Birkhauser, 2019.

[32] M. Ruzhansky, B. Sabitbek, D. Suragan, *Geometric Hardy and Hardy-Sobolev inequalities on Heisenberg groups.* Bulletin of Mathematical Sciences, 2050016 (2020). DOI: 10.1142/S1664360720500162

[33] A. Senouci, N. Azzouz, *Hardy type inequality with sharp constant for* $0 < p < 1$. Eurasian Math. J., 10 (2019), no. 1, 52–58.

[34] A. Senouci, A. Zanou, *Some integral inequalities for quasimonotone functions in weighted variable exponent Lebesgue space with* $0 < p(x) < 1$. Eurasian Math. J., 11 (2020), no. 4, 58–65.

[35] D. Sullivan, *Related aspects of positivity in Riemannian geometry.* J. Differential Geom., 25 (1987), no. 3 , 327–351.

[36] J. Tidblom, *A geometrical version of Hardy's inequality for* $W_0^{1,p}(\Omega)$. Proc. Amer. Math. Soc., 132 (2004), 2265–2271.

[37] J. Tidblom, *A Hardy inequality in the half-space.* Journal of Functional Analysis, 221 (2005), no. 2, 482–495.

[38] K. Tzirakis, *Improving interpolated Hardy and trace Hardy inequalities on bounded domains.* Nonlinear Analysis: Theory, Methods and Applications, 127 (2015), 17–34.

Ramil Gaisaevich Nasibullin
N.I. Lobachevsky Institute of Mathematics and Mechanics,
Kazan Federal University,
18 Kremlevskaya St
420008, Kazan, Tatarstan, Russia,
E-mail: NasibullinRamil@gmail.com

Received: 19.03.2020