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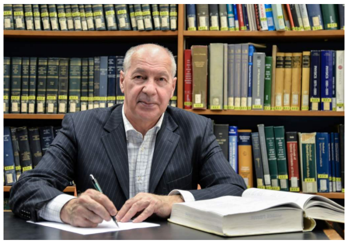
VICTOR IVANOVICH BURENKOV

(to the 80th birthday)



On July 15, 2021 was the 80th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), professor of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (Moscow), chairman of the Dissertation Council at the RUDN University, research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), honorary academician of the National Academy of Sciences of the Republic of Kazakhstan, doctor of physical and mathematical sciences(1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK,2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University. Through 2015-2017 he was head of the Department of Mathematical Analysis and Theory of Functions (RUDN University). He was one of the organisers and the first director of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (2016-2017).

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem on composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 190 scientific papers. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces. In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

V.I. Burenkov is still very active in research. Through 2016-2021 he published 20 papers in leading mathematical journals.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 80th birthday and wishes him good health and new achievements in science and teaching!

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ON MODULAR INEQUALITIES FOR GENERALIZED HARDY OPERATORS ON WEIGHTED ORLICZ SPACES

Kh. Almohammad

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Key words: weighted Orlicz spaces, modular and norm inequalities, cone of decreasing functions, reduction theorems.

AMS Mathematics Subject Classification: 46E30, 42A16.

Abstract. The purpose of this paper is to study the behaviour of Hardy-type operator on weighted Orlicz spaces. The results on modular inequalities for the considered operators of Hardy type are important, since such operators arise in the study of decreasing rearrangements for generalized Bessel and Riesz potentials, in which case Orlicz–Lorentz space serves as an underlying space.

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1 Introduction

The purpose of this paper is to study the behaviour of Hardy-type integral operators

$$\mathcal{F}_\varphi[g](t) = \int_0^\infty f_\varphi(t;\tau) g(\tau)\,\mathrm{d}\tau$$

on weighted Orlicz spaces, see [1, 3, 7, 8, 10].

Here φ is positive, decreasing and continuous on $(0,\infty)$,

$$f_\varphi(t;\tau) = \min\{\varphi(t), \varphi(\tau)\}.$$

Such operators appear when we study integral properties of generalized Bessel and Riesz potentials defined as convolutions of kernels of potentials with functions belonging to Orlicz spaces, see [4, 6, 9]. They were introduced by Z. W. Birnbaum and W. Orlicz in the 1920s and have been widely studied since the 1930s. The book [2] by M. A. Krasnosel'skii and Ya. B. Rutickii in 1961 acted as a catalyst in the theory and gained these spaces wide acceptance in many areas of analysis. This large class of spaces, which includes Lebesgue L_p - spaces, has been effectively used in the theory of differential and integral equations, probability, statistics, and harmonic analysis.

This paper is devoted to studying the behaviour of integral operators on weighted Orlicz spaces with emphasis on new features of this situation as compared with the case of Lebesgue spaces. It is natural to consider embeddings of spaces along with boundedness of operators and we address those topics as well.

2 Auxiliary definitions

Definition 1. (i) A Banach function space, shortly BFS, $E = E(\mathbb{R}^n)$ is a Banach space of Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ with monotone norm, i.e. such that

$$|f| \le g, \quad g \in E \quad \text{implies } f \in E, \quad \|f\|_E \le \|g\|_E, \tag{2.1}$$

and with the Fatou property:

$$0 \le f_n \uparrow f, \quad f_n \in E \quad \text{implies } f \in E, \quad \|f_n\|_E \uparrow \|f\|_E.$$

(ii) A BFS E is called a rearrangement-invariant space, shortly: RIS, if its norm is monotone with respect to rearrangements,

$$f^* \le g^*, \quad g \in E \quad \text{implies } f \in E, \quad \|f\|_E \le \|g\|_E. \tag{2.2}$$

Here f^* is the decreasing rearrangement of the function f, i.e. a nonnegative decreasing right continuous function on $\mathbb{R}_+ = (0, \infty)$, which is equimeasurable with f:

$$\mu_n \{x \in \mathbb{R}^n : \ |f(x)| > y\} = \mu_1 \{t \in \mathbb{R}_+ : \ |f^*(t)| > y\}, \quad y \in \mathbb{R}_+, \tag{2.3}$$

where μ_n is the n–dimensional Lebesgue measure.

Definition 2. The potential space $H_E^G \equiv H_E^G(\mathbb{R}^n)$ on the n-dimensional Euclidean space $\mathbb{R}^n$ is defined by

$$H_E^G(\mathbb{R}^n) = \big\{u = G * f : \ f \in E(\mathbb{R}^n)\big\}. \tag{2.4}$$

Here $E(\mathbb{R}^n)$ is a rearrangement-invariant space, and

$$\|u\|_{H_E^G} = \inf\big\{\|f\|_E : \ f \in E(\mathbb{R}^n), \ G * f = u\big\}. \tag{2.5}$$

We assume that the kernel G of a representation (2.4) is admissible, i.e

$$G \in L_1(\mathbb{R}^n) + E'(\mathbb{R}^n).$$

Here the convolution $G * f$ is defined as the integral

$$(G * f)(x) = \int\limits_{\mathbb{R}^n} G(x - y) f(y) \, \mathrm{d}\mu_n(y). \tag{2.6}$$

Moreover, $E'(\mathbb{R}^n)$ is the associated RIS, i.e. RIS with the norm:

$$\|g\|_{E'} = \sup\bigg\{\int_{\mathbb{R}^n} |fg| \, \mathrm{d}\mu : \ f \in E, \ \|f\|_E \le 1\bigg\}. \tag{2.7}$$

Remark 1.

$$E = L_p, \quad 1 \le p \le \infty \quad \Rightarrow \quad E' = L_{p'}; \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

For the RIS $E(\mathbb{R}^n)$, $E'(\mathbb{R}^n)$, we consider the spaces $\tilde{E} = \tilde{E}(\mathbb{R}_+)$, $\tilde{E}' = \tilde{E}'(\mathbb{R}_+)$ — their Luxemburg representations [1], i.e. RIS for which the following equalities are satisfied

$$\|f\|_E = \|f^*\|_{\tilde{E}}, \quad f \in E(\mathbb{R}^n); \quad \|g\|_{E'} = \|g^*\|_{\tilde{E}'}, \quad g \in E'(\mathbb{R}^n).$$

We denote:

$$f^{**}(t) = \frac{1}{t} \int\limits_0^t f^*(\tau) \, \mathrm{d}\tau; \quad t \in \mathbb{R}_+. \tag{2.8}$$

We introduce the class of monotone functions $\mathfrak{I}_n(R)$, $R > 0$ as follows. A function θ: $(0, R) \to \mathbb{R}_+$ belongs to the class $\mathfrak{I}_n(R)$ if θ satisfies the following conditions:

1. θ is decreasing and continuous on $(0, R)$,

2. there is a constant $c \in \mathbb{R}_+$, such that

$$\int_0^r \theta(\rho)\rho^{n-1}\,\mathrm{d}\rho \le c\theta(r)r^n, \quad r \in (0, R). \tag{2.9}$$

Now, we introduce

$$\varphi(\tau) = \theta\left(\left(\frac{\tau}{V_n}\right)^{\frac{1}{n}}\right) \in \mathfrak{I}_1(T), \quad T = V_n R^n.$$

where V_n is the volume of the unit ball in $\mathbb{R}^n$.

$$f_\varphi(t;\tau) = \min\{\varphi(t), \varphi(\tau)\} = \begin{cases} \varphi(t), & 0 < \tau \le t, \\ \varphi(\tau), & \tau > t. \end{cases} \tag{2.10}$$

Definition 3. Let $\theta \in \mathfrak{I}_n(\infty)$. The potentials $u \in H_E^G(\mathbb{R}^n)$ are called generalized Riesz potentials, if

$$G(x) \cong \theta(|x|), \quad x \in \mathbb{R}^n, \quad (\cong \text{ means two-sided estimate}).$$

Definition 4. Let $\theta \in \mathfrak{I}_n(R)$. The potentials $u \in H_E^G(\mathbb{R}^n)$ are called generalized Bessel potentials, if

$$G(x) = G_R^0(x) + G_R^1(x),$$

$$B_R = \{x \in \mathbb{R}^n : |x| < R\}, \quad R \in \mathbb{R}_+,$$
$$G_R^0(x) = G(x)\chi_{B_R(x)}, \quad G_R^1(x) = G(x)\chi_{B_R^c(x)},$$
$$G_R^0(x) \cong \theta(|x|), \quad x \in B_R, \quad G_R^1(x) \in (L_1 \cap E')(\mathbb{R}^n), \quad \int_{R^n} G\,\mathrm{d}x \ne 0.$$

Definition 5. A function Φ: $[0, +\infty) \to [0, +\infty)$ is called an N-function if $\Phi(t) = \int_0^t \phi(\tau)\,\mathrm{d}\tau$; where ϕ is continuous, $0 < \phi \uparrow$; $\phi(0) = 0$; $\phi(\infty) = \infty$. Let ϕ^{-1} be the right continuous inverse function of ϕ, and define

$$\Psi(t) = \int_0^t \phi^{-1}(\tau)\,\mathrm{d}\tau.$$

Ψ is called the complementary function of Φ.

Definition 6. a) An N-function Φ is said to satisfy the Δ_2 condition (we write $\Phi \in \Delta_2$) if there is a constant $B > 0$, such that

$$\Phi(2t) \le B\Phi(t), \quad \forall t > 0. \tag{2.11}$$

b) We write $\Phi_1 \ll \Phi_2$ if there is a constant $L_0 > 0$, such that the inequality

$$\sum_{i=1}^\infty \Phi_2 \circ \Phi_1^{-1}(a_i) \le L_0 \Phi_2 \circ \Phi_1^{-1}\left(\sum_{i=1}^\infty a_i\right) \tag{2.12}$$

holds for every sequence $\{a_i\}$ with $a_i \ge 0$.

c) Let ω be a positive, measurable weight function and Φ be an N-function. The Orlicz space $L_\Phi(\omega)$ consists of all measurable functions f (modulo equivalence almost everywhere) with

$$\|f\|_{\Phi(\omega)} = \inf\Big\{ \lambda > 0,\ \int_0^\infty \Phi(\lambda^{-1}|f(x)|)\omega(x)\,\mathrm{d}x \le 1 \Big\}. \tag{2.13}$$

We call $\|\cdot\|_{\Phi(\omega)}$ the Luxemburg norm.

The Orlicz norm of a function f is given by

$$\|f\|'_{\Psi(\omega)} = \sup\Big\{ \int_0^\infty |fg|\omega dx : \int_0^\infty \Psi(|g|)\omega dx \le 1 \Big\}, \tag{2.14}$$

where Ψ is the complementary function of Φ.

Remark 2. $L_\Phi(\omega)$ is a Banach space and the Luxemburg and Orlicz norms are equivalent. In fact,

$$\|f\|_{\Phi(\omega)} \le \|f\|'_{\Psi(\omega)} \le 2\|f\|_{\Phi(\omega)}.$$

3 Equivalent descriptions of cones of rearrangements

In this section we show how Hardy-type operators appear in the theory of potentials. Let G be an admissible kernel. We consider the following cones of rearrangements, equipped with a positive homogeneous functional:

$$M \equiv M_E^G = \{h(t) = u^*(t), \quad t \in (0,\infty) : u \in H_E^G \}. \tag{3.1}$$
$$\rho_M(h) = \inf\{\|u\|_{H_E^G} : u \in H_E^G,\ u^*(t) = h(t), \quad t \in (0,\infty)\}.$$

Definition 7. Consider a cone

$$K \equiv K_E^\varphi = \left\{ \begin{array}{c} h(t) = \int\limits_0^\infty f_\varphi(t;\tau) g(\tau)\,\mathrm{d}\tau \\ t \in (0,\infty) \ :\ g \in \tilde{E}_0(0,\infty) \end{array} \right\} \tag{3.2}$$

equipped with the functional

$$\rho_K(h) = \|g\|_{\tilde{E}(0,\infty)}, \tilde{E}_0(0,\infty) = \big\{ g \in \tilde{E}(0,\infty) : 0 \le g \downarrow,\ g(t+0) = g(t),\ t \in (0,\infty) \big\}.$$

A cone K covers a cone M (and we write $M \prec K$), if there exists $c_1 \in \mathbb{R}_+$, such that, for every function $h_1 \in M$, there exists a function $h_2 \in K$, satisfying the conditions

$$\rho_K(h_2) \le c_1 \rho_M(h_1),$$
$$h_1(t) \le h_2(t) \quad t \in (0,\infty).$$

Remark 3. The equivalence of cones means the mutual covering:

$$M \approx K \Leftrightarrow M \prec K \prec M. \tag{3.3}$$

Theorem 3.1. *Let $\theta \in \Im_n(\infty)$, $E(\mathbb{R}^n)$ be an RIS, and the following conditions be fulfilled:*

$$G(x) \cong \theta(|x|), \qquad x \in \mathbb{R}^n; \quad f_\varphi(t;\cdot) \in \tilde{E}'(\mathbb{R}_+), t \in \mathbb{R}_+.$$

Then for the generalized Riesz potenial,

$$M_E^G \approx K_E^\varphi.$$

.

4 Auxiliary theorems

Definition 8. Generalized Hardy operators are operators of the form

$$\mathfrak{R}f(x) = \int_0^x k(x,t)f(t)\,\mathrm{d}t, \quad \mathfrak{R}^*g(t) = \int_t^{+\infty} k(x,t)g(x)\,\mathrm{d}x, \tag{4.1}$$

where
a) $k : \{(x,t) \in \mathbb{R}^2 : 0 < t < x < +\infty\} \to [0,+\infty)$;
b) $k(x,t) \geq 0$ is nondecreasing in x, nonincreasing in t;
c) $k(x,y) \leq D\big(k(x,t) + k(t,y)\big)$, whenever $0 \leq y \leq t < x < +\infty$ for some constant D.

Theorem 4.1. (see [3]). *Let Φ_1, Φ_2 be N-functions, $\Phi_1 \ll \Phi_2$ (see Definition 6), $\mathfrak{R}$ be a generalized Hardy operator defined by (4.1). Let a, b, v and ω be positive weight functions. Then there exists a constant $A > 0$ such that the inequality*

$$\Phi_2^{-1}\left(\int_0^{+\infty} \Phi_2(a\mathfrak{R}f)\omega dx\right) \leq \Phi_1^{-1}\left(\int_0^{+\infty} \Phi_1(Afb)v dx\right) \tag{4.2}$$

holds for all non-negative, measurable functions f if and only if there exists a constant $C > 0$ such that the inequalities

$$\Phi_2^{-1}\left(\int_r^{+\infty} \Phi_2\left(\frac{a(x)}{C}\left\|\frac{k(r;\cdot)\chi_{(0,r)}(.)}{\varepsilon v b}\right\|_{\Psi_1(\varepsilon v)}\right)\omega(x)\,\mathrm{d}x\right) \leq \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right) \tag{4.3}$$

and

$$\Phi_2^{-1}\left(\int_r^{+\infty} \Phi_2\left(\frac{a(x)}{C}\left\|\frac{\chi_{(0,r)}(.)}{\varepsilon v b}\right\|_{\Psi_1(\varepsilon v)} k(x;r)\right)\omega(x)\,\mathrm{d}x\right) \leq \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right) \tag{4.4}$$

hold for all ε, $r > 0$.

Theorem 4.2. (see [3])*: Let Φ_1, Φ_2 be N-functions, $\Phi_1 \ll \Phi_2$, and $\mathfrak{R}^*$ be a generalized Hardy operator defined by (4.1). Then there exists a constant $A > 0$ such that the inequality*

$$\Phi_2^{-1}\left(\int_0^{+\infty} \Phi_2(a\mathfrak{R}^*f\omega dt\right) \leq \Phi_1^{-1}\left(\int_0^{+\infty} \Phi_1(Abf)v dt\right) \tag{4.5}$$

holds for all nonnegative, measurable functions f if and only if there exists a constant $C > 0$ such that the inequalities

$$\Phi_2^{-1}\left(\int_0^{r} \Phi_2\left(\frac{a(t)}{C}\left\|\frac{k(\cdot;r)\chi_{(r,+\infty)}(.)}{\varepsilon v b}\right\|_{\Psi_1(\varepsilon v)}\right)\omega(t)\,\mathrm{d}t\right) \leq \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right) \tag{4.6}$$

and

$$\Phi_2^{-1}\left(\int_0^{r} \Phi_2\left(\frac{a(t)}{C}\left\|\frac{\chi_{(r,+\infty)}(.)}{\varepsilon v b}\right\|_{\Psi_1(\varepsilon v)} k(r;t)\right)\omega(t)\,\mathrm{d}t\right) \leq \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right) \tag{4.7}$$

hold for all ε, $r > 0$.

5 Main result

Recall that

$$K_E^{\varphi} = \left\{ h(x) = \int_0^{\infty} f_{\varphi}(x;\tau) g(\tau)\,\mathrm{d}\tau; \quad x \in (0,\infty) : \, g \in \tilde{E}_0(0,\infty) \right\};$$

$$f_{\varphi}(x;\tau) = \begin{cases} \varphi(x), & 0 < \tau \le x, \\ \varphi(\tau), & \tau > x; \end{cases}$$

$$0 < \varphi \downarrow, \qquad \int_0^x \varphi(\tau)\,\mathrm{d}\tau \le c\varphi(x)x.$$

We consider the operator

$$\mathcal{F}_{\varphi}[g] = \int_0^{\infty} f_{\varphi}(x;\tau) g(\tau)\,\mathrm{d}\tau, \tag{5.1}$$

$$\mathcal{F}_{\varphi}[g] = \varphi(x) \int_0^x g(\tau)\,\mathrm{d}\tau + \int_x^{\infty} \varphi(\tau) g(\tau)\,\mathrm{d}\tau = \mathcal{F}_{\varphi 1}[g] + \mathcal{F}_{\varphi 2}[g]. \tag{5.2}$$

This means that $\mathcal{F}_{\varphi}$ is the sum of two Hardy-type operators.

Theorem 5.1. *Let Φ_1, Φ_2 be N-functions, $\Phi_1 \ll \Phi_2$, a, b, v, ω be positive weight functions, $\mathcal{F}_{\varphi}$ be defined by* (5.1). *Then there exists a constant $A > 0$ such that the inequality*

$$\Phi_2^{-1}\left(\int_0^{+\infty} \Phi_2(a(x)\mathcal{F}_{\varphi} g(x))\omega(x)dx \right) \le \Phi_1^{-1}\left(\int_0^{+\infty} \Phi_1(Ab(x)g(x))v(x)dx \right) \tag{5.3}$$

holds for all nonnegative, measurable functions g if and only if there exists a constant $C > 0$ such that the inequality

$$\Phi_2^{-1}\left(\int_0^{+\infty} \Phi_2\left(\frac{a(x)}{C} . \frac{f_{\varphi}(x;r)}{\varphi(r)} \left\| \frac{f_{\varphi}(\cdot;r)}{\varepsilon v b} \right\|_{\Psi_1(\varepsilon v)} \right) \omega(x)\,\mathrm{d}x \right) \le \Phi_1^{-1}\left(\frac{1}{\varepsilon} \right) \tag{5.4}$$

holds for all ε, $r > 0$.

Proof. **The necessity of condition** (5.4).

Fix r, $\varepsilon > 0$. Since the Orlicz norm does not exceed the Luxemburg norm (see Remark 2) we have for $f \ge 0$

$$\|f\|_{\Psi_1(\varepsilon v)} \le \|f\|'_{\Phi_1(\varepsilon v)} = \sup\left\{ \int_0^{+\infty} f h \varepsilon v \mathrm{d}x : h \ge 0; \int_0^{+\infty} \Phi_1(h)\varepsilon v dx \le 1 \right\},$$

hence,

$$\left\|\frac{f_\varphi(\cdot;r)}{\varepsilon bv}\right\|_{\Psi_1(\varepsilon v)} \leq \sup_{\int\limits_0^{+\infty}\Phi_1(Ag(\tau)b(\tau))\varepsilon v(\tau)d\tau\leq 1} \int\limits_0^{+\infty}\frac{f_\varphi(\tau;r)}{\varepsilon b(\tau)v(\tau)}Ag(\tau)b(\tau)\varepsilon v(\tau)\,\mathrm{d}\tau$$

$$= \sup_{\int\limits_0^{+\infty}\Phi_1(Ag(\tau)b(\tau))\varepsilon v(\tau)d\tau\leq 1} A\int\limits_0^{+\infty} f_\varphi(\tau;r)g(\tau)\,\mathrm{d}\tau.$$

So, for any $\eta<1$, it is possible to choose a nonnegative function g such that

$$\int\limits_0^{+\infty}\Phi_1(Ag(\tau)b(\tau))\varepsilon v(\tau)d\tau\leq 1$$

and

$$A\int\limits_0^{+\infty} f_\varphi(\tau;r)g(\tau)\,\mathrm{d}\tau \geq \eta\left\|\frac{f_\varphi(\cdot;r)}{\varepsilon bv}\right\|_{\Psi_1(\varepsilon v)}$$

Thus, we have

$$\Phi_2^{-1}\left(\int\limits_0^{+\infty}\Phi_2\left(\eta\frac{a(x)}{A}\frac{f_\varphi(x;r)}{\varphi(r)}\left\|\frac{f_\varphi(\cdot;r)}{\varepsilon bv}\right\|_{\Psi_1(\varepsilon v)}\right)\omega(x)\,\mathrm{d}x\right)\leq$$

$$\leq\Phi_2^{-1}\left(\int\limits_0^{+\infty}\Phi_2\left(a(x)\int\limits_0^{+\infty} f_\varphi(\tau;r)g(\tau)\,\mathrm{d}\tau\frac{f_\varphi(x;r)}{\varphi(r)}\right)\omega(x)\,\mathrm{d}x\right). \quad (5.5)$$

Since

$$\frac{f_\varphi(x;r)f_\varphi(\tau;r)}{\varphi(r)}\leq f_\varphi(\tau;x)$$

$\Big($indeed,
if $0<r<\tau<x$, then $\frac{f_\varphi(x,r)\cdot f_\varphi(\tau,r)}{\varphi(r)}=\frac{\varphi(x)\cdot\varphi(\tau)}{\varphi(r)}\leq\varphi(x)=f_\varphi(\tau,x)$,
if $0<\tau<r<x$, then $\frac{f_\varphi(x,r)\cdot f_\varphi(\tau,r)}{\varphi(r)}=\frac{\varphi(x)\cdot\varphi(r)}{\varphi(r)}=\varphi(x)=f_\varphi(\tau,x)\Big)$,
for any r,τ, $t>0$, the right-hand side of (5.5) does not exceed

$$\Phi_2^{-1}\left(\int\limits_0^{+\infty}\Phi_2(a(x)\mathcal{F}_\varphi g(x))\omega(x)dx\right).$$

Hypothesis (5.3) shows that the last expression is not greater than

$$\Phi_1^{-1}\left(\int\limits_0^{+\infty}\Phi_1(Ag(x)b(x))v(x)dx\right)\leq\Phi_1^{-1}\left(\frac{1}{\varepsilon}\right).$$

The necessity of condition (5.4) is proved with the constant $C=\frac{A}{\eta}$.

The sufficiency of condition (5.4).

Since by (5.2)

$$\mathcal{F}_\varphi[g](x) = \int\limits_0^\infty f_\varphi(x;\tau)g(\tau)\,\mathrm{d}\tau = \varphi(x)\int\limits_0^x g(\tau)\,\mathrm{d}\tau + \int\limits_x^\infty \varphi(\tau)g(\tau)\,\mathrm{d}\tau = \mathcal{F}_{\varphi 1}[g] + \mathcal{F}_{\varphi 2}[g],$$

we only need to prove that there exists a constant $A>0$, such that the inequalities

$$\Phi_2^{-1}\left(\int\limits_0^{+\infty}\Phi_2(a(x)\mathcal{F}_{\varphi_i}g(x))\omega(x)dx\right) \le \Phi_1^{-1}\left(\int\limits_0^{+\infty}\Phi_1(Ag(x)b(x))v(x)dx\right)$$

hold for all nonnegative measurable functions g, for $i=1,2$.

For $\mathcal{F}_{\varphi 1}$, we apply Theorem 4.1 with $k\equiv 1$, and $a(x)$ replaced by $a(x)\cdot\varphi(x)$. When $k\equiv 1$ it suffices to show that there is a constant $C>0$ such that the inequality

$$\Phi_2^{-1}\left(\int\limits_r^{+\infty}\Phi_2\left(\frac{a(x)\varphi(x)}{C}\left\|\frac{\chi_{(0,r)}}{\varepsilon bv}\right\|_{\Psi_1(\varepsilon v)}\right)\omega(x)\,\mathrm{d}x\right) \le \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right)$$

holds for any ε, $r>0$. To do this, fix ε, $r>0$, and notice that

$$1 = \frac{f_\varphi(\tau;r)}{\varphi(r)}$$

and

$$\varphi(x) = f_\varphi(x;r)$$

whenever $0\le\tau\le r\le x<+\infty$. We have

$$\begin{aligned}
&\Phi_2^{-1}\left(\int\limits_r^{+\infty}\Phi_2\left(\frac{a(x)}{C}\varphi(x)\left\|\frac{\chi_{(0,r)}}{\varepsilon bv}\right\|_{\Psi_1(\varepsilon v)}\right)\omega(x)\,\mathrm{d}x\right)\\
&\quad= \Phi_2^{-1}\left(\int\limits_r^{+\infty}\Phi_2\left(\frac{a(x)}{C}\cdot f_\varphi(x;r)\left\|\frac{\chi_{(0,r)}(\cdot)f_\varphi(\cdot;r)}{\varepsilon bv\cdot\varphi(r)}\right\|_{\Psi_1(\varepsilon v)}\right)\omega(x)\,\mathrm{d}x\right)\\
&\qquad\le \Phi_2^{-1}\left(\int\limits_0^{+\infty}\Phi_2\left(\frac{a(x)}{C}\cdot\frac{f_\varphi(x;r)}{\varphi(r)}\left\|\frac{\chi_{(0,r)}(\cdot)f_\varphi(\cdot;r)}{\varepsilon bv}\right\|_{\Psi_1(\varepsilon v)}\right)\omega(x)\,\mathrm{d}x\right).
\end{aligned}$$

Condition (5.4) of the theorem implies that the right-hand side of the above inequality is dominated by $\Phi_1^{-1}\left(\frac{1}{\varepsilon}\right)$ as required. For $\mathcal{F}_{\varphi 2}$ we apply Theorem 4.2 with $k\equiv 1$, and $b(\tau)$ replaced by $b(\tau)/\varphi(\tau)$ to the function $g(\tau)\cdot\varphi(\tau)$ by the conditions of the Theorem. When $k\equiv 1$ it is suffices to show that there is a constant $C>0$ such that the inequality

$$\Phi_2^{-1}\left(\int\limits_0^{r}\Phi_2\left(\frac{a(x)}{C}\left\|\frac{\chi_{(r,+\infty)}\cdot\varphi}{\varepsilon bv}\right\|_{\Psi_1(\varepsilon v)}\right)\omega(x)\,\mathrm{d}x\right) \le \Phi_1^{-1}\left(\frac{1}{\varepsilon}\right)$$

holds for any ε, $r>0$. We note that if $0\le x\le r\le\tau<+\infty$, then

$$\frac{f_\varphi(x;r)\cdot f_\varphi(\tau;r)}{\varphi(r)} = \varphi(\tau),$$

and we proceed as we did for $\mathcal{F}_{\varphi 1}$ to deduce the required inequality from condition (5.4) of the Theorem. The details are omitted. This completes the proof of sufficiency. □

6 Application to weighted Lebesgue spaces

Here we apply Theorem 5.1 to the case of weighted Lebesgue spaces. Let $1 \le p, q < \infty$.

$$\Phi_1(t) = t^p; \quad \Phi_2(t) = t^q; \quad \Psi_1(t) = t^{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1;$$

functions a, b, v, ω be as in Theorem 5.1. Then,

$$\Phi_1^{-1}(\tau) = \tau^{\frac{1}{p}}; \quad \Phi_2^{-1}(\tau) = \tau^{\frac{1}{q}};$$
$$\Phi_1 \ll \Phi_2 \Leftrightarrow p \le q.$$

Application of Theorem 5.1 gives the following result.

Let $1 \le p \le q < \infty$. Then there exists a constant $A > 0$ such that the inequality

$$\left(\int\limits_0^{+\infty} a^q(x) (\mathcal{F}_\varphi g)^q(x) \omega(x) \, \mathrm{d}x \right)^{\frac{1}{q}} \le A \left(\int\limits_0^{+\infty} \big(b^p(x) g^p(x) \big) v(x) \, \mathrm{d}x \right)^{\frac{1}{p}}, \tag{6.1}$$

holds for all nonnegative measurable functions g if and only if there exists a constant $C > 0$ such that the inequality

$$\left(\int\limits_0^{+\infty} a(x)^q f_\varphi(x; r)^q \omega(x) \, \mathrm{d}x \right)^{\frac{1}{q}} \left\| \frac{f_\varphi(\cdot; r)}{vb} \right\|^q_{L_{p'}(v)} \le C \varphi(r) \tag{6.2}$$

holds for all $r > 0$.

Remark 4. We consider here the estimates for Hardy-type operators that are applicable for studying the integral properties of generalized Bessel Potentials. Note that the differential properties of such potentials were considered in the paper [5]

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