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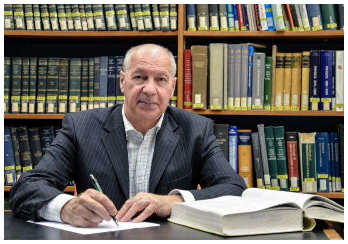
VICTOR IVANOVICH BURENKOV

(to the 80th birthday)



On July 15, 2021 was the 80th birthday of Victor Ivanovich Burenkov, editor-in-chief of the Eurasian Mathematical Journal (together with V.A. Sadovnichy and M. Otelbaev), professor of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (Moscow), chairman of the Dissertation Council at the RUDN University, research fellow (part-time) at the Steklov Institute of Mathematics (Moscow), honorary academician of the National Academy of Sciences of the Republic of Kazakhstan, doctor of physical and mathematical sciences(1983), professor (1986), honorary professor of the L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, 2006), honorary doctor of the Russian-Armenian (Slavonic) University (Yerevan, Armenia, 2007), honorary member of staff of the University of Padua (Italy, 2011), honorary distinguished professor of the Cardiff School of Mathematics (UK,2014), honorary professor of the Aktobe Regional State University (Kazakhstan, 2015).

V.I. Burenkov graduated from the Moscow Institute of Physics and Technology (1963) and completed his postgraduate studies there in 1966 under supervision of the famous Russian mathematician academician S.M. Nikol'skii.He worked at several universities, in particular for more than 10 years at the Moscow Institute of Electronics, Radio-engineering, and Automation, the RUDN University, and the Cardiff University. He also worked at the Moscow Institute of Physics and Technology, the University of Padua, and the L.N. Gumilyov Eurasian National University. Through 2015-2017 he was head of the Department of Mathematical Analysis and Theory of Functions (RUDN University). He was one of the organisers and the first director of the S.M. Nikol'skii Institute of Mathematics at the RUDN University (2016-2017).

He obtained seminal scientific results in several areas of functional analysis and the theory of partial differential and integral equations. Some of his results and methods are named after him: Burenkov's theorem on composition of absolutely continuous functions, Burenkov's theorem on conditional hypoellipticity, Burenkov's method of mollifiers with variable step, Burenkov's method of extending functions, the Burenkov-Lamberti method of transition operators in the problem of spectral stability of differential operators, the Burenkov-Guliyevs conditions for boundedness of operators in Morrey-type spaces. On the whole, the results obtained by V.I. Burenkov have laid the groundwork for new perspective scientific directions in the theory of functions spaces and its applications to partial differential equations, the spectral theory in particular.

More than 30 postgraduate students from more than 10 countries gained candidate of sciences or PhD degrees under his supervision. He has published more than 190 scientific papers. His monograph "Sobolev spaces on domains" became a popular text for both experts in the theory of function spaces and a wide range of mathematicians interested in applying the theory of Sobolev spaces. In 2011 the conference "Operators in Morrey-type Spaces and Applications", dedicated to his 70th birthday was held at the Ahi Evran University (Kirsehir, Turkey). Proceedings of that conference were published in the EMJ 3-3 and EMJ 4-1.

V.I. Burenkov is still very active in research. Through 2016-2021 he published 20 papers in leading mathematical journals.

The Editorial Board of the Eurasian Mathematical Journal congratulates Victor Ivanovich Burenkov on the occasion of his 80th birthday and wishes him good health and new achievements in science and teaching!

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NONEXISTENCE OF NONTRIVIAL WEAK SOLUTIONS OF SOME NONLINEAR INEQUALITIES WITH INTEGER POWER OF THE LAPLACIAN

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AMS Mathematics Subject Classification: 35J30, 35J62, 35R45.

Abstract. In this paper, we make modification of the results obtained by Mitidieri and Pokhozhaev on sufficient conditions for the nonexistence of nontrivial weak solutions of nonlinear inequalities and systems with integer power of the Laplacian with the nonlinearity term of the form $a(x)\,|\Delta^m u|^q + b(x)|u|^s$. We obtain an optimal a priori estimate by employing the nonlinear capacity method under a special choice of test functions. Finally, we prove the nonexistence of nontrivial weak solutions of the considered inequalities and systems by contradiction.

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1 Introduction

In recent decades, great interest was shown to problems of partial differential equations and inequalities. In particular, much attention was paid by researchers to finding solutions of various classes of partial differential equations and inequalities. Nowadays, many authors are interested in proving the nonexistence of solutions of various classes of partial differential equations and inequalities in various function classes.

This paper is inspired by various recent works on proving the nonexistence of nontrivial weak solutions of some nonlinear elliptic partial differential inequalities and systems of such inequalities with integer power of the Laplacian. We obtain sufficient conditions for the nonexistence of nontrivial weak solutions for some nonlinear inequalities and systems with integer power of the Laplacian with the nonlinearity of the form $|\Delta^m u|^q$ by the nonlinear capacity method proposed by S.I. Pokhozhaev [8]. Later on this method was developed in the joint work of E. Mitidieri and S.I. Pokhozhaev [7] (see also [10], [9]). Since the method allowed them to obtain new sharp sufficient unsolvability conditions for wider classes of functions (which cannot be obtained by the comparison method for such classes of functions), the authors are interested in applying it using a special choice of test functions for various operators (see [10]).

The method was based on obtained asymptotically optimal a priori estimates by applying appropriate algebraic inequalities to the integral form of the considered inequality under a special choice of test functions. The method was applied to various types of elliptic equations, inequalities, and systems. It can be found, for instance, in [1] - [5], [11], [12].

In the present work we modify the nonexistence condition obtained in [8] by modifying the nonlinearity term for the inequalities and systems with integer power of the Laplacian considered there.

In Section 2, we prove the nonexistence of nontrivial solutions of the considered inequality and in Section 3, we prove the nonexistence of nontrivial solutions of a system of such inequalities.

2 Scalar inequalities

Consider the inequality of the form

$$\Delta^k u(x) \geq a(x)\,|\Delta^m u(x)|^q + b(x)|u(x)|^s, \qquad x \in \mathbb{R}^N \tag{2.1}$$

with

$$k, m \in \mathbb{N}, m < k, q > 1,\ s > 1, \tag{2.2}$$

where a and b are locally integrable nonnegative functions satisfying the inequalities

$$a(x) \geq c_1(1+|x|)^\alpha,\ \ b(x) \geq c_2(1+|x|)^\beta \tag{2.3}$$

for some $c_1, c_2 > 0$, α, $\beta \in \mathbb{R}$ and all $x \in \mathbb{R}^N$.

Definition 1. By a weak solution to nonlinear inequality (2.1), we mean a function $u \in L^1_{loc}(\mathbb{R}^N)$ such that $a|\Delta^m u|^q \in L^1_{loc}(\mathbb{R}^N)$, $b|u|^s \in L^1_{loc}(\mathbb{R}^N)$, and the inequality

$$\int_{\mathbb{R}^N} \left(a(x)\,|\Delta^m u(x)|^q + b(x)|u(x)|^s\right)\varphi(x)dx \leq \int_{\mathbb{R}^N} u(x)\Delta^k\varphi(x)dx \tag{2.4}$$

holds for any test function $\varphi \in C_0^{2k}(\mathbb{R}^N; \mathbb{R}_+)$.

Theorem 2.1. *Suppose that the exponents q and s satisfy inequalities* (2.1) *and*

$$\begin{aligned} &1 < q \leq \frac{N+\alpha}{N-2(k-m)} \quad \text{if} \quad \mathrm{N \leq 2(k-m)}, \\ &1 < s \leq \frac{N+\beta}{N-2k} \quad \text{if} \quad \mathrm{N \leq 2k}. \end{aligned} \tag{2.5}$$

Then problem (2.1) *has no nontrivial weak solutions.*

Proof. By the definition of a weak solution, integrating by parts, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(a(x)\,|\Delta^m u(x)|^q + b(x)|u(x)|^s\right)\varphi(x)dx \leq \int_{\mathbb{R}^N} u(x)\Delta^k\varphi(x)dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} \Delta^m u(x)\Delta^{k-m}\varphi(x)dx + \frac{1}{2}\int_{\mathbb{R}^N} u(x)\Delta^k\varphi(x)dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} |\Delta^m u(x)|\cdot|\Delta^{k-m}\varphi(x)|dx + \frac{1}{2}\int_{\mathbb{R}^N} |u(x)|\cdot|\Delta^k\varphi(x)|dx. \end{aligned} \tag{2.6}$$

Hence, applying the following Young inequality:

$$AB \leq \frac{A^p}{p} + \frac{B^{p'}}{p'} \ \left(A, B > 0, p > 1, p' = \frac{p}{p-1}\right), \tag{2.7}$$

we estimate the right-hand terms in (2.6) in the following manner:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} |\Delta^m u(x)|\cdot|\Delta^{k-m}\varphi(x)|dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} a(x)\,|\Delta^m u(x)|^q\,\varphi(x)dx + C_1(q)\int_{\mathbb{R}^N} |\Delta^{k-m}\varphi(x)|^{q'} a^{-\frac{q'}{q}}(x)\varphi^{-\frac{q'}{q}}(x)dx, \end{aligned} \tag{2.8}$$

where (2.7) is applied with $p = q$,

$$\begin{cases} A(x) = a^{\frac{1}{q}}\,|\Delta^m u(x)|\,\varphi^{\frac{1}{q}}, \\ B(x) = a^{-\frac{1}{q}}\left|\Delta^{k-m}\varphi\right|\varphi^{-\frac{1}{q}}. \end{cases} \tag{2.9}$$

and

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} |u(x)|\cdot|\Delta^m\varphi(x)|dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} b(x)|u(x)|^s\varphi(x)dx + C_2(s)\int_{\mathbb{R}^N} |\Delta^k\varphi(x)|^{s'}b^{-\frac{s'}{s}}(x)\varphi^{-\frac{s'}{s}}(x)dx, \end{aligned} \tag{2.10}$$

where (2.7) is applied with $p = s$ and

$$\begin{cases} A(x) = b^{\frac{1}{s}}|u(x)|\varphi^{\frac{1}{s}}, \\ B(x) = a^{-\frac{1}{s}}|\Delta^k\varphi|\varphi^{-\frac{1}{s}}, \end{cases} \tag{2.11}$$

Here $C_1(q)$ and $C_2(s)$ are positive constants depending only on q and s, respectively. Then, we obtain the following a priori estimate

$$\frac{1}{2}\int_{\mathbb{R}^N} |\Delta^m u|^q\,\varphi dx \le \int_{\mathbb{R}^N}\left(C_1(q)\frac{|L_{k-m}(\varphi)|^{q'}}{(a\varphi)^{q'-1}} + C_2(q)\frac{|L_k(\varphi)|^{s'}}{(b\varphi)^{s'-1}}\right)dx, \tag{2.12}$$

where $L_j(\varphi) = \Delta^j\varphi$ ($j = k-m$ or k), $q' = \frac{q}{q-1}$.

Now choose the test function φ of the form

$$\varphi(x) = \varphi_0\left(\frac{|x|^2}{R^2}\right), \tag{2.13}$$

where $\varphi_0 \in C_0^{2k}\left(\mathbb{R}^N;\mathbb{R}_+\right)$ is such that

$$\varphi_0(s) = \begin{cases} 1, & 0 \le s \le 1, \\ 0, & s \ge 2. \end{cases} \tag{2.14}$$

Next, let us change the variables,

$$x \to \xi : x = R\xi. \tag{2.15}$$

Then we obtain

$$\int_{\mathbb{R}^N} \frac{|L_{k-m}(\varphi)|^{q'}}{(a\varphi)^{q'-1}}dx = R^{\theta_1}\int_{1\le|\xi|\le\sqrt{2}} \frac{\left|L^*_{k-m}(\varphi_0)\right|^{q'}}{(a_0\varphi_0)^{q'-1}}d\xi, \tag{2.16}$$

where $L^*_{k-m}(\varphi_0) = \Delta^{k-m}\varphi_0$, $a_0(\xi) = a(R\xi)$, $\theta_1 = N - \left(2(k-m) + \frac{\alpha}{q}\right)q'$, and

$$\int_{\mathbb{R}^N} \frac{|L_k(\varphi)|^{s'}}{(b\varphi)^{s'-1}}dx = R^{\theta_2}\int_{1\le|\xi|\le\sqrt{2}} \frac{|L^*_k(\varphi_0)|^{q'}}{(b_0\varphi_0)^{s'-1}}d\xi, \tag{2.17}$$

where $L^*_k(\varphi_0) = \Delta^k\varphi_0$, $b_0(\xi) = b(R\xi)$, $\theta_2 = N - \left(2k + \frac{\beta}{q}\right)q'$.

Now, choose a test function φ_0 such that

$$\int_{1\le|\xi|\le\sqrt{2}} \frac{\left|L^*_{k-m}(\varphi_0)\right|^{q'}}{(a_0\varphi_0)^{q'-1}}d\xi < \infty$$

and

$$\int_{1\le|\xi|\le\sqrt{2}} \frac{|L_k^*(\varphi_0)|^{q'}}{(b_0\varphi_0)^{s'-1}} d\xi < \infty,$$

so that the integral on the right-hand side of (2.12) is finite. Then, (2.12) implies that

$$\int_{\mathbb{R}^N} (a\,|\Delta^m u|^q + bu^s)\varphi dx \le CR^\theta, \tag{2.18}$$

where $\theta = \max(\theta_1, \theta_2)$. Now, assume that $\theta = \theta_1 \ge \theta_2$ (the opposite case can be treated similarly) and consider the following two cases for the values of θ.

Case 1: if $\theta < 0$, passing to the limit as $R \to \infty$ in (2.18) we have

$$\int_{\mathbb{R}^N} (a\,|\Delta^m u|^q + bu^s) dx \le 0. \tag{2.19}$$

Thus, the assertion of theorem is proved for $\theta < 0$, i.e.,

$$1 < q < \frac{N}{N - 2(k-m) - \alpha}. \tag{2.20}$$

Case 2: if $\theta = 0$, then

$$q = \frac{N}{N - 2(k-m) - \alpha}. \tag{2.21}$$

In this case, relation (2.16) implies that

$$\int_{\mathbb{R}^N} \frac{|L_{k-m}(\varphi)|^{q'}}{(a\varphi)^{q'-1}} dx = c_1, \tag{2.22}$$

where

$$c_1 = \int_{1\le|\xi|\le\sqrt{2}} \frac{\left|L_{k-m}^*(\varphi_0)\right|^{q'}}{(a_0\varphi_0)^{q'-1}} d\xi, \tag{2.23}$$

and (since $\theta_2 \le \theta_1 = 0$)

$$\int_{\mathbb{R}^N} \frac{|L_k(\varphi)|^{s'}}{(b\varphi)^{s'-1}} dx = \lim_{R\to\infty} c_2 R^{\theta_2} \le c_2, \tag{2.24}$$

where

$$c_2 = \int_{1\le|\xi|\le\sqrt{2}} \frac{|L_k^*(\varphi_0)|^{s'}}{(b_0\varphi_0)^{s'-1}} d\xi. \tag{2.25}$$

Hence, by (2.12), we have

$$\int_{\mathbb{R}^N} |\Delta^m u|^q \,\varphi dx \le c, \tag{2.26}$$

where $c = c_1 C_1(q) + c_2 C_2(s)$. Passing to the limit as $R \to \infty$, we obtain

$$\int_{\mathbb{R}^N} |\Delta^m u|^q \, dx \le c. \tag{2.27}$$

Now, let us return to inequality (2.6). Note that

$$\operatorname{supp}\{L_m(\varphi)\} \subseteq \{x \in \mathbb{R}^N \mid R \le |x| \le \sqrt{2}R\} = \overline{B}_{\sqrt{2}R} \backslash B_R,$$

where $B_L = \{x \in \mathbb{R}^N : |x| < L\}$.

Then, by the Hölder inequality, relation (2.6) implies

$$\begin{aligned}&\int_{\mathbb{R}^N} (a\,|\Delta^m u|^q + bu^s)\varphi dx\\ &\le \left(\int_{R\le|x|\le\sqrt{2}R} a\,|\Delta^m u|^q\,\varphi dx\right)^{\frac{1}{q}} \left(\int_{R\le|x|\le\sqrt{2}R} \frac{|L_{k-m}(\varphi)|^{q'}}{(a\varphi)^{q'-1}} dx\right)^{\frac{1}{q'}}\\ &+ \left(\int_{R\le|x|\le\sqrt{2}R} bu^s\varphi dx\right)^{\frac{1}{s}} \left(\int_{R\le|x|\le\sqrt{2}R} \frac{|L_{k-m}(\varphi)|^{s'}}{(b\varphi)^{s'-1}} dx\right)^{\frac{1}{s'}}.\end{aligned} \tag{2.28}$$

However, by (2.27) and the absolute convergence of integrals (2.22) and (2.24), we have

$$\int_{R\le|x|\le\sqrt{2}R} a\,|\Delta^m u|^q\,dx \to 0 \tag{2.29}$$

and

$$\int_{R\le|x|\le\sqrt{2}R} b|u|^s dx \to 0 \tag{2.30}$$

as $R \to \infty$.

Passing to the limit as $R \to \infty$ in (2.28) and taking into account (2.22) and (2.24), we obtain (2.19) again. Thus, $u = 0$ almost everywhere in this case as well, i.e., with regard to (2.21), the nonexistence condition for a solution is finally expressed as $1 < q \le \frac{N}{N-2(k-m)}$. □

3 Systems of inequalities

Consider the system of the form

$$\begin{cases} \Delta^{k_1} u(x) \ge a(x)\,|\Delta^{m_1} v(x)|^{q_1} + b(x)|v(x)|^{s_1}, & x \in \mathbb{R}^N,\\ \Delta^{k_2} v(x) \ge c(x)\,|\Delta^{m_2} u(x)|^{q_2} + d(x)|v(x)|^{s_2}, & x \in \mathbb{R}^N \end{cases} \tag{3.1}$$

with

$$\begin{aligned}&k_1, k_2, m_1, m_2 \in \mathbb{N},\ k_1 > m_2,\ \ k_2 > m_1,\ \min(q_1, q_2, s_1, s_2) > 1,\\ &a(x) \ge C(1+|x|)^{\alpha_1},\ b(x) \ge C(1+|x|)^{\beta_1},\\ &c(x) \ge C(1+|x|)^{\alpha_2},\ d(x) \ge C(1+|x|)^{\beta_2}\end{aligned} \tag{3.2}$$

for some $C > 0$ and all $x \in \mathbb{R}^N$.

Definition 2. By a weak solution to system of nonlinear inequalities (3.1), we mean a pair of functions $(u, v) \in L^{s_1}_{loc}(\mathbb{R}^N) \times L^{s_2}_{loc}(\mathbb{R}^N)$ such that the inequalities

$$\int_{\mathbb{R}^N} (a(x)\,|\Delta^{m_1} v(x)|^{q_1} + b(x)|v(x)|^{s_1})\varphi(x) dx \le \int_{\mathbb{R}^N} u(x)\Delta^{k_1}\varphi(x) dx, \tag{3.3}$$

$$\int_{\mathbb{R}^N} (c(x)\,|\Delta^{m_2} u(x)|^{q_2} + d(x)|v(x)|^{s_2})\varphi(x) dx \le \int_{\mathbb{R}^N} v(x)\Delta^{k_2}\varphi(x) dx \tag{3.4}$$

hold for any test function $\varphi \in C^{2k}_0(\mathbb{R}^N; \mathbb{R}_+)$, where $k = \max(k_1, k_2)$.

Theorem 3.1. *Let* (3.2) *hold. Suppose that* $\theta \overset{\text{def}}{=} \max_{i=1,\dots,4} \theta_i \le 0$, *where*

$$\begin{aligned} \theta_1 &= N - \frac{2(k_1 - m_2)q_2 + \alpha_2}{q_2 - 1}, \quad \theta_2 = N - \frac{2(k_1 - m_2)s_2 + \beta_2}{s_2 - 1}, \\ \theta_3 &= N - \frac{2(k_1 - m_2)q_1 + \alpha_1}{q_1 - 1}, \quad \theta_4 = N - \frac{2(k_1 - m_2)s_1 + \beta_1}{s_1 - 1}. \end{aligned} \tag{3.5}$$

Then problem (3.1) *does not have a nontrivial weak solution in* $\mathbb{R}^N$.

Proof. Let us begin by setting $\psi(x) = \varphi(x)$. By the definition of a weak solution, integrating by parts, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} (a(x)\,|\Delta^{m_1} v(x)|^{q_1} + b(x)|v(x)|^{s_1})\varphi(x)dx \le \int_{\mathbb{R}^N} u(x)\Delta^{k_1}\varphi(x)dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} \Delta^{m_2} u(x)\Delta^{k_1-m_2}\varphi(x)dx + \frac{1}{2}\int_{\mathbb{R}^N} u(x)\Delta^{k_1}\varphi(x)dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} |\Delta^{m_2} u(x)|\cdot|\Delta^{k_1-m_2}\varphi(x)|dx + \frac{1}{2}\int_{\mathbb{R}^N} |u(x)|\cdot|\Delta^{k_1}\varphi(x)|dx, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\int_{\mathbb{R}^N} (c(x)\,|\Delta^{m_2} u(x)|^{q_2} + d(x)|v(x)|^{s_2})\varphi(x)dx \le \int_{\mathbb{R}^N} v(x)\Delta\varphi^{k_2}(x)dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} \Delta^{m_1} v(x)\Delta^{k_2-m_1}\varphi(x)dx + \frac{1}{2}\int_{\mathbb{R}^N} v(x)\Delta^{k_2}\varphi(x)dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} |\Delta^{m_1} v(x)|\cdot|\Delta^{k_2-m_1}\varphi(x)|dx + \frac{1}{2}\int_{\mathbb{R}^N} |v(x)|\cdot|\Delta^{k_2}\varphi(x)|dx. \end{aligned} \tag{3.7}$$

Hence, by applying the Young inequality in (3.6) and (3.7), we have

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} |\Delta^{m_2} u(x)|\cdot|\Delta^{k_1-m_2}\varphi(x)|dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} c(x)\,|\Delta^{m_2} u(x)|^{q_2}\,\varphi(x)dx + C_1(q_2)\int_{\mathbb{R}^N} |\Delta^{k_1-m_2}\varphi(x)|^{q'} c^{-\frac{q_2'}{q_2}}(x)\varphi^{-\frac{q_2'}{q_2}}(x)dx, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} |u(x)|\cdot|\Delta^{k_1}\varphi(x)|dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} d(x)|u(x)|^{s_2}\varphi(x)dx + C_2(s_2)\int_{\mathbb{R}^N} |\Delta^{k_1}\varphi(x)|^{s_2'} d^{-\frac{s_2'}{s_2}}(x)\varphi^{-\frac{s_2'}{s_2}}(x)dx, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} |\Delta^{m_1} u(x)|\cdot|\Delta^{k_2-m_1}\varphi(x)|dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} a(x)\,|\Delta^{m} v(x)|^{q_1}\,\varphi(x)dx + C_3(q_1)\int_{\mathbb{R}^N} |\Delta^{k-m}\varphi(x)|^{q_1'} a^{-\frac{q_1'}{q_1}}(x)\varphi^{-\frac{q_1'}{q_1}}(x)dx, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} |u(x)|\cdot|\Delta^{k_2}\varphi(x)|dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} b(x)|u(x)|^{s_1}\varphi(x)dx + C_4(s_1)\int_{\mathbb{R}^N} |\Delta^{k_2}\varphi(x)|^{s_1'} b^{-\frac{s_1'}{s_1}}(x)\varphi^{-\frac{s_1'}{s_1}}(x)dx, \end{aligned} \tag{3.11}$$

where $\frac{1}{q_i} + \frac{1}{q_i'} = 1$ and $\frac{1}{s_i} + \frac{1}{s_i'}$ $(i = 1, 2)$. Let us introduce the following notations

$$X = \int_{\mathbb{R}^N} (a(x)\,|\Delta^{m_1} v(x)|^{q_1} + b(x)|v(x)|^{s_1})\varphi(x)dx$$

and

$$Y = \int_{\mathbb{R}^N} (c(x)\,|\Delta^{m_2} u(x)|^{q_2} + d(x)|u(x)|^{s_2})\varphi(x)dx.$$

Then, it follows from (3.6)–(3.11) that

$$
\begin{aligned}
&X \leq \frac{Y}{2} + C_1(q_2)\int_{\mathbb{R}^N} |\Delta^{k_1-m_2}\varphi(x)|^{q'} c^{-\frac{q_2'}{q_2}}(x)\varphi^{-\frac{q_2'}{q_2}}(x)dx\\
&+C_2(s_2)\int_{\mathbb{R}^N} |\Delta^{k_1}\varphi(x)|^{s_2'} d^{-\frac{s_2'}{s_2}}(x)\varphi^{-\frac{s_2'}{s_2}}(x)dx,
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&Y \leq \frac{X}{2} + C_3(q_1)\int_{\mathbb{R}^N} |\Delta^{k-m}\varphi(x)|^{q_1'} a^{-\frac{q_1'}{q_1}}(x)\varphi^{-\frac{q_1'}{q_1}}(x)dx\\
&+C_4(s_1)\int_{\mathbb{R}^N} |\Delta^{k_2}\varphi(x)|^{s_1'} b^{-\frac{s_1'}{s_1}}(x)\varphi^{-\frac{s_1'}{s_1}}(x)dx.
\end{aligned}
\tag{3.13}
$$

Adding (3.12) and (3.13) and simplifying, we obtain

$$
\begin{aligned}
&X + Y \leq 2C_1(q_2)\int_{\mathbb{R}^N} |\Delta^{k_1-m_2}\varphi(x)|^{q'} c^{-\frac{q_2'}{q_2}}(x)\varphi^{-\frac{q_2'}{q_2}}(x)dx\\
&+2C_2(s_2)\int_{\mathbb{R}^N} |\Delta^{k_1}\varphi(x)|^{s_2'} d^{-\frac{s_2'}{s_2}}(x)\varphi^{-\frac{s_2'}{s_2}}(x)dx\\
&+2C_3(q_1)\int_{\mathbb{R}^N} |\Delta^{k_2-m_1}\varphi(x)|^{q_1'} a^{-\frac{q_1'}{q_1}}(x)\varphi^{-\frac{q_1'}{q_1}}(x)dx\\
&+2C_4(s_1)\int_{\mathbb{R}^N} |\Delta^{k_2}\varphi(x)|^{s_1'} b^{-\frac{s_1'}{s_1}}(x)\varphi^{-\frac{s_1'}{s_1}}(x)dx.
\end{aligned}
\tag{3.14}
$$

Now, introduce the standard test function φ of the form

$$
\varphi(x) = \varphi_0\left(\frac{|x|^2}{R^2}\right),
$$

where $\varphi_0 \in C_0^{2k}(\mathbb{R}_+)$ is such that

$$
\varphi_0(s) = \begin{cases} 1, & 0 \leq s \leq 1,\\ 0, & s \geq 2. \end{cases}
$$

Next, let us change the variables in the right-hand side of inequality (3.14):

$$
x \to \xi : x = R\xi,
$$

which yields

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\Delta^{k_1-m_2}\varphi(x)|^{q_2'} c^{-\frac{q_2'}{q_2}}(x)\varphi^{-\frac{q_2'}{q_2}}(x)dx\\
&= R^{\theta_1}\int_{1\leq|\xi|\leq\sqrt{2}} |\Delta^{k_1-m_2}\varphi_0(\xi)|^{q_2'}(c_0\varphi_0)^{1-q_2'}d\xi,
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\Delta^{k_1}\varphi(x)|^{s_2'} d^{-\frac{s_2'}{s_2}}(x)\varphi^{-\frac{s_2'}{s_2}}(x)dx\\
&= R^{\theta_2}\int_{1\leq|\xi|\leq\sqrt{2}} |\Delta^{k_1}\varphi_0(\xi)|^{s_2'}(d_0\varphi_0)^{1-s_2'}(\xi)d\xi,
\end{aligned}
\tag{3.16}
$$

$$
\int_{\mathbb{R}^N} |\Delta^{k_2-m_1}\varphi(x)|^{q_1'} a^{-\frac{q_1'}{q_1}}(x)\varphi^{-\frac{q_1'}{q_1}}(x)dx
$$

$$= R^{\theta_3} \int_{1 \le |\xi| \le \sqrt{2}} |\Delta^{k_2 - m_1} \varphi(\xi)|^{q_1'} (a_0 \varphi_0)^{1-q_1'}(\xi) d\xi, \tag{3.17}$$

$$\int_{\mathbb{R}^N} |\Delta^{k_2} \varphi(x)|^{s_1'} b^{-\frac{s_1'}{s_1}}(x) \varphi^{-\frac{s_1'}{s_1}}(x) dx$$
$$= R^{\theta_4} \int_{1 \le |\xi| \le \sqrt{2}} |\Delta^{k_2} \varphi_0(\xi)|^{s_1'} (b_0 \varphi_0)^{1-s_1'}(\xi) d\xi. \tag{3.18}$$

Then, inequality (3.14) becomes

$$\begin{aligned} X + Y \le\ & 2C_1(q_2) R^{\theta_1} \int_{1 \le |\xi| \le \sqrt{2}} |\Delta^{k_1 - m_2} \varphi_0(\xi)|^{q_2'} (c_0 \varphi_0)^{1-q_2'} d\xi \\ & + 2C_2(s_2) R^{\theta_2} \int_{1 \le |\xi| \le \sqrt{2}} |\Delta^{k_1} \varphi_0(\xi)|^{s_2'} (d_0 \varphi_0)^{1-s_2'}(\xi) d\xi \\ & + 2C_3(q_1) R^{\theta_3} \int_{1 \le |\xi| \le \sqrt{2}} |\Delta^{k_2 - m_1} \varphi(\xi)|^{q_1'} (a_0 \varphi_0)^{1-q_1'}(\xi) d\xi \\ & + 2C_4(s_1) R^{\theta_4} \int_{1 \le |\xi| \le \sqrt{2}} |\Delta^{k_2} \varphi_0(\xi)|^{s_1'} (b_0 \varphi_0)^{1-s_1'}(\xi) d\xi. \end{aligned} \tag{3.19}$$

Now, choose the test function φ_0 such that

$$\int_{1 \le |\xi| \le \sqrt{2}} |\Delta^{k_1 - m_2} \varphi_0(\xi)|^{q_2'} (c_0 \varphi_0)^{1-q_2'} d\xi < \infty,$$

$$\int_{1 \le |\xi| \le \sqrt{2}} |\Delta^{k_1} \varphi_0(\xi)|^{s_2'} (d_0 \varphi_0)^{1-s_2'}(\xi) d\xi < \infty,$$

$$\int_{1 \le |\xi| \le \sqrt{2}} |\Delta^{k_2 - m_1} \varphi(\xi)|^{q_1'} (a_0 \varphi_0)^{1-q_1'}(\xi) d\xi < \infty,$$

and

$$\int_{1 \le |\xi| \le \sqrt{2}} |\Delta^{k_2} \varphi_0(\xi)|^{s_1'} (b_0 \varphi_0)^{1-s_1'}(\xi) d\xi < \infty.$$

Then, inequality (3.19) implies that

$$X + Y \le C \sum_{i=1}^{r} R^{\theta_i} \tag{3.20}$$

with some $C > 0$. If any of inequalities (3.5) holds, i.e. if $\theta = \max\limits_{i=1,\dots,4} \theta_i \le 0$, we have two cases.

Case 1: $\theta < 0$.

Passing to the limit as $R \to \infty$ in (3.20) we get

$$\int_{\mathbb{R}^N} (a(x)\, |\Delta^{m_1} v(x)|^{q_1} + b(x) |v(x)|^{s_1}) \varphi(x) dx$$
$$+ \int_{\mathbb{R}^N} (c(x)\, |\Delta^{m_2} u(x)|^{q_2} + d(x) |u(x)|^{s_2}) \varphi(x) dx = 0.$$

Hence, $u = 0$ and $v = 0$ almost everywhere in $\mathbb{R}^N$.

Case 2: $\theta = 0$.

In this case inequality (3.20) implies that

$$\begin{aligned} &\int_{\mathbb{R}^N} (a(x)\,|\Delta^{m_1} v(x)|^{q_1} + b(x)|v(x)|^{s_1})\varphi(x)dx \\ &+ \int_{\mathbb{R}^N} (c(x)\,|\Delta^{m_2} u(x)|^{q_2} + d(x)|u(x)|^{s_2})\varphi(x)dx = c < \infty. \end{aligned} \tag{3.21}$$

Now, let us return to inequalities (3.6) and (3.7). Note that

$$\operatorname{supp}\{\Delta^p(\varphi)\} \subseteq \{x \in \mathbb{R}^N \mid R \le |x| \le \sqrt{2}R\} = \overline{B}_{\sqrt{2}R} \backslash B_R \text{ for any } p \in \mathcal{N}.$$

Then, by the Hölder inequality, relations (3.6) and (3.7) imply that

$$\begin{aligned} &\int_{\mathbb{R}^N} (a(x)\,|\Delta^{m_1} v(x)|^{q_1} + b(x)|v(x)|^{s_1})\varphi(x)dx \\ &\le \int_{R\le|x|\le\sqrt{2}R} (c(x)\,|\Delta^{m_2} u(x)|^{q_1} + d(x)|u(x)|^{s_2})\varphi(x)dx, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\int_{\mathbb{R}^N} (c(x)\,|\Delta^{m_2} u(x)|^{q_1} + d(x)|u(x)|^{s_2})\varphi(x)dx \\ &\le \int_{R\le|x|\le\sqrt{2}R} (a(x)\,|\Delta^{m_1} v(x)|^{q_1} + b(x)|v(x)|^{s_1})\varphi(x)dx. \end{aligned} \tag{3.23}$$

However, by (3.21) and the absolute convergence of the integral $\int_{\mathbb{R}^N} |\Delta^m u(x)|^s\,dx$, we have

$$\begin{aligned} &\int_{R\le|x|\le\sqrt{2}R} (a(x)\,|\Delta^{m_1} v(x)|^{q_1} + b(x)|v(x)|^{s_1})\varphi(x)dx \\ &+ \int_{R\le|x|\le\sqrt{2}R} (c(x)\,|\Delta^{m_2} u(x)|^{q_2} + d(x)|u(x)|^{s_2})\varphi(x)dx \to 0 \end{aligned}$$

as $R \to \infty$.

Passing to the limit as $R \to \infty$ in (3.23), we obtain

$$\int_{\mathbb{R}^N} (a(x)\,|\Delta^{m_1} v(x)|^{q_1} + b(x)|v(x)|^{s_1})dx = 0.$$

Then, inequality (3.22) implies

$$\int_{\mathbb{R}^N} (c(x)\,|\Delta^{m_2} u(x)|^{q_1} + d(x)|u(x)|^{s_2})dx = 0.$$

Thus, $u = 0$ and $v = 0$ almost everywhere in this case as well.

$\square$

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