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VLADIMIR MIKHAILOVICH FILIPPOV

(to the 70th birthday)

Vladimir Mikhailovich Filippov was born on 15 April 1951 in the city of Uryupinsk, Stalingrad Region of the USSR. In 1973 he graduated with honors from the Faculty of Physics and Mathematics and Natural Sciences of the Patrice Lumumba University of Peoples' Friendship in the specialty "Mathematics". In 1973-1975 he is a postgraduate student of the University; in 1976-1979 - Chairman of the Young Scientists' Council; in 1980-1987 - Head of the Research Department and the Scientific Department; in 1983-1984 - scientific work at the Free University of Brussels (Belgium); in 1985-2000 - Head of the Mathematical Analysis Department; from 2000 to the present - Head of the Comparative Educational Policy Department; in 1989–1993 - Dean of the Faculty of Physics, Mathematics and Natural Sciences; in 1993–1998 - Rector of the Peoples' Friendship University of Russia; in 1998-2004 - Minister of General and Professional Education, Minister of Education of the Russian Federation; in 2004-2005 - Assistant to the Chairman of the Government of the Russian Federation (in the field of education and culture); from 2005 to May 2020- Rector of the Peoples' Friendship University of Russia, since May 2020 - President of the Peoples' Friendship University of Russia, since 2013 - Chairman of the Higher Attestation Commission of the Ministry of Science and Higher Education of the Russian Federation.

In 1980, he defended his PhD thesis in the V.A. Steklov Mathematical Institute of Academy of Sciences of the USSR on specialty 01.01.01 - mathematical analysis (supervisor - a corresponding member of the Academy of Sciences of the USSR, Professor L.D. Kudryavtsev), and in 1986 in the same Institute he defended his doctoral thesis "Quasi-classical solutions of inverse problems of the calculus of variations in non-Eulerian classes of functionals and function spaces". In 1987, he was awarded the academic title of a professor.

V.M. Filippov is an academician of the Russian Academy of Education; a foreign member of the Ukrainian Academy of Pedagogical Sciences; President of the UNESCO International Organizing Committee for the World Conference on Higher Education (2007-2009); Vice-President of the Eurasian Association of Universities; a member of the Presidium of the Rectors' Council of Moscow and Moscow Region Universities, of the Governing Board of the Institute of Information Technologies in Education (UNESCO), of the Supervisory Board of the European Higher Education Center of UNESCO (Bucharest, Romania),

Research interests: variational methods; non-potential operators; inverse problems of the calculus of variations; function spaces.

In his Ph.D thesis, V.M. Filippov solved a long standing problem of constructing an integral extremal variational principle for the heat equation. In his further research he developed a general theory of constructing extremal variational principles for broad classes of differential equations with non-potential (in classical understanding) operators. He showed that all previous attempts to construct variational principles for non-potential operators "failed" because mathematicians and mechanics from the time of L. Euler and J. Lagrange were limited in their research by functionals of the type Euler - Lagrange. Extending the classes of functionals, V.M. Filippov introduced a new scale of function spaces that generalize the Sobolev spaces, and thus significantly expanded the scope of the variational methods. In 1984, famous physicist, a Nobel Prize winner I.R. Prigogine presented the report of V.M. Filippov to the Royal Academy of Sciences of Belgium. Results of V.M. Filippov's variational principles for non-potential operators are quite fully represented in some of his and his colleagues' monographs.

Honors: Honorary Legion (France), "Commander" (Belgium), Crown of the King (Belgium); in Russia - orders "Friendship", "Honor", "For Service to the Fatherland" III and IV degrees; Prize of the President of the Russian Federation in the field of education; Prize of the Governement of the Russian Federation in the field of education; Gratitude of the President of the Russian Federation; "For Merits in the Social and Labor Sphere of the Russian Federation", "For Merits in the Development of the Olympic Movement in Russia", "For Strengthening the Combat Commonwealth; and a number of other medals, prizes and awards.

He is an author of more than 270 scientific and scientific-methodical works, including 32 monographs, 2 of which were translated and published in the United States by the American Mathematical Society.

V.M. Filippov meets his 70th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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HYPERSPACE OF THE Π-COMPLETE SPACES AND MAPS

A.A. Zaitov, D.I. Jumaev

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AMS Mathematics Subject Classification: 54B20, 54C10.

Abstract. In the present paper we establish that the space $\exp_\beta X$ of compact subsets of a Tychonoff space X is Π-complete if and only if X is so. Further, we prove the Tychonoff map $\exp_\beta f\colon \exp_\beta X \to \exp_\beta Y$ is Π-complete if and only if a given map $f\colon X \to Y$ is Π-complete.

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1 Introduction

In the present paper by a space we mean a topological T_1-space, by a compact a Hausdorff compact space and by a map a continuous map.

A collection ω of subsets of a set X is said to be *star-countable* (respectively, *star-finite*) if each element of ω intersects at most a countable (respectively, finite) set of elements of ω. A collection ω of subsets of a set X *refines* a collection Ω of subsets of X if for each element $A \in \omega$ there is an element $B \in \Omega$ such that $A \subset B$. It is also said that ω is *a refinement* of Ω. For a point $x \in X$ and a natural number n $Kp(x, \omega) \le n$ means that no more than n elements of ω contain x ([1], p. 270), $Kp\,\omega \le n$ if $Kp(x, \omega) \le n$ for every $x \in X$.

A finite sequence of subsets $M_0, \ldots, M_s$ of a set X is [4] a *chain* connecting sets M_0 and M_s, if $M_{i-1} \cap M_i \ne \varnothing$ for $i = 1, \ldots, s$. A collection ω of subsets of a set X is said to be *connected* if for any pair of sets $M, M' \subset X$ there exists a chain in ω connecting the sets M and M'. The maximal connected subcollections of ω are called *components* of ω. A star-finite open cover of a space X is said to be *a finite-component cover* if the number of elements of each component is finite.

For a collection $\omega = \{O_\alpha : \alpha \in A\}$ of subsets of a space X we put $[\omega] = [\omega]_X = \{[O_\alpha]_X : \alpha \in A\}$. For a space X, its subspace W and a point $x \in X \setminus W$ an open cover λ of the space W pricks out the point x in X if $x \notin \cup[\lambda]_X$ [4].

For a Tychonoff space X let βX be its the Stone-Cěch compactification (i. e. the maximal compact extension).

A Tychonoff space X is said to be Π-*complete* if for every point $x \in \beta X \setminus X$ there exists a finite-component cover ω of X which pricks out the point x in βX [4].

Recall the notion of a perfect compactification. For a topological space X and its subset A a set $Fr_X A = [A]_X \cap [X \setminus A]_X = [A]_X \setminus Int_X A$ is called a boundary of A.

Let vX be a compact extension of a Tychonoff space X. If $H \subset X$ is an open set in X, then by $O(H)$ (or by $O_{vX}(H)$) we denote the maximal open set in vX satisfying $O_{vX}(H) \cap X = H$. It is easy to see that

$$O_{vX}(H) = \bigcup_{\substack{\Gamma \in \tau_{vX}, \\ \Gamma \cap X = H}} \Gamma,$$

where τ_{vX} is the topology of the space vX.

A compactification vX of a Tychonoff space X is called *perfect with respect to an open set H* in X, if the equality $[Fr_X H]_{vX} = Fr_{vX} O_{vX}(H)$ holds. If vX is perfect with respect to every open set in X, then it is called *a perfect compactification* of the space X ([1], p. 232). A compactification vX of a space X is perfect if and only if for any two disjoint open sets U_1 and U_2 in X the equality $O(U_1 \bigcup U_2) = O(U_1) \bigcup O(U_2)$ holds. The Stone-Cěch compactification βX of X is a perfect compactification of X. The equality $O(U_1 \bigcup U_2) = O(U_1) \bigcup O(U_2)$ holds for every pair of open sets U_1 and U_2 in X if and only if X is normal, and the compactification vX coincides with the Stone-Cěch compactification βX, i. e. $vX \cong \beta X$.

The following criterion plays a key role in investigation the class of Π-complete spaces ([4], Theorem 1.1 Π), see pages 16-17).

Theorem 1.1. *A Tychonoff space X is* Π*-complete if and only if for every $x \in bX \setminus X$ of an arbitrary perfect compatification bX there exists a cover ω of X with $Kp\,\omega = 1$, pricking out x in bX (i. e. $x \notin \cup[\omega]_{bX}$).*

Since the Stone-Cěch compactification βX of a Tychonoff space X is a perfect compactification of X then Theorem 1.1 implies the following assertion.

Corollary 1.1. *A Tychonoff space X is* Π*-complete if and only if for every $x \in \beta X \setminus X$ there exists a cover ω of X with $Kp\,\omega = 1$, pricking out x in βX (i. e. $x \notin \cup[\omega]_{\beta X}$).*

Note that every compact ($\cong$ Hausdorff compact space) is Π-complete. The square of the Sorgenfrey line is Π-complete, but it is not paracompact (hence, it is not compact). The space $T(\omega_1)$ of all ordinal numbers less than the first uncountable number is a normal space but it is not Π-complete.

Π-complete spaces have the following properties.

(a) A closed subset of a Π-complete space is Π-complete ([4], p. 19).

(b) If $f\colon X \to Y$ is a perfect map onto a Π-complete space Y then X is also Π-complete ([4], p. 26).

(c) A Π-complete space is complete in the Dieudonné sense ([4], p. 18).

It is well-known that the action of functors on various categories of topological spaces and their continuous maps is one of the main problems of the theory of covariant functors (see, for example, [6], [2]). In the present paper we investigate the action of the functor exp (the construction of taking of a hyperspace of a given space) on Π-complete spaces (section 2) and Π-complete maps (section 3).

2 Hyperspace of Π-complete spaces

Let X be a space. By exp X we denote a set of all nonempty closed subsets of X. A family of sets

$$O\langle U_1, \ldots, U_n\rangle = \{F \in \exp X : F \subset \bigcup_{i=1}^{n} U_n, F \cap U_1 \neq \varnothing, \ldots, F \cap U_n \neq \varnothing\}$$

forms a base of a topology on exp X, where $U_1, \ldots, U_n$ are open nonempty sets in X. This topology is called *the Vietoris topology.* A space exp X equipped with the Vietoris topology is called a *hyperspace* of X. For a compact X its hyperspace exp X is also a compact.

Note for any space X it is well known [3] that

$$[O\langle U_1, \ldots, U_n\rangle]_{\exp X} = O\left\langle [U_1]_X, \ldots, [U_n]_X\right\rangle. \tag{2.1}$$

Let $f\colon X \to Y$ be a continuous map of compacts, $F \in \exp X$. We put

$$(\exp f)(F) = f(F).$$

This equality defines a map $\exp f\colon \exp X \to \exp Y$. For a continuous map f the map $\exp f$ is continuous. Indeed, this follows from the equality

$$(\exp f)^{-1} O\langle U_1, \ldots, U_m\rangle = O\langle f^{-1}(U_1), \ldots, f^{-1}(U_m)\rangle$$

which can be checked directly. Note that if $f\colon X \to Y$ is an epimorphism, then $\exp f$ is also an epimorphism.

For a Tychonoff space X we put

$$\exp_\beta X = \{F \in \exp \beta X : F \subset X\}.$$

It is clear, that $\exp_\beta X \subset \exp X$. Consider the set $\exp_\beta X$ as a subspace of the space $\exp X$. For a Tychonoff spaces X the space $\exp_\beta X$ is also a Tychonoff space with respect to the induced topology.

For a continuous map $f\colon X \to Y$ of Tychonoff spaces we put

$$\exp_\beta f = (\exp \beta f)|_{\exp_\beta X},$$

where $\beta f\colon \beta X \to \beta Y$ is the Stone-Cěch compactification of f (it is unique).

It is well known that for a Tychonoff space X the set $\exp_\beta X$ is everywhere dense in $\exp \beta X$, i. e. $\exp \beta X$ is a compactification of the space $\exp_\beta X$. We claim $\exp \beta X$ is a perfect compactification of $\exp_\beta X$. First we will prove the following technical statement.

Lemma 2.1. *Let γX be a compact extension of a space X and, V and W be disjoint open sets in γX. Let $V^X = X \cap V$ and $W^X = X \cap W$. Then the following equality is true:*

$$[X \setminus V^X]_{\gamma X} \cap [X \setminus W^X]_{\gamma X} = [X \setminus (V^X \cup W^X)]_{\gamma X}.$$

Proof. It is clear that $[X\setminus V^X]_{\gamma X}\cap[X\setminus W^X]_{\gamma X} \supset [X\setminus(V^X\cup W^X)]_{\gamma X}$. Let $x \in [X\setminus V^X]_{\gamma X}\cap[X\setminus W^X]_{\gamma X}$. Then each open neighbourhood Ox in γX of x intersects with the sets $X \setminus V^X$ and $X \setminus W^X$. Hence, $Ox \not\subset V^X$ and $Ox \not\subset W^X$. Therefore, since $V^X \cap W^X = \varnothing$, we have $Ox \not\subset V^X \cup W^X$, i. e. $Ox \cap X \setminus (V^X \cup W^X) \neq \varnothing$. By virtue of arbitrariness of the neighbourhood Ox we conclude that $x \in [X \setminus (V^X \cup W^X)]_{\gamma X}$. □

Theorem 2.1. *For a Tychonoff space X the space* $\exp \beta X$ *is a perfect compactification of the space* $\exp_\beta X$.

Proof. It is enough to consider basic open sets. Let U_1 and U_2 be disjoint open sets in X. Since βX is perfect compactification of X we have $O_{\beta X}(U_1 \cup U_2) = O_{\beta X}(U_1) \cup O_{\beta X}(U_2)$. Consider open sets

$$O\langle U_i\rangle = \{F : F \in \exp_\beta X, F \subset U_i\}, \qquad i = 1,\ 2$$

in $\exp_\beta X$. It is clear, that $O\langle U_1\rangle \cap O\langle U_2\rangle = \varnothing$. We will show that

$$O_{\exp \beta X}(O\langle U_1\rangle \cup O\langle U_2\rangle) = O_{\exp \beta X}(O\langle U_1\rangle) \cup O_{\exp \beta X}(O\langle U_2\rangle).$$

The inclusion $\supset$ follows from the definition of the set $O(H)$ (see [1], p. 234). That is why it is enough to show the inverse inclusion. Let $\Phi \subset \beta X$ be a closed set such that $\Phi \notin O_{\exp \beta X}(O\langle U_1\rangle) \cup O_{\exp \beta X}(O\langle U_2\rangle)$. Then $\Phi \in \exp \beta X \setminus O_{\exp \beta X}(O\langle U_i\rangle)$, $i = 1,\ 2$. From [1] (see, p. 234) we have

$$\exp \beta X \setminus O_{\exp \beta X}(O\langle U_i\rangle) = [\exp_\beta X \setminus O\langle U_i\rangle]_{\exp \beta X}, \qquad i = 1,\ 2.$$

Hence $\Phi \in [\exp_\beta X \setminus O\langle U_i\rangle]_{\exp \beta X}$, $i = 1, 2$. Since $O\langle U_1\rangle \cap O\langle U_2\rangle = \varnothing$ by Lemma 2.1 we have

$$[\exp_\beta X \setminus O\langle U_1\rangle]_{\exp \beta X} \cap [\exp_\beta X \setminus O\langle U_2\rangle]_{\exp \beta X} = [\exp_\beta X \setminus O(\langle U_1\rangle \cup O\langle U_2\rangle)]_{\exp \beta X}.$$

Therefore, $\Phi \in [\exp_\beta X \setminus O_{\exp \beta X}(O\langle U_1\rangle \cup O\langle U_2\rangle)]_{\exp \beta X}$, what is equivalent to the belonging $\Phi \in \exp \beta X \setminus O_{\exp \beta X}(\langle U_1\rangle \cup \langle U_2\rangle)$ (see [1], p. 234). In other words, we have $\Phi \notin O_{\exp \beta X}(\langle U_1\rangle \cup \langle U_2\rangle)$. Thus, we have established that the inclusion $O_{\exp \beta X}(\langle U_1\rangle \cup \langle U_2\rangle) \subset O_{\exp \beta X}(O\langle U_1\rangle \cup O_{\exp \beta X}(O\langle U_2\rangle)$ is also true. □

Lemma 2.2. *Let $U_1, \ldots, U_n$; $V_1, \ldots, V_m$ be open subsets of a space X. Then*

$$O\langle U_1, \ldots, U_n\rangle \cap O\langle V_1, \ldots, V_m\rangle \neq \varnothing$$

if and only if for each $i \in \{1, \ldots, n\}$ and for each $j \in \{1, \ldots, m\}$ there exists, respectively $j(i) \in \{1, \ldots, m\}$ and $i(j) \in \{1, \ldots, n\}$, such that $U_i \cap V_{j(i)} \neq \varnothing$ and $U_{i(j)} \cap V_j \neq \varnothing$.

Proof. Assume that for every $i \in \{1, \ldots, n\}$ there exists $j(i) \in \{1, \ldots, m\}$ such that $U_i \cap V_{j(i)} \neq \varnothing$ and for every $j \in \{1, \ldots, m\}$ there exists $i(j) \in \{1, \ldots, n\}$ such that $U_{i(j)} \cap V_j \neq \varnothing$. For any pair $(i,\ j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ for which $U_i \cap V_j \neq \varnothing$, choose a point $x_{ij} \in U_i \cap V_j$ and make a closed set F consisting of these points. Then $F \subset \bigcup\limits_{i=1}^{n} U_i$ and $F \subset \bigcup\limits_{j=1}^{m} V_j$. Besides, $F \cap U_i \neq \varnothing$, $i = 1, \ldots, n$, and $F \cap V_j \neq \varnothing$, $j = 1, \ldots, m$. Therefore, $F \in O\langle U_1, \ldots, U_n\rangle \cap O\langle V_1, \ldots, V_m\rangle$.

We suppose exists $i_0 \in \{1, \ldots, n\}$ such that $U_{i_0} \cap V_j = \varnothing$ for all $j \in \{1, \ldots, m\}$. Then $U_{i_0} \cap \bigcup\limits_{j=1}^{m} V_j = \varnothing$ and for each $F \in O\langle U_1, \ldots, U_n\rangle$ we have $F \not\subset \bigcup\limits_{j=1}^{m} V_j$. Hence, $F \notin O\langle V_1, \ldots, V_m\rangle$. Similarly, every $\Gamma \in O\langle V_1, \ldots, V_m\rangle$ lies in $\bigcup\limits_{j=1}^{m} V_j$. Consequently, $\Gamma \cap U_{i_0} = \varnothing$. Then $\Gamma \notin O\langle U_1, \ldots, U_n\rangle$. Thus, $O\langle U_1, \ldots, U_n\rangle \cap O\langle V_1, \ldots, V_m\rangle = \varnothing$.

□

Corollary 2.1. *Let U, V be open subsets of a space X. Then $O\langle\, U\,\rangle \cap O\langle\, V\,\rangle \neq \varnothing$ if and only if $U \cap V \neq \varnothing$.*

Lemma 2.3. *Let υ be an open cover of a Tychonoff space X with $Kp\,\upsilon = 1$. Then the family* $\exp_\beta \upsilon = \{O\langle U_1, \ldots,\ U_n\rangle : U_i \in \upsilon,\ i = 1, \ldots,\ n;\ n \in \mathbb{N}\}$ *is an open cover of the space* $\exp_\beta X$ *with* $Kp \exp_\beta \upsilon = 1$.

Proof. Let $O\langle G_1, \ldots, G_k\rangle$ be an element of $\exp_\beta \upsilon$. Owing to $Kp\,\upsilon = 1$, Lemma 2.2 implies $O\langle G_1, \ldots, G_k\rangle \cap O\langle U_1, \ldots, U_l\rangle \neq \varnothing$ if and only if $k = l$ and for every $i \in \{1, \ldots k\}$ the equality $G_i = U_j$ holds for some unique $j \in \{1, \ldots k\}$. In other words $O\langle G_1, \ldots, G_k\rangle \cap O\langle U_1, \ldots, U_l\rangle \neq \varnothing$ if and only if $\{G_1, \ldots, G_k\} = \{U_1, \ldots, U_l\}$. Hence, $Kp \exp_\beta \upsilon = 1$.

Let $F \in \exp_\beta X$. There is a subfamily $\upsilon_F \subset \upsilon$ such that $F \subset \bigcup\limits_{U \in \upsilon_F} U$. From a cover $\{F \cap U : U \in \upsilon_F,\ F \cap U \neq \varnothing\}$ of the compact F one can allocate a finite subcover $\{F \cap U_i : i = 1, \ldots, m\}$. We have $F \in O\langle U_1, \ldots, U_m\rangle$. So, the family $\exp_\beta \upsilon$ is a cover of $\exp_\beta X$. On the other hand by the definition of Vietoris topology the cover $\exp_\beta \upsilon$ is open. Thus, $\exp_\beta \upsilon$ is an open cover of $\exp_\beta X$ with $Kp \exp_\beta \upsilon = 1$. □

The following statement is the main result of the section.

Theorem 2.2. *For a Tychonoff space X its hyperspace* $\exp_\beta X$ *is* Π*-complete if and only if X is* Π*-complete.*

Proof. Property (a) implies the Π-completeness of the closed subset $X \subset \exp_\beta X$.

Let X be a Π-complete space and $F \in \exp \beta X \setminus \exp_\beta X$. Owing to Theorem 2.1 $\exp \beta X$ is a perfect compactification of $\exp_\beta X$. That is why it is enough to show the existence an open cover ω of $\exp_\beta X$ with $Kp\,\omega = 1$, pricking out F in $\exp \beta X$. We have $F \not\subset X$. By Corollary 1.1 for every point $x \in F \setminus X \subset \beta X \setminus X$ there exists an open cover ω_x with $Kp\,\omega_x = 1$, pricking out x in βX, i. e. $x \notin \cup[\omega_x]_{\beta X}$. Fix a point $x_0 \in F \setminus X$. Consequently $F \not\subset [U]_{\beta X}$ for every $U \in \omega_{x_0}$. Hence $F \notin O\langle [U_1]_{\beta X}, \ldots, [U_n]_{\beta X}\rangle$ for every n-tuple $\{U_1, \ldots, U_n\} \subset \omega_{x_0}$. Therefore by equality (2.1) we have

$$F \notin \cup[\exp \omega_{x_0}]_{\exp \beta X} = \{[O\langle U_1, \ldots, U_n\rangle]_{\exp \beta X} : U_i \in \omega_{x_0},\, i = 1, \ldots, n;\, n \in \mathbb{N}\}.$$

Now, the using of Theorem 1.1 and Lemma 2.3 completes the proof. □

3 Π-completeness of the map $\exp_\beta f$

For a continuous map $f\colon (X, \tau_X) \to (Y, \tau_Y)$ and $O \in \tau_Y$ a preimage $f^{-1}O$ is called *a tube* (above O). Remind, a continuous map $f\colon X \to Y$ is called [4] a *T_0-map*, if for each pair of distinct points $x, x' \in X$, such that $f(x) = f(x')$, at least one of these points has an open neighbourhood in X which does not contain another point. A continuous map $f\colon X \to Y$ is called *totally regular*, if for each point $x \in X$ and every closed set F in X not containing x there exists an open neighbourhood O of $f(x)$ such that in the tube $f^{-1}O$ the sets $\{x\}$ and F are functional separable. Totally regular T_0-map is said to be *a Tychonoff map*.

Obviously, each continuous map $f\colon X \to Y$ of a Tychonoff space X into a topological space Y is a Tychonoff map. In this case for every Tychonoff space X owing to the fact that the set $\exp_\beta X$ is a Tychonoff space with respect to the Vietoris topology, the map $\exp_\beta f\colon \exp_\beta X \to \exp_\beta Y$ is a Tychonoff map.

A continuous, closed map $f\colon X \to Y$ is said to be *compact* if the preimage $f^{-1}y$ of each point $y \in Y$ is compact. A continuous map $f\colon X \to Y$ is compact if and only if for each point $y \in Y$ and every cover ω of the fibre $f^{-1}y$, consisting of open sets in X, there is an open neighbourhood O of y in Y such that the tube $f^{-1}O$ can be covered with a finite subfamily of ω.

A compact map $bf\colon b_fX \to Y$ is said to be *a compactification* of a continuous map $f\colon X \to Y$ if X is everywhere dense in b_fX and $bf|_X = f$. On the set of all compactifications of the map f it is possible to introduce a partial order: for the compactifications $b_1f\colon b_{1f}X \to Y$ and $b_2f\colon b_{2f}X \to Y$ of f we put $b_1f \le b_2f$ if there is a natural map of $b_{2f}X$ onto $b_{1f}X$. B. A. Pasynkov showed that for each Tychonoff map $f\colon X \to Y$ there exists its maximal compactification $g\colon Z \to Y$, which he denoted by βf, and the space Z where this maximal compactification defines by β_fX. For a given Tychonoff map f its maximal compactification βf is unique.

Remark 1. Note that the maps b_1f, b_2f, βf are compactifications of the map f. The spaces $b_{1f}X$, b_{2f}, β_fX are some extensions of X but they are not obliged to be compactifications.

A Tychonoff map $f: X \to Y$ is said to be Π-*complete*, if for every point $x \in \beta_f X \setminus X$ there exists a disjoint clopen (≅closed-open) cover of X pricking out x in $\beta_f X$ ([5], pp. 120 – 121).

We introduce the following notion.

Definition 1. [7] A compactification $bf\colon b_f X \to Y$ of a Tychonoff map $f\colon X \to Y$ is said to be a *perfect compactification* of f if for each point $y \in Y$ and for every disjoint open sets U_1 and U_2 in X there exists an open neighbourhood $O \subset Y$ of y such that the equality

$$O_{b_fX}(U_1 \cup U_2) \cap bf^{-1}O = \big(O_{b_fX}(U_1) \cup O_{b_fX}(U_2)\big) \cap bf^{-1}O$$

holds.

Let $f\colon X \to Y$ be a continuous map of a Tychonoff space X into a space Y. It is well known there exists a compactification vX of X such that f has a continuous extension $vf\colon vX \to Y$ on vX. It is clear, vf is a perfect compactification of f.

The following result is an analog of Theorem 1.1 for the case of maps.

Theorem 3.1. *Let $bf\colon b_f X \to Y$ be a perfect compactification of a Tychonoff map $f\colon X \to Y$. The map f is Π-complete if and only if for every point $x \in b_f X \setminus X$ there exists a disjoint clopen cover of X pricking out x in $b_f X$.*

Proof. The proof is carried out similar to the proof of Theorem 1.1 Π from [4]. □

The following result is a variant of Theorem 2.1 for the case of maps.

Theorem 3.2. *Let $f\colon X \to Y$ be a Tychonoff map. Then*

$$\exp_\beta \beta f\colon \exp_\beta \beta_f X \to \exp_\beta Y$$

is a perfect compactification of

$$\exp_\beta f\colon \exp_\beta X \to \exp_\beta Y.$$

Proof. The proof is similar to the proof of Theorem 2.1. Here the equality

$$(\exp_\beta \beta f)^{-1} O\langle U_1, \ldots, U_m\rangle = O\langle \beta f^{-1}(U_1), \ldots, \beta f^{-1}(U_m)\rangle$$

is used. □

The following statement is the main result of this section.

Theorem 3.3. *The Tychonoff map $\exp_\beta f\colon \exp_\beta X \to \exp_\beta Y$ is Π-complete if and only if a map $f\colon X \to Y$ is Π-complete.*

Proof. Let $\exp_\beta f\colon \exp_\beta X \to \exp_\beta Y$ be a Π-complete map. It implies that $f\colon X \to Y$ is a Π-complete map since $X \cong \exp_1 X$ is a closed set in $\exp_\beta X$.

Let now $f\colon X \to Y$ be a Π-complete map. We consider an arbitrary point

$$F \in \exp_\beta \beta_f X \setminus \exp_\beta X$$

and using Theorems 3.1 and 3.2 show that there exists a disjoint clopen cover of $\exp_\beta X$ pricking out the point F in $\exp_\beta \beta_f X$.

By definition for every point $x \in F \setminus X \subset \beta_f X \setminus X$ there exists a disjoint clopen cover ω_x of X pricking out x in $\beta_f X$. Fix a point $x_0 \in F \setminus X$. Then $x_0 \notin \cup[\omega_{x_0}]_{\beta_f X}$. Hence $F \not\subset [U]_{\beta_f X}$ for all $U \in \omega_{x_0}$. Consequently, $F \notin O\langle [U_1]_{\beta_f X}, \ldots, [U_n]_{\beta_f X}\rangle$ for every finite subcollection $\{U_1, \ldots, U_n\} \subset \omega_{x_0}$. Then $F \notin \cup[\exp_\beta \omega_{x_0}]$ owing to (2.1). Applying Lemma 2.3 and equality (2.1) one more time we conclude that $\exp_\beta \omega_{x_0}$ is a disjoint clopen cover of $\exp_\beta X$ pricking out the considered point F in $\exp_\beta \beta_f X$. □

Corollary 3.1. *The functor $\exp_\beta$ lifts onto category of Π-complete spaces and their continuous maps.*

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