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INTERPOLATION THEOREM FOR STOCHASTIC PROCESSES

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Abstract. In this paper the class of stochastic processes $N_{p,q}(F)$ is introduced and an interpolation theorem for a quasilinear transform is proved. This theorem is a generalization of the Marcinkiewicz interpolation theorem.

1 Introduction

Assume that $(\Omega, \mathfrak{F}, P)$ is a complete probability space. A family $F = \{\mathfrak{F}_n\}_{n\geq 1}$ of σ-algebras $\mathfrak{F}_n$ such that $\mathfrak{F}_1 \subseteq ... \subseteq \mathfrak{F}_n \subseteq ... \subseteq \mathfrak{F}$ is called a filtration.

Let F be a filtration and a sequence $\{X_n\}_{n\geq 1}$ of random variables X_n be such that for any $n \geq 1$ X_n is a measurable function with respect to the σ-algebra $\mathfrak{F}_n$. Then we say that the set $X = (X_n, \mathfrak{F}_n)_{n\geq 1}$ is a stochastic process.

We consider the nondecreasing sequence of numbers $\overline{X}(F) = \{\overline{X}_n(F)\}_n$, where

$$\overline{X}_k(F) = \sup_{A\in\mathfrak{F}_k, P(A)>0} \frac{1}{P(A)} \left| \int_A X_k P(d\omega) \right|, \quad k \in \mathbb{N}.$$

By $N_{p,q}(F)$, $0 < p < \infty$, $0 < q \leq \infty$ we denote the set of all stochastic processes X, defined on F for which

$$\|X\|_{N_{p,q}(F)} = \left(\sum_{k=1}^{\infty} k^{-1-\frac{q}{p}} \overline{X}_k^q \right)^{\frac{1}{q}} < \infty \tag{1}$$

if $0 < q < \infty$ and

$$\|X\|_{N_{p,\infty}(F)} = \sup_k k^{-\frac{1}{p}} \overline{X}_k < \infty \tag{2}$$

if $q = \infty$.

These classes are similar to the net spaces, which were introduced in [4], [5].

In this paper we prove a Marcinkiewicz-type interpolation theorem for the introduced spaces. An interpolation method, essentially related to the properties of the Markov stopping times, is introduced.

Let us note that the interpolation properties of a quasilinear transform in the space of martingales are studied in [2], [3], [6], [7] and other papers.

We write $A \lesssim B$ (or $A \gtrsim B$) if $A \leq cB$ (or $cA \geq B$) for some positive constant c independent of appropriate quantities involved in the expressions A and B. Notation $A \asymp B$ means that $A \lesssim B$ and $A \gtrsim B$.

2 Properties of the spaces $N_{p,q}(F)$

It is said that a stochastic process $(X_n, \mathfrak{F}_n)_{n\geq 1}$, is a martingale if for every $n \in \mathbb{N}$ the following conditions hold: 1) $E|X_n| < \infty$; 2) $E(X_{n+1}|\mathfrak{F}_n) = X_n$ (a.p.). If instead of property 2) it is assumed that $E(X_{n+1}|\mathfrak{F}_n) \geq X_n$ $(E(X_{n+1}|\mathfrak{F}_n) \leq X_n)$, then it is said that the process $X = (X_n, \mathfrak{F}_n)_{n=1}^{\infty}$ is a submartingale (supermartingale).

Definition. *We say that a stochastic process X belongs to the class $W(F)$ if there exists a constant c such that for every $k \leq m$ and for every $A \in \mathfrak{F}_k$*

$$\left|\int_A X_k P(d\omega)\right| \leq c\left|\int_A X_m P(d\omega)\right|.$$

This inequality implies that $\overline{X}_k(F) \leq c\overline{X}_m(F)$ for every $k \leq m$. The class $W(F)$ contains martingales, nonnegative submartingales, nonpositive supermartingales. The property of a process which is determined by belonging of the process to the class $W(F)$ we call the generalized monotonicity.

Lemma 1. *Let $X \in W(F)$. Then*

1) for $0 < q \leq q_1 \leq \infty$,

$$\|X\|_{N_{p,q_1}(F)} \leq c_{p,q,q_1}\|X\|_{N_{p,q}(F)},$$

2) for $0 < p_1 < p < \infty$, $0 < q, q_1 \leq \infty$,

$$\|X\|_{N_{p_1,q_1}(F)} \leq c_{p,q,p_1,q_1}\|X\|_{N_{p,q}(F)},$$

where $c_{p,q,q_1}, c_{p,q,p_1,q_1} > 0$ depend only on the indicated parameters.

Remark. *Here and in the sequel constants c, c_1 etc. may be different in different formulas.*

Proof. Let $\varepsilon > 0$. By Minkowski's inequality and by the generalized monotonicity of a process $X = (X_n, \mathfrak{F}_n)_{n\geq 1}$ we get

$$\|X\|_{N_{p,q_1}(F)} = \left(\sum_{k=1}^{\infty} k^{\varepsilon q_1 - 1} k^{-\varepsilon q_1 - \frac{q_1}{p}} \overline{X}_k^{q_1}\right)^{\frac{1}{q_1}} \lesssim$$

$$\lesssim \left(\sum_{k=1}^{\infty} k^{\varepsilon q_1 - 1}\left(\sum_{r=k}^{\infty} r^{-q\left(\varepsilon + \frac{1}{p}\right) - 1}\right)^{\frac{q_1}{q}} \overline{X}_k^{q_1}\right)^{\frac{1}{q_1}} \lesssim$$

$$\lesssim \left(\sum_{k=1}^{\infty} k^{\varepsilon q_1 - 1} \left(\sum_{r=k}^{\infty} r^{-q(\varepsilon + \frac{1}{p}) - 1} \overline{X}_r^q \right)^{\frac{q_1}{q}} \right)^{\frac{1}{q_1}} \lesssim$$

$$\lesssim \left(\sum_{r=1}^{\infty} r^{-q(\varepsilon + \frac{1}{p}) - 1} \overline{X}_r^q \left(\sum_{k=1}^{r} k^{\varepsilon q_1 - 1} \right)^{\frac{q}{q_1}} \right)^{\frac{1}{q}} \lesssim$$

$$\lesssim \left(\sum_{r=1}^{\infty} r^{-\frac{q}{p} - 1} \overline{X}_r^q \right)^{\frac{1}{q}} = \|X\|_{N_{p,q}(F)}.$$

To prove the second statement it is enough to show that $\|X\|_{N_{p_1,q_1}(F)} \leq \|X\|_{N_{p,\infty}(F)}$ and apply the first statement. Since $p_1 < p$, we have

$$\|X\|_{N_{p_1,q_1}(F)} = \left(\sum_{k=1}^{\infty} k^{-\frac{q_1}{p_1} - 1} \overline{X}_k^{q_1} \right)^{\frac{1}{q_1}} \leq$$

$$\leq \left(\sum_{k=1}^{\infty} k^{\frac{q_1}{p} - \frac{q_1}{p_1} - 1} \right)^{\frac{1}{q_1}} = \|X\|_{N_{p,\infty}(F)}.$$

□

Lemma 2. *Let $0 < p < \infty$, $a > 1$. If $X \in W(F)$, then for $0 < q < \infty$*

$$\|X\|_{N_{p,q}(F)} \asymp \left(\sum_{k=0}^{\infty} \left(a^{-\frac{k}{p}} \overline{X_{a^k}} \right)^q \right)^{\frac{1}{q}}, \tag{3}$$

and for $q = \infty$

$$\|X\|_{N_{p,\infty}(F)} \asymp \sup_{k \in \mathbb{N}} a^{-\frac{k}{p}} \overline{X_{a^k}}.$$

Here by $\overline{X_{a^k}}$ we mean $\overline{X_{[a^k]}}$, where $[a^k]$ is the integer part of the number a^k. Moreover, expressions X^α and $\sum_{k=\alpha}^{\beta} b_k$ will be understood as $X^{[\alpha]}$, $\sum_{k=[\alpha]}^{[\beta]} b_k$ respectively.

Proof. Using the generalized monotonicity of process X, we have

$$\|X\|_{N_{p,q}(F)} = \left(\sum_{k=1}^{\infty} k^{-\frac{q}{p} - 1} \overline{X_k}^q \right)^{\frac{1}{q}} = \left(\sum_{k=0}^{\infty} \sum_{l=a^k}^{a^{k+1}-1} l^{-\frac{q}{p} - 1} \overline{X_l}^q \right)^{\frac{1}{q}} \gtrsim$$

$$\gtrsim \left(\sum_{k=0}^{\infty} a^{-\frac{kq}{p}} \overline{X_{a^k}}^q \sum_{l=a^k}^{a^{k+1}-1} \frac{1}{l} \right)^{\frac{1}{q}} \gtrsim \left(\sum_{k=0}^{\infty} \left(a^{-\frac{k}{p}} \overline{X_{a^k}} \right)^q \right)^{\frac{1}{q}}.$$

One can prove the reverse estimate in a similar way.

We will need the following Hardy-type inequalities. □

Lemma 3. *Let $s \geq 1,\ \nu > 0,\ \alpha > 0,\ \beta > 0,\ \gamma > 0$, then for a nonnegative sequence $a = \{a_k\}_k$ the following inequalities hold:*

$$\left(\sum_{k=1}^{\infty} k^{-\alpha s-1}\left(\sum_{l=1}^{(\gamma k)^{\nu}} l^{\beta-1}a_l\right)^s\right)^{1/s} \leq \gamma^{\alpha} C_{\alpha,\beta,s,\nu}\left(\sum_{k=1}^{\infty} k^{(\beta-\frac{\alpha}{\nu})s-1}a_k^s\right)^{1/s},$$

$$\left(\sum_{k=1}^{\infty} k^{\alpha s-1}\left(\sum_{l=(\gamma k)^{\nu}}^{\infty} l^{\beta-1}a_l\right)^s\right)^{1/s} \leq \gamma^{-\alpha} C_{\alpha,\beta,s,\nu}\left(\sum_{k=1}^{\infty} k^{(\beta+\frac{\alpha}{\nu})s-1}a_k^s\right)^{1/s},$$

$$\left(\sum_{k=0}^{\infty}\left(2^{-\alpha k}\sum_{m=0}^{\gamma k} 2^{\beta m}a_m\right)^s\right)^{1/s} \leq C_{\alpha,\beta,s,\gamma}\left(\sum_{k=0}^{\infty}\left(2^{(\beta-\frac{\alpha}{\nu})k}a_k\right)^s\right)^{1/s},$$

$$\left(\sum_{k=0}^{\infty}\left(2^{\alpha k}\sum_{m=\gamma k}^{\infty} 2^{\beta m}a_m\right)^s\right)^{1/s} \leq C_{\alpha,\beta,s,\gamma}\left(\sum_{k=0}^{\infty}\left(2^{(\beta+\frac{\alpha}{\nu})k}a_k\right)^s\right)^{1/s}.$$

3 Interpolation theorem

Let $\mathbf{T} = \{T_n\}_{n=1}^{\infty}$ be a transform that transforms a stochastic process X, which is defined on the system $F = \{\mathfrak{F}\}_{n=1}^{\infty}$, to the stochastic process $\mathbf{T}(X) = \{T_n(X), \Phi_n\}_{n=1}^{\infty}$, which is defined on the system $R = \{\mathfrak{R}\}_{n=1}^{\infty}$. It is said that the transform T is quasilinear if there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ the following inequality is almost probably true

$$|T_n(X) - T_n(Y)| \leq C|T_n(X-Y)|. \tag{4}$$

A random variable τ, which takes values in the set $(1, 2, ..., \infty)$, is called the Markov time of the filtration $G = \{\mathfrak{G}_n\}_{n\geq 1}$ if $\{\omega : \tau(\omega) = n\} \in \mathfrak{G}_n$ for any $n \in \mathbb{N}$. The Markov time τ, for which $\tau(\omega) < \infty$ (a.p.), is called the stopping time.

Let $X = (X_n, \mathfrak{G}_n)_{n\geq 1}$ be a stochastic process, τ be Markov time. By X^{τ} we denote the stopped process $X^{\tau} = (X_{n\wedge\tau}, \mathfrak{G}_n)$, where $X_{n\wedge\tau} = \sum\limits_{m=1}^{n-1} X_m\chi_{\tau=m}(\omega) + X_n\chi_{\tau\geq n}(\omega)$ and $\chi_A(\omega)$ is the characteristic function of the set A.

It is known that if a process $X = (X_n, \mathfrak{G}_n)_{n\geq 1}$ is a martingale (submartingale), then the process $X^{\tau} = (X_{n\wedge\tau}, \mathfrak{G}_n)_{n\geq 1}$ is also a martingale (submartingale) [2].

Denote $X_n^*(\omega) = \max\limits_{1\leq k\leq n} |X_k(\omega)|$ and $X^* = (X_n^*, \mathfrak{G}_n)_{n\geq 1}$.

The transforms X^{τ} and X^* of the stochastic process X are examples of quasilinear transforms.

Theorem. *Let $0 < p_0 < p_1 < \infty,\ 0 < q_0 < q_1 < \infty,\ 0 < \theta < 1,\ 1 \leq s \leq \infty$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $X \in W(F)$ and $T = \{T_n\}_{n=1}^{\infty}$ be a quasilinear transform. If for any $k \in \mathbb{N}$ the following conditions hold*

$$\|T(X^k)\|_{N_{q_1,\infty}(R)} \leq M_1\|X^k\|_{N_{p_1,1}(F)}, \tag{5}$$

$$\|T(X - X^k)\|_{N_{q_0,\infty}(R)} \le M_0 \|X - X^k\|_{N_{p_0,1}(F)}, \tag{6}$$

then

$$\|T(X)\|_{N_{q,s}(R)} \le C M_0^{1-\theta} M_1^{\theta} \|X\|_{N_{p,s}(F)}, \tag{7}$$

where $C > 0$ depends only on $p_0, p_1, q_0, q_1, \theta$.

Proof. Let $\gamma > 0$, $\nu > 0$. Using quasilinearity of the transform T and Minkowski's inequality, we have

$$\|T(X)\|_{N_{q,s}(R)} = \left(\sum_{k=1}^{\infty} k^{-1-\frac{s}{q}} \overline{T_k(X)}^s\right)^{\frac{1}{s}} \lesssim$$

$$\lesssim \left(\left(\sum_{k=1}^{\infty} k^{-1-\frac{s}{q}} \overline{T_k(X^{(\gamma k)^\nu})}^s\right)^{\frac{1}{s}} + \left(\sum_{k=1}^{\infty} k^{-1-\frac{s}{q}} \overline{T_k(X - X^{(\gamma k)^\nu})}^s\right)^{\frac{1}{s}}\right). \tag{8}$$

Denote $\lambda_k = (\gamma k)^\nu$ for any $k \in \mathbb{N}$. Using the definition of functional (2) and conditions (5) and (6), we have

$$\overline{T_k(X^{\lambda_k})} \le k^{\frac{1}{q_1}} M_1 \|X^{\lambda_k}\|_{N_{p_1,1}(F)}, \qquad \overline{T_k(X - X^{\lambda_k})} \le k^{\frac{1}{q_0}} M_0 \|X - X^{\lambda_k}\|_{N_{p_0,1}(F)}. \tag{9}$$

By definitions of processes X^k and $X - X^k$, taking into account the generalized monotonicity of the sequences $\{\overline{X_k}\}$ and $\{\overline{X_\infty - X_k}\}$, we have

$$\|X^{\lambda_k}\|_{N_{p_1,1}(F)} = \sum_{l=1}^{\lambda_k - 1} l^{-1-\frac{1}{p_1}} \overline{X}_l + \overline{X}_{\lambda_k} \sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_1}} \lesssim$$

$$\lesssim \left(\sum_{l=1}^{\lambda_k} l^{-1-\frac{1}{p_1}} \overline{X_l} + \lambda_k^{\frac{1}{p_0} - \frac{1}{p_1}} \sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_1}} \overline{X_l}\right),$$

$$\|X - X^{\lambda_k}\|_{N_{p_0,1}(F)} = \sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_0}} (\overline{X_l - X_{\lambda_k}}) \lesssim \sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_0}} \overline{X_l}. \tag{10}$$

Substituting estimates (9) and (10) in the first summand of the right-hand side of (8) and using Minkowski's inequality we have

$$\left(\sum_{k=1}^{\infty} k^{-1-\frac{s}{q}} \overline{T_k(X^{\lambda_k})}^s\right)^{\frac{1}{s}} \lesssim$$

$$\lesssim M_1 \left(\sum_{k=1}^{\infty} k^{-1-\frac{s}{q}+\frac{s}{q_1}} \left(\sum_{l=1}^{\lambda_k} l^{-1-\frac{1}{p_1}} \overline{X_l} + \lambda_k^{\frac{1}{p_0} - \frac{1}{p_1}} \sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_0}} \overline{X_l}\right)^s\right)^{\frac{1}{s}} \lesssim$$

$$\lesssim M_1 \left(\left(\sum_{k=1}^{\infty} \frac{1}{k} \left(k^{-\frac{1}{q}+\frac{1}{q_1}} \sum_{l=1}^{\lambda_k} l^{-1-\frac{1}{p_1}} \overline{X_l}\right)^s\right)^{\frac{1}{s}} + \right. \tag{11}$$

$$+\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(k^{-\frac{1}{q}+\frac{1}{q_1}}\lambda_k^{\frac{1}{p_0}-\frac{1}{p_1}}\sum_{l=\lambda_k}^{\infty}l^{-1-\frac{1}{p_0}}\overline{X_l}\right)^s\right)^{\frac{1}{s}}\right). \tag{12}$$

Let us estimate the first summand in the right-hand side of (12):

$$\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(k^{-\frac{1}{q}+\frac{1}{q_1}}\sum_{l=1}^{\lambda_k}l^{-1-\frac{1}{p_1}}\overline{X_l}\right)^s\right)^{\frac{1}{s}}=\left(\sum_{k=1}^{\infty}k^{(\frac{1}{q_1}-\frac{1}{q})-1}\left(\sum_{l=1}^{(\gamma k)^{\nu}}l^{\frac{1}{p_1'}-1}\left(\frac{\overline{X_l}}{l}\right)\right)^s\right)^{\frac{1}{s}}.$$

By applying Lemma 3 and using equalities

$$\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1},\quad \frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$$

and

$$\nu=\frac{\left(\frac{1}{q_0}-\frac{1}{q_1}\right)}{\left(\frac{1}{p_0}-\frac{1}{p_1}\right)},$$

we obtain

$$\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(k^{-\frac{1}{q}+\frac{1}{q_1}}\sum_{l=1}^{\lambda_k}l^{-1-\frac{1}{p_1}}\overline{X_l}\right)^s\right)^{\frac{1}{s}}\lesssim\gamma^{\frac{1}{q}-\frac{1}{q_1}}\|X\|_{N_{p,s}(F)}. \tag{13}$$

By using Lemma 3 we estimate the second summand of the right-hand side of (12):

$$\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(k^{-\frac{1}{q}+\frac{1}{q_1}}\lambda_k^{\frac{1}{p_0}-\frac{1}{p_1}}\sum_{l=\lambda_k}^{\infty}l^{-1-\frac{1}{p_1}}\overline{X_l}\right)^s\right)^{\frac{1}{s}}=$$

$$=\gamma^{\frac{1}{q_0}-\frac{1}{q_1}}\left(\sum_{k=1}^{\infty}k^{(\frac{1}{q_0}-\frac{1}{q})s-1}\left(\sum_{l=(\gamma k)^{\nu}}^{\infty}l^{\frac{1}{p_0'}-1}\frac{\overline{X_l}}{l}\right)^s\right)^{\frac{1}{s}}\lesssim$$

$$\lesssim\gamma^{\frac{1}{q}-\frac{1}{q_1}}\|X\|_{N_{p,s}(F)}. \tag{14}$$

Substituting estimates (13) and (14) in (12), we get

$$\left(\sum_{k=1}^{\infty}k^{-1-\frac{s}{q}}\overline{T_k(X^{\lambda_k})}^s\right)^{\frac{1}{s}}\lesssim M_1\gamma^{\frac{1}{q}-\frac{1}{q_1}}\|X\|_{N_{p,s}(F)}. \tag{15}$$

Applying the second estimates of (9) and (10) to the second summand of the right hand side of (8), we have

$$\left(\sum_{k=1}^{\infty}k^{-1-\frac{s}{q}}\overline{T_k(X-X^{\lambda_k})}^s\right)^{\frac{1}{s}}\lesssim$$

$$\lesssim M_0\left(\sum_{k=1}^{\infty}k^{(\frac{1}{q_0}-\frac{1}{q})s-1}\left(\sum_{l=\lambda_k}^{\infty}l^{\frac{1}{p_0'}-1}\frac{\overline{X}_l}{l}\right)^s\right)^{\frac{1}{s}}\lesssim M_0\gamma^{\frac{1}{q}-\frac{1}{q_0}}\|X\|_{N_{p,s}(F)}. \qquad (16)$$

Choosing $\gamma = M_0^{\frac{q_0q_1}{q_1-q_0}}M_1^{\frac{q_0q_1}{q_0-q_1}}$ and substituting (15) and (16) in (8), we obtain (7). □

The classical interpolation Marcinkievicz - Calderon theorem follows from this theorem. Let f be a measurable function on (Ω,μ) and

$$m(\sigma,f)=\mu\{x:|f(x)|>\sigma\}$$

its distribution function. The function

$$f^*(t)=\inf\{\sigma:m(\sigma,f)\le t\}$$

is called the nonincreasing rearrangement of the function f.

Let $0<p<\infty$ and $0<q\le\infty$. The Lorentz space $L_{pq}(\Omega,\mu)$ is defined as the set of all measurable functions f such that

$$\|f\|_{L_{p,q}}=\left(\int_0^\infty(t^{1/p}f^*(t))^q\frac{dt}{t}\right)^{1/q}<\infty$$

for $0<q<\infty$ and

$$\|f\|_{L_{p,\infty}}=\sup_t\ t^{1/p}f^*(t)<\infty$$

for $q=\infty$.

Corollary (Marcinkievicz–Calderon theorem)**.** *Let* $1\le s\le\infty$, $0<p_0<p_1<\infty, 0<q_0<q<q_1\le\infty$, $\theta\in(0,1), 1/p=(1-\theta)/p_0+\theta/p_1$, $1/q=(1-\theta)/q_0+\theta/q_1$. *If T is a quasilinear map and*

$$T:L_{p_i,1}(D,\nu)\to L_{q_i,\infty}(\Omega,\mu)\quad \textit{with the norm } M_i,\quad i=0,1,$$

then

$$T:L_{p,s}(D,\nu)\to L_{q,s}(\Omega,\mu)\quad \textit{and}\quad \|T\|\le M_0^{1-\theta}M_1^\theta.$$

Proof. Let $f\ge 0$, $f\in L_{p,s}(\Omega,\mu)$ and $f^*(t)$ be the nonincreasing rearrangement of f. Let us define sets $\Omega_n=\{w\in\Omega: f(w)\le f^*(1/n)\}$, $n\in\mathbb{N}$. The sequence $X=(X_n,\mathfrak{F}_n)_{n\ge1}$ is a stochastic process of class $W(F)$, where $X_n(w)=\min\{f(w),f^*(1/n)\}$, $\mathfrak{F}_n$ is the minimal $\sigma-$ algebra containing set Ω and system of all measurable subsets Ω_n. By the property of monotonicity of f^* we have

$$\|f\|_{L_{p,s}(\Omega,\mu)}\asymp\left(\sum_{k=1}^{\infty}\left(k^{-\frac{1}{p}}f^*(1/k)\right)^s\frac{1}{k}\right)^{1/s}=\left(\sum_{k=1}^{\infty}\left(k^{-\frac{1}{p}}\overline{X_k}\right)^s\frac{1}{k}\right)^{1/s}=\|X\|_{N_{p,s}(F)}.$$

Let X^n, $X-X^n$ be the stopping time, the starting time respectively, of the process X corresponding to the time n. Then we also have

$$\|X_n\|_{L_{p,s}(\Omega,\mu)}\asymp\|X^n\|_{N_{p,s}(F)}, \qquad (17)$$

$$\|f - X_n\|_{L_{p,s}(\Omega,\mu)} \asymp \|X - X^n\|_{N_{p,s}(F)}. \tag{18}$$

For a quasilinear operator T we will define a transform $\mathbf{T}$ which transforms a sequence $X = \{X_k\}_{k=1}^{\infty}$ to the sequence $\mathbf{T}X = \{TX_k\}_{k=1}^{\infty}$. Define the system of sets $\Phi = \{\Phi_k\}_{k=1}^{\infty}, \Phi_k = \{\Omega_m\}_{m=1}^{k}$. Then for the stopped sequence $X^n = \left(X_{\min(k,n)}, F_k\right)_{k\geq 1}$, we have

$$\|TX^n\|_{N_{q_0,\infty}(F)} = \sup_k k^{\frac{1}{q_0}} \sup_{A\in\Phi_k} \frac{1}{P(A)} \left| \int_A TX_{\min(k,n)}(y)dP(y) \right| \leq$$

$$\leq \sup_{k\leq r\leq n} \sup_{k\in\mathbb{N}} k^{\frac{1}{q_0}} \sup_{P(A)\geq\frac{1}{k}} \frac{1}{P(A)} \left| \int_A TX_r(y)dP(y) \right| \lesssim$$

$$\lesssim \sup_{1\leq r\leq n} \|TX_r\|_{L_{q_0,\infty}} \leq \sup_{1\leq r\leq n} M_0 \|X_r\|_{L_{p_0,1}} \lesssim M_0 \|X_n\|_{L_{p_0,1}}.$$

According to (17) we have

$$\|TX^n\|_{N_{q_0,\infty}(\Phi)} \lesssim M_0 \|X\|_{N_{p_1,1}(F)}$$

and

$$\|T(X - X^n)\|_{N_{q_1,\infty}} = \sup_{k\geq n} k^{\frac{1}{q_1}} \sup_{A\in\Phi_k} \frac{1}{P(A)} \left| \int_{P(A)} T\left(X_k - X_n\right)(y)dP(y) \right| \leq$$

$$\leq \sup_{r\geq n} \sup_{k\in\mathbb{N}} k^{\frac{1}{q_1}} \sup_{P(A)\geq\frac{1}{k}} \frac{1}{P(A)} \left| \int_{P(A)} T\left(X_r - X_n\right)dP \right| \lesssim$$

$$\lesssim \sup_{r\geq n} \|T(X_r - X_n)\|_{L_{q_1,\infty}} \leq M_1 \sup_{r\geq n} \|X_r - X_n\|_{L_{p,1}} = M_1 \|f - X_n\|_{L_{p,1}}.$$

Thus, taking into account (18), we get

$$\|T(X - X^n)\|_{N_{q_1,\infty}(\Phi)} \lesssim M_1 \|X - X_n\|_{N_{p_1,1}(F)},$$

then, by the Theorem, we have

$$\|TX\|_{N_{q,s}(\Phi)} \lesssim M_0^{1-\theta} M_1^{\theta} \|X\|_{N_{p,s}(F)},$$

which is equivalent to

$$\|Tf\|_{L_{q,s}} \lesssim M_0^{1-\theta} M_1^{\theta} \|f\|_{L_{p,s}}.$$

In the case when f changes sign, we consider the representation $f = f_+ - f_-$, where $f_+ = \max(f(x), 0) \geq 0, f_- = f_+ - f \geq 0$. Using the quasilinearity of the operator T we obtain the statement of the corollary. $\square$

References

[1] J. Bergh, J. Löfström, *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschaften, No. 223. Springer Verlag, Berlin – New York, 1976.

[2] D.L. Burkholder, R.F. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales.* Acta mathematica, 124 (1970), 250 – 304.

[3] M. Milman, *On interpolation of martingale L^p spaces.* Indiana university journal, 30, no. 2 (1981), 313 – 318.

[4] E.D. Nursultanov, *Interpolation properties of some anisotropic spaces and Hardy–Littlewood type inequalities.* East J. App., 3 (1998), 243 – 275.

[5] E.D. Nursultanov, *Network spaces and inequalities of Hardy–Littlewood type.* Sb. Math., 189, no. 3-4 (1998), 399 – 419 (in Russian).

[6] F. Weisz, *Interpolation between martingale Hardy and BMO spaces, the real method.* Bull. Sc. math., 116 (1992), 145 – 158.

[7] F. Weisz, *Interpolation between two-parameter martingale Hardy spaces, the real method.* Bull. Sc. math., 115 (1991), 253 – 264.

[8] F. Weisz, *Interpolation between continuous parameter martingale spaces, the real method.* Acta Math. Hungar., 68 (1995), 37 – 54.

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