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**VLADIMIR MIKHAILOVICH FILIPPOV**

(to the 70th birthday)

Vladimir Mikhailovich Filippov was born on 15 April 1951 in the city of Uryupinsk, Stalingrad Region of the USSR. In 1973 he graduated with honors from the Faculty of Physics and Mathematics and Natural Sciences of the Patrice Lumumba University of Peoples' Friendship in the specialty "Mathematics". In 1973-1975 he is a postgraduate student of the University; in 1976-1979 - Chairman of the Young Scientists' Council; in 1980-1987 - Head of the Research Department and the Scientific Department; in 1983-1984 - scientific work at the Free University of Brussels (Belgium); in 1985-2000 - Head of the Mathematical Analysis Department; from 2000 to the present - Head of the Comparative Educational Policy Department; in 1989–1993 - Dean of the Faculty of Physics, Mathematics and Natural Sciences; in 1993–1998 - Rector of the Peoples' Friendship University of Russia; in 1998-2004 - Minister of General and Professional Education, Minister of Education of the Russian Federation; in 2004-2005 - Assistant to the Chairman of the Government of the Russian Federation (in the field of education and culture); from 2005 to May 2020- Rector of the Peoples' Friendship University of Russia, since May 2020 - President of the Peoples' Friendship University of Russia, since 2013 - Chairman of the Higher Attestation Commission of the Ministry of Science and Higher Education of the Russian Federation.

In 1980, he defended his PhD thesis in the V.A. Steklov Mathematical Institute of Academy of Sciences of the USSR on specialty 01.01.01 - mathematical analysis (supervisor - a corresponding member of the Academy of Sciences of the USSR, Professor L.D. Kudryavtsev), and in 1986 in the same Institute he defended his doctoral thesis "Quasi-classical solutions of inverse problems of the calculus of variations in non-Eulerian classes of functionals and function spaces". In 1987, he was awarded the academic title of a professor.

V.M. Filippov is an academician of the Russian Academy of Education; a foreign member of the Ukrainian Academy of Pedagogical Sciences; President of the UNESCO International Organizing Committee for the World Conference on Higher Education (2007-2009); Vice-President of the Eurasian Association of Universities; a member of the Presidium of the Rectors' Council of Moscow and Moscow Region Universities, of the Governing Board of the Institute of Information Technologies in Education (UNESCO), of the Supervisory Board of the European Higher Education Center of UNESCO (Bucharest, Romania),

Research interests: variational methods; non-potential operators; inverse problems of the calculus of variations; function spaces.

In his Ph.D thesis, V.M. Filippov solved a long standing problem of constructing an integral extremal variational principle for the heat equation. In his further research he developed a general theory of constructing extremal variational principles for broad classes of differential equations with non-potential (in classical understanding) operators. He showed that all previous attempts to construct variational principles for non-potential operators "failed" because mathematicians and mechanics from the time of L. Euler and J. Lagrange were limited in their research by functionals of the type Euler - Lagrange. Extending the classes of functionals, V.M. Filippov introduced a new scale of function spaces that generalize the Sobolev spaces, and thus significantly expanded the scope of the variational methods. In 1984, famous physicist, a Nobel Prize winner I.R. Prigogine presented the report of V.M. Filippov to the Royal Academy of Sciences of Belgium. Results of V.M. Filippov's variational principles for non-potential operators are quite fully represented in some of his and his colleagues' monographs.

Honors: Honorary Legion (France), "Commander" (Belgium), Crown of the King (Belgium); in Russia - orders "Friendship", "Honor", "For Service to the Fatherland" III and IV degrees; Prize of the President of the Russian Federation in the field of education; Prize of the Governement of the Russian Federation in the field of education; Gratitude of the President of the Russian Federation; "For Merits in the Social and Labor Sphere of the Russian Federation", "For Merits in the Development of the Olympic Movement in Russia", "For Strengthening the Combat Commonwealth; and a number of other medals, prizes and awards.

He is an author of more than 270 scientific and scientific-methodical works, including 32 monographs, 2 of which were translated and published in the United States by the American Mathematical Society.

V.M. Filippov meets his 70th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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# OPTIMAL REARRANGEMENT-INVARIANT BANACH FUNCTION RANGE FOR THE HILBERT TRANSFORM

**K.S. Tulenov**

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**Key words:** rearrangement-invariant Banach function space, Hilbert transform, optimal range.

**AMS Mathematics Subject Classification:** 46E30, 44A15, 47L20, 47C15.

**Abstract.** We describe the optimal rearrangement-invariant Banach function range of the classical Hilbert transform acting on a rearrangement-invariant Banach function space. We also show the existence of such optimal range for the Lorentz and Marcinkiewicz spaces.

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## 1 Introduction

The classical Hilbert transform $\mathcal{H}$ (for measurable functions on $\mathbb{R}$) is defined, in the sense of principal value, by the following formula

$$(\mathcal{H}x)(t) = p.v.\frac{1}{\pi}\int_{\mathbb{R}}\frac{x(s)}{t-s}ds.$$

However, it may be undefined for some measurable functions. Later we will show that the maximal domain for $\mathcal{H}$ is the Lorentz space $\Lambda_{\varphi_0}(\mathbb{R})$ (see Remark 2.1), where $\varphi_0$ is defined in (2.6) below. Over the last few years, the characterization of optimal domain and range spaces, has been considered for many different kinds of operators and function spaces as applications to Sobolev embeddings [7, 9]; classical results in Fourier analysis such as the Hausdorff-Young inequality and Fourier multipliers [2, 16, 17]; vector measures [19]. For instance, in [18] the optimal domain and range spaces for $L_p$ (among solid Banach spaces), and in [8] the optimal domains in the class of rearrangement-invariant Banach function spaces were described. Operators with a similar bihaviour to the Hilbert transform were studied in the papers [4], [10]. The symmetric quasi-Banach optimal range of the classical Hilbert transform acting on a symmetric quasi-Banach function spaces were studied in [22] (see also [12] [23], [24]), where authors described the optimal range for its noncommutative counterparts including the triangular truncation operator with applications to the theory of operator Lipschitz functions and commutator estimates in ideals of compact operators.

Let $E$ and $F$ be rearrangement-invariant Banach function spaces on $\mathbb{R}$. In this paper, we are considering the problem of what is the least rearrangement-invariant Banach function space $F(\mathbb{R})$ such that $\mathcal{H} : E(\mathbb{R}) \to F(\mathbb{R})$ is bounded for a fixed rearrangement-invariant Banach function space $E(\mathbb{R})$. We shall be referring to the space $F(\mathbb{R})$ as the optimal range space for the operator $\mathcal{H}$ restricted to the domain $E(\mathbb{R}) \subseteq \Lambda_{\varphi_0}(\mathbb{R})$. Similar constructions have been considered in [20] and [6] (see also [21]) for the optimal range and domain spaces for the Hardy and Hardy type operators. We use their methods to obtain similar results for the Hilbert transform. This problem reduces to a familiar problem settled by D. Boyd [5] in 1967. Indeed, in this special case [5, Theorem 2.1] (see also [1] for the discrete case) asserts that $\mathcal{H} : E(\mathbb{R}) \to F(\mathbb{R})$ if and only if $S : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$, where the operator $S$, known as the Calderòn operator, is defined by the formula

$$(Sx)(t) = \frac{1}{t}\int_0^t x(s)ds + \int_t^\infty \frac{x(s)}{s}ds, \quad x \in \Lambda_{\varphi_0}(\mathbb{R}_+).$$

Effectively, the problem reduces to describing the optimal range of the operator $S$. Addressing precisely this framework, one of our main results, Theorem 3.1, provides a description of the optimal range $F(\mathbb{R})$ among the rearrangement-invariant Banach function spaces for a given rearrangement-invariant Banach function space $E(\mathbb{R})$, thereby complementing [5, Theorem 2.1]. The main results will be proved in Section 3. We also obtain some results on the existence of optimal rearrangement-invariant Banach function range of the Hilbert transform on the Lorentz and Marcinkiewicz spaces in Section 4.

## 2 Preliminaries

### 2.1 Rearrangement-invariant Banach function spaces

Let $(I, m)$ denote the measure space $I := \mathbb{R}_+ = (0, +\infty)$ (resp. $I := \mathbb{R} = (-\infty, +\infty)$) equipped with the Lebesgue measure $m$. Let $L(I)$ be the space of all measurable real-valued functions on $I$ equipped with the Lebesgue measure $m$, i.e. functions which coincide almost everywhere are considered identical. Let $L(I)^+$ be the cone of $m$-measurable functions on $\mathbb{R}$ whose values lie in $[0, \infty]$. The characteristic function or indicator of a $m$-measurable subset $\Delta$ of $\mathbb{R}$ will be denoted by $\chi_\Delta$.

**Definition 1.** [3, Definition I. 1.1, p. 2] A mapping $\rho : L(I)^+ \to [0, \infty]$ is called a Banach function norm if, for all $x, y, x_n$, $(n = 1, 2, 3, ...)$, in $L(I)^+$, for all $m$-measurable subsets $\Delta$ of $\mathbb{R}$, the following properties hold:

(i) $\rho$ is a norm

(ii) $0 \le y \le x$ a.e. $\Rightarrow \rho(y) \le \rho(x)$

(iii) $0 \le x_n \uparrow x$ a.e. $\Rightarrow \rho(x_n) \uparrow \rho(x)$

(iv) $\rho(\Delta) < \infty \Rightarrow \rho(\chi_\Delta) < \infty$

(v) $\rho(\Delta) < \infty \Rightarrow \int_\Delta x dm \le c_\Delta \rho(x)$

for some constant $c_\Delta$, $0 < c_\Delta < \infty$, depending on $\Delta$ and $\rho$ but independent of $x$.

Let $\rho$ be a function norm. The set $E = E(\rho)$ of all functions $x$ in $L(I)$ for which $\rho(|x|) < \infty$ is called a Banach function space. For any $x \in E$, define

$$\|x\|_E = \rho(|x|).$$

Define $L_0(I)$ to be the subset of $L(I)$ which consists of all functions $x$ such that

$$m(\{t : |x(t)| > s\})$$

is finite for some $s > 0$. Two functions $x$ and $y$ are called equimeasurable, if

$$m(\{t : |x(t)| > s\}) = m(\{t : |y(t)| > s\}).$$

For $x \in L_0(I)$, we denote by $\mu(x)$ the decreasing rearrangement of the function $|x|$. That is,

$$\mu(t,x) = \inf\{s \geq 0 : \ m(\{|x| > s\}) \leq t\}, \quad t > 0.$$

We say that $y$ is submajorized by $x$ in the sense of Hardy–Littlewood–Pólya (written $y \prec\prec x$) if

$$\int_0^t \mu(s,y)ds \leq \int_0^t \mu(s,x)ds, \quad t \geq 0.$$

**Definition 2.** [3, Definition 4.1, p. 59] A Banach function space $E$ is called rearrangement-invariant if, whenever $x$ belongs to $E$ and $y$ is equimeasurable with $x$, then $y$ also belongs to $E$ and $\|y\|_E = \|x\|_E$.

Throughout this paper we use RIBF space instead of rearrangement-invariant Banach function space. It is well known that $L_p(I)$, $(1 \leq p \leq \infty)$ and the Lorentz spaces $L_{p,q}(I)$, $(1 < p, q < \infty)$ are basic examples of RIBF spaces. For the general theory of RIBF spaces, we refer the reader to [3, 11, 12].

The dilation operator on $L_0(I)$ is defined by

$$\sigma_s x(t) := x\left(\frac{t}{s}\right), \quad s > 0.$$

It is obvious that the dilation operator $\sigma_s$ is continuous on $L_0(I)$ (see [11, Chapter II.3, p. 96]).

Let $E$ be a RIBF space on $I$. The upper and lower Boyd indices of $E(I)$ are numbers $\overline{\beta}_E$ and $\underline{\beta}_E$ defined by

$$\overline{\beta}_E := \lim_{t\to 0+} \frac{\log \|\sigma_s\|_{E\to E}}{\log t}, \quad \underline{\beta}_E := \lim_{t\to\infty} \frac{\log \|\sigma_s\|_{E\to E}}{\log t}.$$

Moreover, they satisfy $0 \leq \underline{\beta}_E \leq \overline{\beta}_E \leq 1$ (see [3, Definition III.5.12 and Proposition III.5.13, p. 149]).

## 2.2 Köthe dual of RIBF spaces

Next we define the Köthe dual (or associate) space of RIBF spaces. Given a RIBF space $E$ on $I$, equipped with the Lebesgue measure $m$ the Köthe dual space $E^\times$ on $I$ is defined by

$$E(I)^\times = \left\{y \in L_0(I) : \int_I |x(t)y(t)|dt < \infty, \ \forall x \in E(I)\right\}.$$

$E^\times$ is a Banach space with the norm

$$\|y\|_{E(I)^\times} := \sup\left\{\int_I |x(t)y(t)|dt : x \in E(I), \ \|x\|_{E(I)} \leq 1\right\}. \tag{2.1}$$

If $E(I)$ is a RIBF space, then $(E^\times(I), \|\cdot\|_{E^\times(I)})$ is also a RIBF space (cf. [3, Section 2.4]). For more details we refer to [3, 12]).

## 2.3 $L_1 \cap L_\infty$ and $L_1 + L_\infty$ spaces

Two examples below are of particular interest. Consider the separated topological vector space $L_0(I)$ consisting of all measurable functions $x$ such that $m(\{t : |x(t)| > s\})$ is finite for some $s > 0$ with the topology of convergence in measure. Then the spaces $L_1(I)$ and $L_\infty(I)$ are algebraically and topologically imbedded in the topological vector space $L_0(I)$, and so these spaces form a Banach couple (see [11, Chapter I] for more details). The space $(L_1 \cap L_\infty)(I) = L_1(I) \cap L_\infty(I)$ consists of all bounded summable functions $x$ on $I$ with norm

$$\|x\|_{(L_1 \cap L_\infty)(I)} = \max\{\|x\|_{L_1(I)}, \|x\|_{L_\infty(I)}\}, x \in (L_1 \cap L_\infty)(I).$$

The space $(L_1 + L_\infty)(I) = L_1(I) + L_\infty(I)$ consists of functions which are sums of bounded measurable and summable functions $x \in L_0(I)$ equipped with the norm given by

$$\|x\|_{(L_1 + L_\infty)(I)} = \inf\{\|x_1\|_{L_1(I)} + \|x_2\|_{L_\infty(I)} : x = x_1 + x_2,$$
$$x_1 \in L_1(I),\ x_2 \in L_\infty(I)\}.$$

For more details we refer the reader to [3], [11]. We recall that that every RIBF space on $I$ (with respect to Lebesgue measure) satisfies

$$(L_1 \cap L_\infty)(I) \subset E(I) \subset (L_1 + L_\infty)(I)$$

equipped with the norm given by with continuous embeddings (see for instance [11, Theorem II. 4.1. p. 91]).

## 2.4 Lorentz and Marcinkiewicz spaces

**Definition 3.** [11, Definition II. 1.1, p. 49] A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be quasiconcave if

(i) $\varphi(t) = 0 \Leftrightarrow t = 0$;

(ii) $\varphi(t)$ is positive and increasing for $t > 0$;

(iii) $\frac{\varphi(t)}{t}$ is decreasing for $t > 0$.

Observe that every nonnegative concave function on $[0, \infty)$ that vanishes only at origin is quasiconcave. The reverse, however, is not always true. But, we may replace, if necessary, a quasiconcave function $\varphi$ by its least concave majorant $\widetilde{\varphi}$ such that

$$\frac{1}{2}\widetilde{\varphi} \le \varphi \le \widetilde{\varphi}$$

(see [3, Proposition 5.10, p. 71]).

Let $\Omega$ denote the set of all increasing concave functions $\varphi$ such that $\varphi(+0) = 0$. For a function $\varphi$ in $\Omega$, the Lorentz space $\Lambda_\varphi(I)$ is defined by setting

$$\Lambda_\varphi(I) := \left\{x \in L_0(I) : \int_{\mathbb{R}_+} \mu(s, x) d\varphi(s) < \infty\right\}$$

equipped with the norm

$$\|x\|_{\Lambda_\varphi(I)} := \int_{\mathbb{R}_+} \mu(s, x) d\varphi(s). \tag{2.2}$$

Let $\psi \in \Omega$. Define the Marcinkiewicz space $M_\psi(I)$ as follows:

$$M_\psi(I) := \left\{x \in L_0(I) : \sup_{t>0} \frac{1}{\psi(t)} \int_0^t \mu(s, x) ds < \infty\right\} \tag{2.3}$$

with the norm

$$\|x\|_{M_\psi(I)} := \sup_{t>0} \frac{1}{\psi(t)} \int_0^t \mu(s, x) ds < \infty. \tag{2.4}$$

These spaces are examples of RIBF spaces. For more details on the Lorentz and Marcinkiewicz spaces, we refer the reader to [3, Chapter II.5] and [11, Chapter II.5].

## 2.5 Calderón operator and Hilbert transform

Let $E(\mathbb{R}_+)$ be a RIBF space. For a function $x \in E(\mathbb{R}_+)$, the operator $S$ is defined as follows:

$$(Sx)(t) := \frac{1}{t}\int_0^t x(s)ds + \int_t^\infty x(s)\frac{ds}{s}. \tag{2.5}$$

It is obvious that $S$ is a linear operator. Next, it is easy to see that if $0 < t < t'$, then

$$\min\left(1, \frac{s}{t'}\right) \le \min\left(1, \frac{s}{t}\right) \le \frac{t'}{t}\cdot\min\left(1, \frac{s}{t'}\right), \ (s > 0).$$

So, if $x$ is nonnegative, it follows from the first of these inequalities that $(Sx)(t)$ is a decreasing function of $t$. The operator $S$ is often applied to the decreasing rearrangement $\mu(x)$ of a function $x$ defined on some other measure space. Since $S\mu(x)$ is itself decreasing, it is easy to see that $\mu(S\mu(x)) = S\mu(x)$. Throughout this paper, we shall use the symbol $\mathcal{A} \lesssim \mathcal{B}$ to indicate that there exists a universal positive constant $c_{abs}$, independent of all important parameters, such that $\mathcal{A} \le c_{abs}\mathcal{B}$. $\mathcal{A} \approx \mathcal{B}$ means that $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$.

The next proposition, which gives the exact domain of the operator $S$, was proved in [23]. For the convenience, we recall its proof here.

**Proposition 2.1.** *Let $S$ be the operator defined by the above formula* (2.5)*. If*

$$\varphi_0(t) := \begin{cases} t\log(\frac{e^2}{t}), & 0 < t < 1, \\ 2\log(et), & 1 \le t < \infty, \end{cases} \tag{2.6}$$

*then the Lorentz space $\Lambda_{\varphi_0}(\mathbb{R}_+)$ is the maximal among the RIBF spaces $E(\mathbb{R}_+)$ such that*

$$S : E(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+).$$

*Proof.* Let $E(\mathbb{R}_+)$ be a RIBF space such that $S : E(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$. If

$$\|S\mu(x)\|_{(L_1+L_\infty)(\mathbb{R}_+)} \approx \|x\|_{\Lambda_{\varphi_0}(\mathbb{R}_+)}, \tag{2.7}$$

then, for any $x \in E(\mathbb{R}_+)$, we have

$$\|x\|_{\Lambda_{\varphi_0}(\mathbb{R}_+)} \lesssim c_{abs}\|S\mu(x)\|_{(L_1+L_\infty)(\mathbb{R}_+)} \lesssim c_{abs}\|x\|_{E(\mathbb{R}_+)}.$$

This shows that $E(\mathbb{R}_+) \subset \Lambda_{\varphi_0}(\mathbb{R}_+)$. Therefore, it is sufficient to show (2.7). Indeed, if $x \in \Lambda_{\varphi_0}(\mathbb{R}_+)$, then by using Fubini's theorem and (2.6), we obtain

$$\begin{aligned}
\|S\mu(x)\|_{(L_1+L_\infty)(\mathbb{R}_+)} &= \int_0^1 S\mu(t,x)dt \overset{(2.5)}{=} \int_0^1 \frac{1}{t}\int_0^t \mu(s,x)dsdt + \int_0^1\int_t^\infty \frac{\mu(s,x)}{s}dsdt \\
&= \int_0^1 \mu(s,x)\big(1-\log(s)\big)ds + \int_1^\infty \mu(s,x)\frac{1}{s}ds \\
&\le \int_0^1 \mu(s,x)\big(1-\log(s)\big)ds + 2\int_1^\infty \mu(s,x)\frac{1}{s}ds \overset{(2.2)}{=} \|x\|_{\Lambda_{\varphi_0}(\mathbb{R}_+)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|x\|_{\Lambda_{\varphi_0}(\mathbb{R}_+)} &\overset{(2.2)}{=} \int_0^1 \mu(s,x)\big(1-\log(s)\big)ds + 2\int_1^\infty \mu(s,x)\frac{1}{s}ds \\
&\le 2\int_0^1 \mu(s,x)\big(1-\log(s)\big)ds + 2\int_1^\infty \mu(s,x)\frac{1}{s}ds \\
&\overset{(2.5)}{=} 2\|S\mu(x)\|_{(L_1+L_\infty)(\mathbb{R}_+)}
\end{aligned}$$

□

Let $x \in \Lambda_{\varphi_0}(I)$. Since for each $t > 0$, the kernel $k_t(s) = \frac{1}{s} \cdot \min\left\{1, \frac{s}{t}\right\}$ is a decreasing function of $s$, it follows from [3, Theorem II.2.2, p. 44] that

$$\begin{aligned}|(Sx)(t)| &\overset{(2.5)}{=} \left| \int_{\mathbb{R}_+} x(s) \min\left\{1, \frac{s}{t}\right\} \frac{ds}{s} \right| \\ &\leq \int_{\mathbb{R}_+} |x(s)| \min\left\{1, \frac{s}{t}\right\} \frac{ds}{s} \\ &\leq \int_{\mathbb{R}_+} \mu(s,x) \min\left\{1, \frac{s}{t}\right\} \frac{ds}{s} \overset{(2.5)}{=} (S\mu(x))(t).\end{aligned} \tag{2.8}$$

For more information about these operators, we refer to [3, Chapter III] and [11, Chapter II].

If $x \in \Lambda_{\varphi_0}(\mathbb{R})$, then the classical Hilbert transform $\mathcal{H}$ is defined by the principal-value integral

$$(\mathcal{H}x)(t) = p.v. \frac{1}{\pi} \int_{\mathbb{R}} \frac{x(s)}{t-s} ds, \quad x \in \Lambda_{\varphi_0}(\mathbb{R}). \tag{2.9}$$

**Remark 2.1.** Let $x = x\chi_{(0,\infty)}$ be such that $x$ is a non-negative decreasing function on $\mathbb{R}_+$. Then, it is easy to see that

$$\begin{aligned}|(\mathcal{H}x)(-t)| &\overset{(2.9)}{=} \frac{1}{\pi} \left| \int_{\mathbb{R}_+} \frac{x(s)}{-t-s} ds \right| \\ &= \frac{1}{\pi} \int_{\mathbb{R}_+} \frac{x(s)}{t+s} ds = \frac{1}{\pi} \left( \int_0^t \frac{x(s)}{t+s} ds + \int_t^\infty \frac{x(s)}{t+s} ds \right) \\ &\geq \frac{1}{\pi} \left( \int_0^t \frac{x(s)}{2t} ds + \int_t^\infty \frac{x(s)}{2s} ds \right) \\ &= \frac{1}{2\pi} \cdot \left( \frac{1}{t} \int_0^t x(s) ds + \int_t^\infty \frac{x(s)}{s} ds \right) \overset{(2.5)}{=} \frac{1}{2\pi} (Sx)(t), \ t > 0.\end{aligned}$$

If $(\mathcal{H}x)(-t)$ exists for any $t > 0$, then it follows that $S\mu(x)$ exists, and it means $x$ belongs to the domain of $S$, i.e. $x \in \Lambda_{\varphi_0}(\mathbb{R}_+)$ (see (2.5)). On the other hand, if $x \in \Lambda_{\varphi_0}(\mathbb{R}_+)$, then by [3, Theorem III.4.8, p. 138], we have

$$\mu(\mathcal{H}x) \lesssim S\mu(x),$$

which shows the existence of $\mathcal{H}x$.

## 3 Optimal range for the Hilbert transform

In this section, we describe the optimal RIBF range space for the classical Hilbert transform. So, we shall say optimal RIBF range instead of optimal rearrangement-invariant Banach function range space. Let $E$ and $F$ be RIBF spaces on $\mathbb{R}_+$ and let $E^\times$ and $F^\times$ be their Köthe duals on $\mathbb{R}_+$, respectively.

First, we need the following lemma.

**Lemma 3.1.** *Let $S$ be the operator defined in* (2.5)*. Then $S$ is a self-adjoint operator in the following sense:*

$$\int_{\mathbb{R}_+} (Sx)(s)y(s)ds = \int_{\mathbb{R}_+} x(s)(Sy)(s)ds, \tag{3.1}$$

*for all nonnegative functions $x, y \in \Lambda_{\varphi_0}(\mathbb{R}_+)$.*

*Furthermore, if $S : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$, then $S : F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$ and we have*

$$\|S\|_{F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)} \leq \|S\|_{E(\mathbb{R}_+) \to F(\mathbb{R}_+)}. \tag{3.2}$$

*Proof.* Equality (3.1) follows immediately from (6.31) in [11, Chapter II.7, p. 138]. Let us prove (3.2). Since $S: E(\mathbb{R}_+) \to F(\mathbb{R}_+)$ and $S$ is a positive operator, it follows from [15, Proposition 1.3.5, p. 27] that $S$ is bounded from $E(\mathbb{R}_+)$ to $F(\mathbb{R}_+)$. Then by the definition of the Köthe duality (see (2.1)) and (3.1), we have

$$\begin{aligned}
\|Sy\|_{E^\times(\mathbb{R}_+)} &\overset{(2.1)}{=} \sup\{\int_{\mathbb{R}_+} x(s)(Sy)(s)ds : \ x \in E(\mathbb{R}_+),\ \|x\|_{E(\mathbb{R}_+)} \le 1\} \\
&\overset{(3.1)}{=} \sup\{\int_{\mathbb{R}_+} (Sx)(s)y(s)ds : \ x \in E(\mathbb{R}_+),\ \|x\|_{E(\mathbb{R}_+)} \le 1\} \\
&\le \sup\{\|Sx\|_{F(\mathbb{R}_+)}\|y\|_{F^\times(\mathbb{R}_+)} : \ x \in E(\mathbb{R}_+),\ \|x\|_{E(\mathbb{R}_+)} \le 1\} \\
&\le \sup\{\|S\|_{E(\mathbb{R}_+)\to F(\mathbb{R}_+)}\|x\|_{E(\mathbb{R}_+)}\|y\|_{F^\times(\mathbb{R}_+)} : \ x \in E(\mathbb{R}_+),\ \|x\|_{E(\mathbb{R}_+)} \le 1\} \\
&\le \|S\|_{E(\mathbb{R}_+)\to F(\mathbb{R}_+)}\|y\|_{F^\times(\mathbb{R}_+)}.
\end{aligned}$$

Since $\|S\|_{E(\mathbb{R}_+)\to F(\mathbb{R}_+)}$ is finite, this concludes the proof. □

**Definition 4.** Let $E$ be a RIBF space on $\mathbb{R}$ such that $E(\mathbb{R}) \subset \Lambda_{\varphi_0}(\mathbb{R})$, where $\varphi_0$ defined in (2.6). Let us define the set

$$F(\mathbb{R}_+) := \{x \in (L_1 + L_\infty)(\mathbb{R}) : \|x\|_{F(\mathbb{R})} < \infty\},$$

where

$$\|x\|_{F(\mathbb{R})} := \sup_{\mu(y)\in(L_1\cap L_\infty)(\mathbb{R}_+)} \frac{1}{\|S\mu(y)\|_{E^\times(\mathbb{R}_+)}} \int_{\mathbb{R}_+} \mu(t,x)\mu(t,y)dt,$$

and the operator $S$ is defined as in Proposition 2.1.

It was proved in [20, Theorem 3.2] that $(F(\mathbb{R}_+), \|\cdot\|_{F(\mathbb{R}_+)})$ is a RIBF space. Moreover, it was shown that the space $(F(\mathbb{R}_+), \|\cdot\|_{F(\mathbb{R}_+)})$ is the optimal range for the operator $S$. The following theorem describes the optimal range for the Hilbert transform $\mathcal{H}$ defined on $\mathbb{R}$ among the RIBF spaces.

**Theorem 3.1.** *Let $E$ be a RIBF space on $\mathbb{R}$ such that $E(\mathbb{R}) \subset \Lambda_{\varphi_0}(\mathbb{R})$, where $\varphi_0$ defined in* (2.6)*. Then, the space $F(\mathbb{R})$ defined in Definition 4 is the optimal RIBF range among the RIBF spaces for the Hilbert transform $\mathcal{H}$ defined on $E(\mathbb{R})$.*

*Proof.* Let $E(\mathbb{R}) \subset \Lambda_{\varphi_0}(\mathbb{R})$. an argument similar to the one in [20, Theorem 3.2] shows that $(F(\mathbb{R}), \|\cdot\|_{F(\mathbb{R})})$ is a RIBF space. Let us show that the Hilbert transform is bounded from $E(\mathbb{R})$ into $F(\mathbb{R})$. Now, let $x \in E(\mathbb{R})$, then by [3, Theorem III.4.8, p. 138] and by (3.1) in Lemma 3.1, we have

$$\begin{aligned}
\|\mathcal{H}x\|_{F(\mathbb{R})} &= \sup_{\mu(y)\in(L_1\cap L_\infty)(\mathbb{R}_+)} \frac{1}{\|S\mu(y)\|_{E^\times(\mathbb{R}_+)}} \int_{\mathbb{R}_+} \mu(s,\mathcal{H}x)\mu(s,y)ds \\
&\le c_{abs} \sup_{\mu(y)\in(L_1\cap L_\infty)(\mathbb{R}_+)} \frac{1}{\|S\mu(y)\|_{E^\times(\mathbb{R}_+)}} \int_{\mathbb{R}_+} S\mu(x)(s)\mu(s,y)ds \\
&\overset{(3.1)}{=} c_{abs} \sup_{\mu(y)\in(L_1\cap L_\infty)(\mathbb{R}_+)} \frac{1}{\|S\mu(y)\|_{E^\times(\mathbb{R}_+)}} \int_{\mathbb{R}_+} \mu(s,x)(S\mu(y))(s)ds \\
&\le c_{abs}\|\mu(x)\|_{E(\mathbb{R}_+)} = c_{abs}\|x\|_{E(\mathbb{R})}.
\end{aligned}$$

Hence, $\mathcal{H}: E(\mathbb{R}) \to F(\mathbb{R})$ is bounded. Now, suppose that $G(\mathbb{R})$ is another RIBF space such that $\mathcal{H}: E(\mathbb{R}) \to G(\mathbb{R})$ is bounded. Let us show that $F(\mathbb{R}) \subset G(\mathbb{R})$. If $x \in E(\mathbb{R}_+)$, then by [3, Proposition

III. 4.10, p. 140] there is a function $y \in E(\mathbb{R})$ equimeasurable with $x$ such that $S\mu(x) \le 2\mu(\mathcal{H}y)$. Thus

$$\begin{aligned}\|S\mu(x)\|_{F(\mathbb{R}_+)} &\le 2\|\mu(\mathcal{H}y)\|_{F(\mathbb{R}_+)} = 2\|\mathcal{H}y\|_{F(\mathbb{R})}\\ &\le c_{abs}\|y\|_{E(\mathbb{R})} = c_{abs}\|x\|_{E(\mathbb{R}_+)},\end{aligned}$$

which shows that

$$S : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$$

is bounded. Therefore, Lemma 3.1 implies that $S$ is self-adjoint and $S : F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$ is bounded. So, for any $y \in L_0(\mathbb{R}_+)$ such that $\mu(y) \in F^\times(\mathbb{R}_+)$, we have

$$\|S\mu(y)\|_{E^\times(\mathbb{R}_+)} \le \|S\|_{G^\times(\mathbb{R}_+)\to E^\times(\mathbb{R}_+)}\|y\|_{G^\times(\mathbb{R}_+)} \le \|S\|_{E(\mathbb{R}_+)\to G(\mathbb{R}_+)}\|y\|_{G^\times(\mathbb{R}_+)}.$$

Hence, for any $x \in G(\mathbb{R}_+)$, we obtain

$$\begin{aligned}\|x\|_{F(\mathbb{R}_+)} &= \sup_{\mu(y)\in(L_1\cap L_\infty)(\mathbb{R}_+)} \frac{1}{\|y\|_{G^\times(\mathbb{R}_+)}}\int_{\mathbb{R}_+}\mu(s,x)\mu(s,y)ds\\ &\le \|S\|_{E(\mathbb{R}_+)\to G(\mathbb{R}_+)} \sup_{\mu(y)\in(L_1\cap L_\infty)(\mathbb{R}_+)} \frac{1}{\|S\mu(y)\|_{E^\times(\mathbb{R}_+)}}\int_{\mathbb{R}_+}\mu(s,x)\mu(s,y)ds\\ &= \|S\|_{E(\mathbb{R}_+)\to G(\mathbb{R}_+)}\|x\|_{F(\mathbb{R}_+)}.\end{aligned}$$

Therefore, we have that $F(\mathbb{R}) \subset G(\mathbb{R})$ as claimed. So, the space $F(\mathbb{R})$ is the optimal RIBF space for the Hilbert transform $\mathcal{H}$ on $E(\mathbb{R})$. This completes the proof. □

Similarly to [20, Proposition 3.9], we obtain the following result for the Hilbert transform.

**Proposition 3.1.** *Let $E$ be a RIBF space on $\mathbb{R}_+$ (respectively $\mathbb{R}$) such that $E(\mathbb{R}_+) \subset \Lambda_{\varphi_0}(\mathbb{R}_+)$ (respectively $E(\mathbb{R}) \subset \Lambda_{\varphi_0}(\mathbb{R})$), where $\varphi_0$ defined in* (2.6). *Let $S$ be the operator defined in* (2.5). *Then, the following are equivalent:*

(i) *there exists an optimal RIBF range $F(\mathbb{R})$ for the Hilbert transform $\mathcal{H}$ on $E(\mathbb{R})$;*

(ii) *$S : E(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$ is a bounded operator;*

(iii) *$S\chi_{(0,1)} \in E^\times(\mathbb{R}_+)$.*

*Moreover, if any of these conditions holds, then the optimal RIBF range for the Hilbert transform on $E(\mathbb{R})$ is given by*

$$F(\mathbb{R}) := \{x \in (L_1 + L_\infty)(\mathbb{R}) :$$
$$\|x\|_{F(\mathbb{R})} = \sup_{\mu(y)\in(L_1\cap L_\infty)(\mathbb{R}_+)} \frac{1}{\|S\mu(y)\|_{E^\times(\mathbb{R}_+)}}\int_{\mathbb{R}_+}\mu(s,x)\mu(s,y)ds < \infty\}.$$

*Proof.* *(i)* ⇒ *(ii)* Let $F(\mathbb{R})$ be the optimal RIBF range for the Hilbert transform. Then $\mathcal{H} : E(\mathbb{R}) \to F(\mathbb{R})$ is a bounded operator. Since the embedding $F(\mathbb{R}) \hookrightarrow (L_1 + L_\infty)(\mathbb{R})$ is continuous, it follows that $\mathcal{H} : E(\mathbb{R}) \to (L_1 + L_\infty)(\mathbb{R})$ is bounded. On the other hand, [3, Proposition III. 4.10, p. 140] shows that to each $x \in E(\mathbb{R}_+)$, there corresponds a function $y \in E(\mathbb{R})$ equimeasurable with $x$ such that $S\mu(x) \le 2\mu(\mathcal{H}y)$. Then

$$\begin{aligned}\|S\mu(x)\|_{(L_1+L_\infty)(\mathbb{R}_+)} &\le 2\|\mu(\mathcal{H}y)\|_{(L_1+L_\infty)(\mathbb{R}_+)} = 2\|\mathcal{H}y\|_{(L_1+L_\infty)(\mathbb{R})}\\ &\le c_{abs}\|y\|_{E(\mathbb{R})} = c_{abs}\|x\|_{E(\mathbb{R}_+)},\end{aligned}$$

which shows that

$$S : E(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$$

is bounded.

$(ii) \Rightarrow (iii)$ Since $S : E(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$ is bounded and self-adjoint, by Lemma 3.1 we also have that $S : (L_1 \cap L_\infty)(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$ is bounded. Now, since $\chi_{(0,t)} \in (L_1 \cap L_\infty)(\mathbb{R}_+)$ for every $t > 0$, it follows that $S\chi_{(0,t)} \in E^\times(\mathbb{R}_+)$.

$(iii) \Rightarrow (ii)$ Let $x \in E(\mathbb{R}_+)$. Since $\mu(Sx) \leq S\mu(x)$ by (2.8), it follows from the Hölder inequality [3, Theorem I.2.4, p.9] that

$$\begin{aligned}
\|Sx\|_{(L_1+L_\infty)(\mathbb{R}_+)} &= \int_0^1 \mu(s, Sx)ds \\
&\overset{(2.8)}{\leq} \int_0^1 (S\mu(x))(s)ds \\
&= \int_{\mathbb{R}_+} \mu(s,x)S\chi_{(0,1)}(s)ds \\
&\leq \|x\|_{E(\mathbb{R}_+)}\|S\chi_{(0,1)}\|_{E^\times(\mathbb{R}_+)}.
\end{aligned}$$

Hence, since $S\chi_{(0,1)} \in E^\times(\mathbb{R}_+)$, it follows that $S : E(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$ is bounded.

$(ii) \Rightarrow (i)$ As $\mathcal{H}$ satisfies the hypothesis of Theorem 3.1, we obtain that $F(\mathbb{R})$ is the optimal RIBF range for the Hilbert transform $\mathcal{H}$ on $E(\mathbb{R})$. □

**Remark 3.2.** Since $S\chi_{(0,1)}$ does not belong to $L_1(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$, one direct application of Proposition 3.1 shows that there are no optimal RIBF ranges $F(\mathbb{R})$ and $G(\mathbb{R})$ which are Banach such that $\mathcal{H} : L_1(\mathbb{R}) \to F(\mathbb{R})$ and $\mathcal{H} : L_\infty(\mathbb{R}) \to G(\mathbb{R})$, respectively.

## 4 Existence of optimal RIBF range for the Lorentz and Marcinkiewicz spaces

In this section, we will show the existence of the optimal range of the operator $S$ and Hilbert transform for the Lorentz and Marcinkiewicz spaces.

Similar result to the following was obtained in [21, Lemma 2.1] for the Hardy operator.

**Lemma 4.1.** *Let $\varphi$ be an increasing concave function such that $\varphi(0+) = 0$. Let $E$ be a symmetric space on $\mathbb{R}_+$ and let $S$ be the operator defined in (2.5). The following conditions are equivalent:*

(i) $\|S\chi_{(0,t)}\|_{E(\mathbb{R}_+)} \lesssim \varphi(t)$, $t > 0$;

(ii) $S : \Lambda_\varphi(\mathbb{R}_+) \to E(\mathbb{R}_+)$ *is bounded.*

*Proof.* Since the fundamental function of $\Lambda_\varphi(\mathbb{R}_+)$ is equal to $\varphi(t)$ (see [3, Chapter II.5, pp. 65-73]), $(ii) \Rightarrow (i)$ part is clear. Let us now prove that $(i)$ implies $(ii)$. If $x \in \Lambda_{\varphi_0}(\mathbb{R})$, where $\varphi_0$ defined in (2.6) is a positive function on $\mathbb{R}_+$, and $x$ equal to zero on the negative semiaxis, then by the definition of Hilbert transform (2.9), we have

$$\mathcal{H}(x)(-t) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{x(-s)}{t-s}ds = p.v.\frac{1}{\pi}\int_{\mathbb{R}_+} \frac{x(s)}{t+s}ds.$$

Hence, if $t > 0$, then

$$\mathcal{H}(x)(-t) = \int_{\mathbb{R}_+} \frac{x(s)}{t+s}ds \geq \frac{\pi}{2}(Sx)(t).$$

Taking decreasing rearrangement $\mu$, we obtain

$$\mu(t, Sx) \lesssim \mu(t, \mathcal{H}x), \quad t > 0. \tag{4.1}$$

Let $\Delta$ be a Lebesgue measurable subset of $\mathbb{R}_+$ and let $x = \chi_\Delta$. By [3, Theorem III. 4.8, p. 138], we have

$$\mu(t, \mathcal{H}\chi_\Delta) \lesssim S\chi_{(0,m(\Delta))}(t),\ t > 0.$$

Combining (4.1) and preceding inequality, we obtain

$$\mu(t, S\chi_\Delta) \lesssim S\chi_{(0,m(\Delta))}(t),\ t > 0. \tag{4.2}$$

Note that for a measurable set $\Delta$ with measure $m(\Delta)$, by (4.2), we have

$$\|S\chi_\Delta\|_{E(\mathbb{R}_+)} \lesssim \|S\chi_{(0,m(\Delta))}\|_{E(\mathbb{R}_+)} \lesssim \|\chi_\Delta\|_{\Lambda_\varphi(\mathbb{R}_+)}.$$

For a given $x \in \Lambda_\varphi(\mathbb{R}_+)$, denote $\Delta_n = \{s : 2^n < |x(s)| \le 2^{n+1}\}$, for $n \in \mathbb{Z}$. Using (5.4) in [11, Chapter II.5, p. 111]

$$\|x\|_{\Lambda_\varphi(\mathbb{R}_+)} = \int_{\mathbb{R}_+} \varphi(d_x(t))dt,$$

where $d_x(t) = m(\{s : |x(t)| > s\})$ is the distribution function of $x$, we have

$$\begin{aligned}\|Sx\|_{E(\mathbb{R}_+)} = \|\sum_{n\in\mathbb{Z}} S(x\chi_{\Delta_n})\|_{E(\mathbb{R}_+)} &\lesssim \sum_{n\in\mathbb{Z}} 2^{n+1}\|\chi_{\Delta_n}\|_{\Lambda_\varphi(\mathbb{R}_+)} \\ &\lesssim \sum_{n\in\mathbb{Z}} 2^{n+1}\varphi(d_x(2^n)) \lesssim \|x\|_{\Lambda_\varphi(\mathbb{R}_+)}.\end{aligned}$$

This concludes the proof. □

The following theorem yields a necessary and sufficient condition for the existence of optimal RIBF range of the Hilbert transform $\mathcal{H}$ for a given Lorentz space $\Lambda_\varphi(\mathbb{R})$.

**Theorem 4.1.** *Let $\varphi$ be an increasing concave function on $[0, \infty)$ such that $\varphi(0+) = 0$. Then*

(i) $S : \Lambda_\varphi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$ *if and only if $\varphi$ satisfies*

$$\varphi(t) \gtrsim \varphi_0(t),\ t > 0, \tag{4.3}$$

*where $\varphi_0$ defined in* (2.6).

(ii) *If $\varphi(t) \gtrsim \varphi_0(t),\ t > 0$, then the optimal range of the Hilbert transform on $\Lambda_\varphi(\mathbb{R})$ coincides with*

$$G(\mathbb{R}) := \{x \in (L_1 + L_\infty)(\mathbb{R}) : \mu(x) \prec\prec S\mu(y),\ \exists y \in \Lambda_\varphi(\mathbb{R})\}, \tag{4.4}$$

*endowed with the norm*

$$\|x\|_{G(\mathbb{R})} := \inf\{\|y\|_{\Lambda_\varphi(\mathbb{R})} : \mu(x) \prec\prec S\mu(y))\}.$$

*Proof.* First, let us prove (i). Let $S : \Lambda_\varphi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$. Since $S$ maps positive function to a positive function, i.e. $S$ is a positive operator (see [3, Chapter III, p. 134]), it follows from [15, Proposition 1.3.5, p. 27] that $S$ is bounded from $\Lambda_\varphi(\mathbb{R}_+)$ into $(L_1 + L_\infty)(\mathbb{R}_+)$. We know that

$\|\chi_{(0,t)}\|_{\Lambda_\varphi(\mathbb{R}_+)} = \varphi(t),\ t > 0$ (see (5.21) in [3, Chapter II.6, p. 73]). Therefore, by using (2.7) in the proof of Proposition 2.1, we obtain

$$\varphi_0(t) = \|\chi_{(0,t)}\|_{\Lambda_{\varphi_0}(\mathbb{R}_+)} \approx \|S\chi_{(0,t)}\|_{(L_1+L_\infty)(\mathbb{R}_+)} \lesssim \|\chi_{(0,t)}\|_{\Lambda_\varphi(\mathbb{R}_+)} = \varphi(t),\ \ t > 0.$$

On the other hand, let $\varphi$ satisfies

$$\varphi(t) \gtrsim \varphi_0(t),\ \ t > 0.$$

We will show $S : \Lambda_\varphi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$. Indeed, again using (2.7) in the proof of Proposition 2.1, we have

$$\varphi(t) \gtrsim \varphi_0(t) = \|\chi_{(0,t)}\|_{\Lambda_{\varphi_0}(\mathbb{R}_+)} \approx \|S\chi_{(0,t)}\|_{(L_1+L_\infty)(\mathbb{R}_+)},\ \ t > 0.$$

Then applying Lemma 4.1 [$(i) \Rightarrow (ii)$] when $E(\mathbb{R}_+) = (L_1 + L_\infty)(\mathbb{R}_+)$, we obtain the desired result.

Next, we prove the second part of the theorem. If

$$\varphi(t) \gtrsim \varphi_0(t),\ \ t > 0,$$

then by the first part of the theorem, we have $S : \Lambda_\varphi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$. Hence, it follows from Proposition 3.1 [$(ii) \Rightarrow (i)$] that the optimal range for the operator $S$ on $\Lambda_\varphi(\mathbb{R}_+)$ exists. Let us show that the optimal range coincides with $G(\mathbb{R}_+)$ defined in (4.4). For this, first we have to show that $G(\mathbb{R}_+)$ is a RIBF space. However, it was proved in [20, Theorem 3.5] that the space $G(\mathbb{R}_+)$ defined as above is a minimal RIBF space such that $S : \Lambda_\varphi(\mathbb{R}_+) \to G(\mathbb{R}_+)$ is bounded. Therefore, by [3, Theorem III. 4.8, p. 138], $\mathcal{H} : \Lambda_\varphi(\mathbb{R}) \to G(\mathbb{R})$ is bounded. It follows from the definition of optimal range $F(\mathbb{R})$ (see Definition 4) that $F(\mathbb{R}) \subset G(\mathbb{R})$. For the converse inclusion, pick any RIBF space $Y(\mathbb{R})$ such that $\mathcal{H} : \Lambda_\varphi(\mathbb{R}) \to Y(\mathbb{R})$ is bounded. Then by [3, Proposition III. 4.10, p. 140], we obtain that $S : \Lambda_\varphi(\mathbb{R}_+) \to Y(\mathbb{R}_+)$ is bounded. If $x \in G(\mathbb{R}_+)$, then for every decreasing $y \in \Lambda_\varphi(\mathbb{R}_+)$ with $\mu(x) \prec\prec Sy$, by [3, Theorem II.4.6] we have

$$\|x\|_{Y(\mathbb{R}_+)} \leq \|Sy\|_{Y(\mathbb{R}_+)} \leq \|S\|_{\Lambda_\varphi(\mathbb{R}_+)\to Y(\mathbb{R}_+)}\|y\|_{\Lambda_\varphi(\mathbb{R}_+)},$$

which implies via taking infimum over all such $y$'s that

$$\|x\|_{Y(\mathbb{R}_+)} \leq \|S\|_{\Lambda_\varphi(\mathbb{R}_+)\to Y(\mathbb{R}_+)}\|x\|_{G(\mathbb{R}_+)},$$

i.e. $G(\mathbb{R}) \subset Y(\mathbb{R})$. This proves the minimality condition, and hence $F(\mathbb{R})$ coincides with $G(\mathbb{R})$. □

**Corollary 4.1.** *Given an increasing concave function $\varphi$ such that $\varphi(0+) = 0$ and satisfying condition* (4.3)*, we have*

$$\Lambda_\varphi(\mathbb{R}_+) = G(\mathbb{R}_+),$$

*where the space $G(\mathbb{R}_+)$ is defined in* (4.4)*, if and only if the upper and lower Boyd indices satisfy* $0 < \underline{\beta}_{\Lambda_\varphi} \leq \overline{\beta}_{\Lambda_\varphi} < 1$.

*Proof.* It is easy to see that, under condition (4.3), we always have $\Lambda_\varphi(\mathbb{R}_+) \subset G(\mathbb{R}_+)$. Indeed, if $x \in \Lambda_\varphi(\mathbb{R}_+)$, then taking $y = \mu(x) \in \Lambda_\varphi(\mathbb{R}_+)$ and by [20, Lemma 2.4 (ii)] we get that $\mu(x) = y \prec\prec Sy$. So, $\Lambda_\varphi(\mathbb{R}_+) = G(\mathbb{R}_+)$ holds if and only if $G(\mathbb{R}_+) \subset \Lambda_\varphi(\mathbb{R}_+)$, which is by Theorem 4.1 equivalent to the boundedness $S : \Lambda_\varphi(\mathbb{R}_+) \to \Lambda_\varphi(\mathbb{R}_+)$. However, the latter condition is known to be equivalent to the condition $0 < \underline{\beta}_{\Lambda_\varphi} \leq \overline{\beta}_{\Lambda_\varphi} < 1$ (see, e.g. [3, Theorem III.5.18, p. 154]). This completes the proof. □

The next theorem gives a necessary and sufficient condition for the existence of optimal RIBF range of the operator $S$ for a given Marcinkiewicz space $M_\varphi(\mathbb{R}_+)$.

**Theorem 4.2.** *Let $\varphi$ be an increasing concave function on $[0,\infty)$ such that $\varphi(0+) = 0$. Then*

(i) $S : M_\varphi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$ *if and only if $\varphi$ satisfies*

$$\int_0^1 \frac{\varphi(t)}{t} dt + \int_0^\infty \frac{\varphi'(t)}{1+t} dt < \infty; \tag{4.5}$$

(ii) *If $\varphi$ satisfies* (4.5), *then the optimal range of the Hilbert transform on $M_\varphi(\mathbb{R})$ coincides with*

$$G(\mathbb{R}) := \{x \in (L_1 + L_\infty)(\mathbb{R}) : \mu(x) \prec\prec S\mu(y), \ \ \exists y \in M_\varphi(\mathbb{R})\}, \tag{4.6}$$

*endowed with the norm*

$$\|x\|_{G(\mathbb{R})} := \inf\{\|y\|_{M_\varphi(\mathbb{R})} : \mu(x) \prec\prec S\mu(y))\}.$$

*Proof.* First, let us prove (i). It is easy to see by a calculation that

$$(S\chi_{(0,1)})(s) \approx \frac{1}{1+s} + \chi_{(0,1)}(s) \log\left(\frac{1}{s}\right), \ \ s > 0. \tag{4.7}$$

By formula (4.7) condition (4.5) is equivalent to $S\chi_{(0,1)} \in \Lambda_\varphi(\mathbb{R}_+)$. Since $M_\varphi^\times(\mathbb{R}_+) = \Lambda_\varphi(\mathbb{R}_+)$ (see [11, Chapter II.5]), it follows from Proposition 3.1 [$(iii) \Rightarrow (ii)$] that $S\chi_{(0,1)} \in \Lambda_\varphi(\mathbb{R}_+)$ is equivalent to the boundedness of the operator $S$ from $M_\varphi(\mathbb{R}_+)$ to $(L_1 + L_\infty)(\mathbb{R}_+)$. Next, let us prove (ii). If $\varphi$ satisfies (4.5), then by the first part of this theorem $S : M_\varphi(\mathbb{R}_+) \to (L_1 + L_\infty)(\mathbb{R}_+)$ is bounded. But, by Proposition 3.1 [$(ii) \Rightarrow (i)$] this is equivalent to the existence of the optimal range of the Hilbert transform on $M_\varphi(\mathbb{R}_+)$. The other part is proved similarly to the proof of Theorem 4.1 (ii). □

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