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VLADIMIR MIKHAILOVICH FILIPPOV

(to the 70th birthday)

Vladimir Mikhailovich Filippov was born on 15 April 1951 in the city of Uryupinsk, Stalingrad Region of the USSR. In 1973 he graduated with honors from the Faculty of Physics and Mathematics and Natural Sciences of the Patrice Lumumba University of Peoples' Friendship in the specialty "Mathematics". In 1973-1975 he is a postgraduate student of the University; in 1976-1979 - Chairman of the Young Scientists' Council; in 1980-1987 - Head of the Research Department and the Scientific Department; in 1983-1984 - scientific work at the Free University of Brussels (Belgium); in 1985-2000 - Head of the Mathematical Analysis Department; from 2000 to the present - Head of the Comparative Educational Policy Department; in 1989–1993 - Dean of the Faculty of Physics, Mathematics and Natural Sciences; in 1993–1998 - Rector of the Peoples' Friendship University of Russia; in 1998-2004 - Minister of General and Professional Education, Minister of Education of the Russian Federation; in 2004-2005 - Assistant to the Chairman of the Government of the Russian Federation (in the field of education and culture); from 2005 to May 2020- Rector of the Peoples' Friendship University of Russia, since May 2020 - President of the Peoples' Friendship University of Russia, since 2013 - Chairman of the Higher Attestation Commission of the Ministry of Science and Higher Education of the Russian Federation.

In 1980, he defended his PhD thesis in the V.A. Steklov Mathematical Institute of Academy of Sciences of the USSR on specialty 01.01.01 - mathematical analysis (supervisor - a corresponding member of the Academy of Sciences of the USSR, Professor L.D. Kudryavtsev), and in 1986 in the same Institute he defended his doctoral thesis "Quasi-classical solutions of inverse problems of the calculus of variations in non-Eulerian classes of functionals and function spaces". In 1987, he was awarded the academic title of a professor.

V.M. Filippov is an academician of the Russian Academy of Education; a foreign member of the Ukrainian Academy of Pedagogical Sciences; President of the UNESCO International Organizing Committee for the World Conference on Higher Education (2007-2009); Vice-President of the Eurasian Association of Universities; a member of the Presidium of the Rectors' Council of Moscow and Moscow Region Universities, of the Governing Board of the Institute of Information Technologies in Education (UNESCO), of the Supervisory Board of the European Higher Education Center of UNESCO (Bucharest, Romania),

Research interests: variational methods; non-potential operators; inverse problems of the calculus of variations; function spaces.

In his Ph.D thesis, V.M. Filippov solved a long standing problem of constructing an integral extremal variational principle for the heat equation. In his further research he developed a general theory of constructing extremal variational principles for broad classes of differential equations with non-potential (in classical understanding) operators. He showed that all previous attempts to construct variational principles for non-potential operators "failed" because mathematicians and mechanics from the time of L. Euler and J. Lagrange were limited in their research by functionals of the type Euler - Lagrange. Extending the classes of functionals, V.M. Filippov introduced a new scale of function spaces that generalize the Sobolev spaces, and thus significantly expanded the scope of the variational methods. In 1984, famous physicist, a Nobel Prize winner I.R. Prigogine presented the report of V.M. Filippov to the Royal Academy of Sciences of Belgium. Results of V.M. Filippov's variational principles for non-potential operators are quite fully represented in some of his and his colleagues' monographs.

Honors: Honorary Legion (France), "Commander" (Belgium), Crown of the King (Belgium); in Russia - orders "Friendship", "Honor", "For Service to the Fatherland" III and IV degrees; Prize of the President of the Russian Federation in the field of education; Prize of the Governement of the Russian Federation in the field of education; Gratitude of the President of the Russian Federation; "For Merits in the Social and Labor Sphere of the Russian Federation", "For Merits in the Development of the Olympic Movement in Russia", "For Strengthening the Combat Commonwealth; and a number of other medals, prizes and awards.

He is an author of more than 270 scientific and scientific-methodical works, including 32 monographs, 2 of which were translated and published in the United States by the American Mathematical Society.

V.M. Filippov meets his 70th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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ON THE COMPLETENESS AND MINIMALITY OF DOUBLE AND UNITARY SYSTEMS IN MORREY-TYPE SPACES

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Key words: completeness, minimality, system of functions, Morrey spaces.

AMS Mathematics Subject Classification: 33B10, 46E30, 54D70.

Abstract. In this work, double and unitary systems of functions in Morrey type spaces $M^{p,\alpha}(-a,a)$ and $M^{p,\alpha}(0,a)$ are considered. A relationship between the completeness and minimality properties of these systems in these spaces is established.

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1 Introduction

Consider the following unitary system of functions of the form

$$v_n^{\pm}(t) \equiv a(t)\,\omega_n^{+}(t) \pm b(t)\,\omega_n^{-}(t), \ t \in [0,a], n \in N, \tag{1.1}$$

where $a; b; \omega_n^{\pm} : [0,a] \to \mathbb{C}$ are some Lebesgue measurable, generally speaking, complex valued functions on a finite segment $[0,a]$. Let us associate this system with the following double system

$$\{A(t)\ W_n(t);\ A(-t)\ W_n(-t)\}_{n\in N}, \tag{1.2}$$

where

$$A(t) = \begin{cases} a(t), & t \in [0,a], \\ b(-t), & t \in [-a,0), \end{cases}$$

$$W_n(t) = \begin{cases} \omega_n^{+}(t), & t \in [0,a], \\ \omega_n^{-}(-t), & t \in [-a,0). \end{cases}$$

These systems are modifications of the perturbed systems of sines

$$\{\sin(n+\beta)t\}_{n\in N}, \tag{1.3}$$

cosines

$$\{\cos(n+\beta)t\}_{n\in N}, \tag{1.4}$$

and exponents

$$\left\{e^{i(n+\beta\, sign\, n)t}\right\}_{n\in Z}, \tag{1.5}$$

$$\left\{e^{i(n+\beta\, sign\, n)t}\right\}_{n\neq 0}, \tag{1.6}$$

where $\beta \in \mathbb{C}$ is a complex parameter. Systems of forms (1.3) and (1.4) arise when solving partial differential equations of mixed (or elliptic) type by Fourier method. Regarding the related works, you can see, e.g. [19, 20, 11, 12, 15]. The study of basis properties of systems (1.3)-(1.6) has a long history. It dates back to the works by Paley-Wiener [18] and B.Y. Levin [10]. In L_2, the final result regarding the Riesz basicity of system (1.5) follows from the results obtained by B.Y. Levin [10] and M.I. Kadets [9]. For the first time, basicity criterion for the system (1.5) in L_p, has been obtained by A.M. Sedletski [21]. For other methods, the same and other results concerning systems (1.3) - (1.6) were obtained by E.I. Moiseev [13, 14]. Wherein E.I. Moiseev first studies the basis properties of (1.3) and (1.4) in $L_p(0,\pi)$ and then using these results he obtains similar results with respect to systems (1.5) and (1.6) in $L_p(-\pi,\pi)$. Further development of this approach, used in establishing basis properties of systems of form (1.1) is due to B.T. Bilalov [3]-[6]. He deduced the basis properties of unitary systems (1.1) in $L_p(0,a)$ from similar properties of double systems (1.2) in $L_p(-a,a)$. This method was applied by T.R. Muradov in the Lebesgue space with a variable summability exponent in [16, 17]. We will also follow the scheme proposed by B.T. Bilalov. The basis properties (completeness, minimality and basicity) of the system of exponents in weighted Morrey spaces, where the weight function is defined as a product of power functions are investigated in [7]. Some approximation problems have been investigated in the Morrey-Smirnov classes in [8].

In this paper, the Morrey space $L^{p,\alpha}(I)$ is considered on some segment $I \subset R$. Using the shift operator, its subspace $M^{p,\alpha}(I)$ in which continuous functions are dense is defined. A relationship between the completeness properties of systems (1.1) and (1.2) in the spaces $M^{p,\alpha}(0,a)$ and $M^{p,\alpha}(-a,a)$, respectively, is established.

2 Preliminaries

Let us first define the Morrey space $L^{p,\alpha}(a,b)$, $-\infty < a < b < \infty, 1 \le p \le \infty, 0 \le \alpha \le 1$. It is a Banach space of all measurable functions on (a,b) with the finite norm

$$\|f\|_{L^{p,\alpha}(a,b)} = \sup_{I\subset(a,b)} \left(|I|^{\alpha-1} \int_I |f(t)|^p \, dt \right)^{\frac{1}{p}},$$

where sup is taken over all intervals $I \subset (a,b)$. For $0 \le \alpha_1 \le \alpha_2 \le 1$ the following continuous embedding holds: $L^{p,\alpha_1} \subset L^{p,\alpha_2}$. It is easy to notice that $L^{p,1}(a,b) = L_p(a,b)$ and $L^{p,0}(a,b) = L_\infty(a,b)$. Moreover $L^{p,\alpha}(a,b) \subset L_1(a,b)$, $\forall p > 1$, $\forall \alpha \in [0,1]$. It is known that $L^{p,\alpha}(a,b)$, $1 \le p < +\infty$, $\alpha \in (0,1)$, is not separable and $C[a,b]$ is not dense in it. Let

$$M^{p,\alpha}(a,b) = \left\{ f \in L^{p,\alpha}(a,b) : \|f(\cdot+\delta) - f(\cdot)\|_{L^{p,\alpha}(a,b)} \to 0,\ \delta \to 0 \right\}.$$

As shown in [2], $M^{p,\alpha}(a,b)$, for $1 \le p < +\infty$, $0 \le \alpha < 1$, is a separable Banach space and $C_0^\infty(a,b)$ (the space of all infinitely differentiable functions on (a,b) with compact support) is dense in it. When defining the space $M^{p,\alpha}(a,b)$, the function $f(\cdot)$ is assumed to be extended outside the interval (a,b) by zero.

We will also need the following elementary

Lemma 2.1. Let $f \in M^{p,\alpha}(a,b)$, $1 \le p < +\infty$, $0 \le \alpha < 1$, be an arbitrary function. Then $\|f\chi_E\|_{L^{p,\alpha}(a,b)} \to 0$, as $|E| \to 0$, where $E \subset (a,b)$ is an arbitrary interval, $|E|$ is the length of this interval.

Proof. Indeed, let $f \in M^{p,\alpha}(a,b)$ be an arbitrary function and $\varepsilon > 0$ be an arbitrary number. Since, $C[a,b]$ is dense in $M^{p,\alpha}(a,b)$, then it is clear that $\exists g \in C[a,b] : \|f-g\|_{L^{p,\alpha}(a,b)} < \varepsilon$. If $|E|$ is sufficiently small we have

$$\|f\chi_E\|_{L^{p,\alpha}(a,b)} \le \|(f-g)\chi_E\|_{L^{p,\alpha}(a,b)} + \|g\chi_E\|_{L^{p,\alpha}(a,b)}$$

$$\le \|f-g\|_{L^{p,\alpha}(a,b)} + \|g\|_\infty \|\chi_E\|_{L^{p,\alpha}(a,b)} < \varepsilon + \|g\|_\infty |E|^{\frac{\alpha}{p}} < 2\varepsilon.$$

From the arbitrariness of ε we obtain what is required. □

We define the space $(M^{p,\alpha}(a,b))'$ associated with $M^{p,\alpha}(a,b)$ and, for brevity, denote it by M'. Let S be the unit ball in $M^{p,\alpha}(a,b)$, i.e.

$$S=\left\{f\in M^{p,\alpha}(a,b):\|f\|_{L^{p,\alpha}(a,b)}\leq 1\right\}.$$

M' is the Banach space of all measurable functions on (a,b) for which the norm

$$\|g\|_{M'}=\sup_{f\in S}\left|\int_a^b fg\,dt\right|<+\infty$$

is finite.

Now we state the following known (see, for example, [1])

Proposition 2.1. [1] *The conjugate space X^* to a Banach space of functions X is isometrically isomorphic to the associated space X' if and only if X has an absolutely continuous norm.*

Taking into account Lemma 2.1, in particular, from this statement we obtain that M' is isometrically isomorphic to the conjugate to the space $M^{p,\alpha}(a,b)$ for $1\leq p<+\infty$, $0\leq\alpha<1$, and denote it by $M^*=(M^{p,\alpha}(a,b))^*$.

We will also need the following completeness criterion of a system in a Banach space X.

Proposition 2.2. *A system $\{x_n\}_{n\in N}\subset X$ in a Banach space X is complete if and if only if from the relation $v\in X^*:v(x_n)=0,\ \forall n\in N$, follows that $v=0$.*

3 Main results

If $\operatorname{Re}f;\operatorname{Im}f\in M^{p,\alpha}(a,b)$, we say that a complex-valued function f belongs to the space $M^{p,\alpha}(a,b)$. Similarly, $g\in M^*\Leftrightarrow\operatorname{Re}g;\operatorname{Im}g\in M^*$. Each functional $g\in M^*$ is generated by some function (we will also denote it by $g(\cdot)$) $g\in M'$ by the expression

$$<g,f>=\int_a^b f(t)\,\overline{g(t)}\,dt,\ \forall f\in M^{p,\alpha}(a,b).$$

So, the following theorem is true.

Theorem 3.1. *Let $\{a\,\omega_n^+;b\,\omega_n^-:\forall n\}\subset M^{p,\alpha}(0,a)$, $1\leq p<+\infty$, $0<\alpha\leq 1$, and the double system*

$$V_{n;m}\equiv(A(t)\,W_n(t);A(-t)\,W_m(-t)),\ \ n,m\in N,$$

is defined by expressions (1.2). *Then this system is complete in $M^{p,\alpha}(-a,a)$ if and only if the unitary systems $\{v_n^+\}_{n\in N}$ and $\{v_n^-\}_{n\in N}$ are complete in $M^{p,\alpha}(0,a)$ simultaneously.*

Proof. Let a system $\{V_{n;m}\}_{n;m\in N}$ be complete in $M^{p,\alpha}(-a,a)$. Let us prove that the systems $\{v_n^+\}_{n\in N}$ and $\{v_n^-\}_{n\in N}$ are complete in $M^{p,\alpha}(0,a)$. Based on the completeness criterion (Proposition 2.1), we propose the opposite, i.e. let for some functions $f_k(\cdot)\in M'(a,b)$, $k=1,2$,

$$\int_0^a v_n^+(t)\,\overline{f_1(t)}dt=0,\forall n\in N;$$

$$\int_0^a v_n^-(t)\,\overline{f_2(t)}dt=0,\forall n\in N,$$

hold. Define

$$f^{+}(t)=\left\{\begin{array}{l} f_1(t),\ \ t\in[0,a]\ ,\\ f_1(-t),\ t\in[-a,0)\ ,\end{array}\right. \quad f^{-}(t)=\left\{\begin{array}{l} f_2(t)\ ,\ t\in[0,a]\ ,\\ -f_2(-t)\ ,\ t\in[-a,0)\ .\end{array}\right.$$

We have

$$\int_{-a}^{a}A(t)\ W_n(t)\,\overline{f^{+}(t)}dt$$

$$=\int_0^a a(t)\,\omega_n^{+}(t)\,\overline{f_1(t)}dt+\int_{-a}^{0}b(-t)\,\omega_n^{-}(-t)\,\overline{f_1(-t)}dt$$

$$=\int_0^a v_n^{+}(t)\,\overline{f_1(t)}dt=0,\forall n\in N.$$

Since, $f^{+}(\cdot)$ is a even function on $(-a,a)$, then it is clear that

$$\int_{-a}^{a}A(-t)\,W_n(-t)\,\overline{f^{+}(t)}dt=\int_{-a}^{a}A(t)\,W_n(t)\,\overline{f^{+}(t)}dt=0,\ \ \forall n\in N.$$

From the completeness of the system $\{V_{n;n}\}_{n\in N}$ in $M^{p,\alpha}(-a,a)$, it follows that $f_1^{+}(t)=0$, for almost all $t\in(-a,a)$, and $f_1(t)=0$, for almost all $t\in(0,a)$. So, we obtain the completeness of the system $\{v_n^{+}\}_{n\in N}$ in $M^{p,\alpha}(0,a)$.

In the same way the completeness of system $\{v_n^{-}\}_{n\in N}$ is proved in $M^{p,\alpha}(0,a)$.

Next, let systems $\{v_n^{+}\}_{n\in N}$ and $\{v_n^{-}\}_{n\in N}$ be complete in $M^{p,\alpha}(0,a)$. Suppose that for some function $F\in M'(-a,a)$

$$\int_{-a}^{a}A(t)\ W_n(t)\,\overline{F(t)}dt=0,$$

$$\int_{-a}^{a}A(-t)\ W_n(-t)\,\overline{F(t)}dt=0,\forall n\in N,$$

holds. Transforming, we have

$$I_n^{+}=\int_{-a}^{a}A(t)\ W_n(t)\,\overline{F(t)}dt=\int_0^a a(t)\,\omega_n^{+}(t)\,\overline{F(t)}dt$$

$$+\int_0^a b(t)\,\omega_n^{-}(t)\,\overline{F(-t)}dt=0,\forall n\in N;$$

$$I_n^{-}=\int_{-a}^{a}A(-t)\ W_n(-t)\,\overline{F(t)}dt=\int_0^a a(t)\,\omega_n^{+}(t)\,\overline{F(-t)}dt$$

$$+\int_0^a b(t)\,\omega_n^{-}(t)\,\overline{F(t)}dt=0,\forall n\in N.$$

Consequently,

$$I_n^{+}+I_n^{-}=\int_0^a v_n^{+}(t)\left(\overline{F(t)}+\overline{F(-t)}\right)dt=0,\forall n\in N. \tag{3.1}$$

It is clear that $g\in M'(0,a)$, where $g(t)=F(t)+F(-t)$. Indeed, let $\varphi\in M^{p,\alpha}(0,a)$ be an arbitrary function. We have

$$\int_0^a\varphi(t)\,g(t)\,dt=\int_0^a\varphi(t)\,F(t)\,dt+\int_0^a\varphi(t)\,F(-t)\,dt=\int_{-a}^{a}\Phi(t)\,F(t)\,dt, \tag{3.2}$$

where

$$\Phi(t) = \begin{cases} \varphi(t), & t \in [0,a] \\ \varphi(-t), & t \in [-a,0) \end{cases}.$$

Let $I \subset [-a,a]$ be an arbitrary interval. Assume $I_1 = [-a,0] \bigcap I$, $I_2 = [0,a] \bigcap I$. We have

$$\frac{1}{|I|^{1-\alpha}} \int_I |\Phi(t)|^p \, dt = \frac{1}{|I|^{1-\alpha}} \left(\int_{I_1} |\varphi(-t)|^p \, dt + \int_{I_2} |\varphi(t)|^p \right) dt$$

$$= \frac{1}{|I|^{1-\alpha}} \left(\int_{-I_1} |\varphi(t)|^p \, dt + \int_{I_2} |\varphi(t)|^p \, dt \right)$$

$$\leq \frac{1}{|-I_1|^{1-\alpha}} \int_{-I_1} |\varphi(t)|^p \, dt + \frac{1}{|I_2|^{1-\alpha}} \int_{I_2} |\varphi(t)|^p \, dt \leq 2 \|\varphi\|_{L^{p,\alpha}(0,a)}.$$

Hence it follows that $\Phi \in M^{p,\alpha}(-a,a)$. Since $F \in M'(-a,a)$, then from (3.1) it follows that

$$\left| \int_0^a \varphi(t) \, g(t) \, dt \right| < +\infty, \ \forall \varphi \in M^{p,\alpha}(0,a),$$

i.e. $g \in M'(0,a)$. As the system $\{v_n^+\}_{n \in N}$ is complete in $M^{p,\alpha}(0,a)$, then from relation (3.2) it follows that $g(t) = 0$, for almost all $t \in (0,a)$, i.e. $F(t) = -F(-t)$, for almost all $t \in (0,a)$.

In exactly the same way, from the relation

$$I_n^+ - I_n^- = \int_0^a v_n^-(t) \left(\overline{F(t)} - \overline{F(-t)} \right) dt = 0, \forall n \in N,$$

and from the completeness of the system $\{v_n^-\}_{n \in N}$ in $M^{p,\alpha}(0,a)$ it follows that $F(t) = F(-t)$, for almost all $t \in (0,a)$. From these two relations we obtain that $F(t) = 0$, for almost all $t \in (-a,a)$. □

The following statement is proved in a similar way.

Theorem 3.2. *Let $\{a\,\omega_n^+; b\,\omega_n^- : \forall n\} \subset M^{p,\alpha}(0,a)$, $1 \leq p < +\infty$, $0 < \alpha \leq 1$, and the double system*

$$V_{n;m} \equiv (A(t) \, W_n(t); A(-t) \, W_m(-t)), \ \ n, m \in N,$$

is defined by expressions (1.2). *The system $1 \bigcup \{V_{n;n}\}_{n \in N}$ is complete in the space $M^{p,\alpha}(-a,a)$ if and only if the systems $1 \bigcup \{v_n^+\}_{n \in N}$ and $\{v_n^-\}_{n \in N}$ are complete in $M^{p,\alpha}(0,a)$.*

As a particular case, we consider the systems of sines

$$\{\sin(nt + \beta(t))\}_{n \in N}, \tag{3.3}$$

and cosines

$$1 \bigcup \{\cos(nt + \beta(t))\}_{n \in N}, \tag{3.4}$$

where $\beta : [0,\pi] \to \mathbb{C}$ generally speaking, is a complex-valued function. Let us extend the function $\beta(\cdot)$ to the segment $[-\pi,\pi]$ by parity and also denote it by $\beta(\cdot)$. Applying the above notation to this case, we obtain the following corollaries.

Corollary 3.1. The double system of exponents

$$\left\{ e^{i(nt + \beta(t) \, sign \, n)} \right\}_{n \neq 0},$$

is complete in $M^{p,\alpha}(-\pi,\pi)$, $1 \leq p < +\infty$, $0 < \alpha \leq 1$, if and only if the systems of sines (3.3) and cosines $\{\cos(nt + \beta(t))\}_{n \in N}$ are complete in $M^{p,\alpha}(0,\pi)$.

Corollary 3.2. The system of exponents

$$1\bigcup\left\{e^{i(nt+\beta(t)\,sign\,n)}\right\}_{n\neq 0},$$

is complete in $M^{p,\alpha}(-\pi,\pi)$, $1\leq p<+\infty$, $0<\alpha\leq 1$, if and only if systems (3.3) and (3.4) are complete in $M^{p,\alpha}(0,\pi)$.

Using the result of [2], in particular, we also obtain the following

Corollary 3.3. Let $\beta\in\mathbb{C}$ be a complex parameter and $2Re\beta+\frac{\alpha}{p}\notin Z$, where $1<p<+\infty$, $0<\alpha<1$ (Z is the set of integers). If $\left[2Re\beta+\frac{\alpha}{p}\right]\leq 0$ ($[\cdot]$ is the integer part), then the systems of sines $\{\sin(n+\beta)\,t\}_{n\in N}$ and cosines $1\bigcup\{\cos(n+\beta)\,t\}_{n\in N}$ are complete in $M^{p,\alpha}(0,\pi)$.

Now let us establish the relationship between the minimality of systems (1.1) and (1.2). Let the system $\{V_{n;n}\}_{n\in N}$ be minimal in $M^{p,\alpha}(-a,a)$ and $\{h_n^+;h_n^-\}_{n\in N}\subset M'(-a,a)$ be a biorthogonal system to it. Define

$$\vartheta_k^+(t)\equiv h_k^+(t)+h_k^-(-t),\forall k\in N.$$

Let us show that $\{\vartheta_k^+\}_{k\in N}\subset M'(0,a)$. Let $f\in M^{p,\alpha}(0,a)$ be an arbitrary function. We have

$$\int_0^a\vartheta_k^+(t)\,f(t)\,dt=\int_0^a h_k^+(t)\,f(t)\,dt+\int_{-a}^0 h_k^-(t)\,f(-t)\,dt.$$

The function $f(\cdot)$ on $(-a,0)$ is assumed to be extended by zero. Then it is clear that $f\in M^{p,\alpha}(-a,a)$ and from the previous relation $\left|\int_0^a\vartheta_k^+(t)\,f(t)\,dt\right|<+\infty$, $\forall f\in M^{p,\alpha}(0,a)$, and as a result, $\vartheta_k^+\in M'(0,a)$, $\forall k\in N$. We have

$$\int_0^a\vartheta_n^+(t)\,\overline{\vartheta_k^+(t)}dt=\int_0^a\left[a(t)\,\omega_n^+(t)+b(t)\,\omega_n^-(t)\right]\left[\overline{h_k^+(t)}+\overline{h_k^+(-t)}\right]dt$$

$$=\int_0^a a(t)\,\omega_n^+(t)\,\overline{h_k^+(t)}dt+\int_{-a}^0 b(-t)\,\omega_n^-(-t)\,\overline{h_k^+(t)}dt$$

$$+\int_{-a}^0 a(-t)\,\omega_n^+(-t)\,\overline{h_k^+(t)}dt$$

$$+\int_0^a b(t)\,\omega_n^-(t)\,\overline{h_k^+(t)}dt=\int_{-a}^a A(t)\,W_n(t)\,\overline{h_k^+(t)}dt$$

$$+\int_{-a}^a A(-t)\,W_n(-t)\,\overline{h_k^+(t)}dt=\delta_{nk},\forall n,k\in N.$$

Consequently, the system $\{\vartheta_k^+\}_{k\in N}$ is biorthogonal to $\{\vartheta_n^+\}_{n\in N}$ and, therefore, it is minimal in $M^{p,\alpha}(0,a)$.

Absolutely similarly one can show that the system

$$\vartheta_k^-(t)\equiv h_k^-(-t)+h_k^-(t),\forall k\in N,$$

is biorthogonal to $\{\vartheta_n^-\}_{n\in N}$, i.e. $\{\vartheta_n^-\}_{n\in N}$ is minimal in $M^{p,\alpha}(0,a)$.

Next, let systems $\{\vartheta_n^\pm\}_{n\in N}$ be minimal in $M^{p,\alpha}(0,a)$ and $\{\vartheta_n^\pm\}_{n\in N}\subset M'(0,a)$ be the corresponding biorthogonal systems. Define

$$\tilde{\vartheta}_k^\pm(t)\equiv\begin{cases}\vartheta_k^\pm(t)\;,\;t\in(0,a)\\ \pm\vartheta_k^\pm(-t)\;,\;t\in(-a,0)\,,\end{cases}$$

and

$$h_k^{\pm}(t) \equiv \frac{1}{2}\left[\tilde{\vartheta}_k^{+}(t) \pm \tilde{\vartheta}_k^{-}(t)\right], \forall k \in N.$$

By simple transformations we get

$$\int_{-a}^{a} A(t)\ W_n(t)\, \overline{h_k^{\pm}(t)} dt = \frac{1}{2}\int_0^a \vartheta_n^{+}(t)\, \overline{\vartheta_k^{+}(t)} dt$$

$$\pm\frac{1}{2}\int_0^a \vartheta_n^{-}(t)\, \overline{\vartheta_k^{-}(t)} dt = \frac{1}{2}\left[\delta_{nk} \pm \delta_{nk}\right], \forall n, k \in N;$$

$$\int_{-a}^{a} A(-t)\ W_n(-t)\, \overline{h_k^{\pm}(t)} dt = \frac{1}{2}\left[\delta_{nk} \mp \delta_{nk}\right], \forall n, k \in N.$$

It is easy to see that $\left\{h_k^{\pm}\right\}_{k\in N} \subset M'(-a,a)$. Then from the previous relations it follows that the system $\{V_{n;n}\}_{n\in N}$ is minimal in $M^{p,\alpha}(-a,a)$. Thus, the following theorem is valid.

Theorem 3.3. *Let* $\{a\,\omega_n^{+}; b\,\omega_n^{-} : \forall n\} \subset M^{p,\alpha}(0,a)$, $1 \le p < +\infty$, $0 < \alpha \le 1$, *and the double system*

$$V_{n;m} \equiv (A(t)\ W_n(t); A(-t)\ W_m(-t)), \quad n, m \in N,$$

is defined by expressions (1.2). *The system* $\{V_{n;n}\}_{n\in N}$ *is minimal in* $M^{p,\alpha}(-a,a)$ *if and only if the systems* $\{\vartheta_n^{+}\}_{n\in N}$ *and* $\{\vartheta_n^{-}\}_{n\in N}$ *are minimal in* $M^{p,\alpha}(0,a)$.

Quite similarly the following statement is proved.

Theorem 3.4. *Let* $\{a\,\omega_n^{+}; b\,\omega_n^{-} : \forall n\} \subset M^{p,\alpha}(0,a)$, $1 \le p < +\infty$, $0 < \alpha \le 1$, *and the double system*

$$V_{n;m} \equiv (A(t)\ W_n(t); A(-t)\ W_m(-t)), \quad n, m \in N,$$

is defined by expressions (1.2). *The system* $1\bigcup\{V_{n;n}\}_{n\in N}$ *is minimal in* $M^{p,\alpha}(-a,a)$ *if and only if the systems* $1\bigcup\{\vartheta_n^{+}\}_{n\in N}$ *and* $\{\vartheta_n^{-}\}_{n\in N}$ *are minimal in* $M^{p,\alpha}(0,a)$.

Remark 1. It is not difficult to see that if the functions $a; b$ and $\omega_n^{\pm}; \forall n$, are bounded on $[0,a]$, then all conditions of above cited theorems are satisfied and therefore all the assertions of these theorems are true for such systems.

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