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VLADIMIR MIKHAILOVICH FILIPPOV

(to the 70th birthday)

Vladimir Mikhailovich Filippov was born on 15 April 1951 in the city of Uryupinsk, Stalingrad Region of the USSR. In 1973 he graduated with honors from the Faculty of Physics and Mathematics and Natural Sciences of the Patrice Lumumba University of Peoples' Friendship in the specialty "Mathematics". In 1973-1975 he is a postgraduate student of the University; in 1976-1979 - Chairman of the Young Scientists' Council; in 1980-1987 - Head of the Research Department and the Scientific Department; in 1983-1984 - scientific work at the Free University of Brussels (Belgium); in 1985-2000 - Head of the Mathematical Analysis Department; from 2000 to the present - Head of the Comparative Educational Policy Department; in 1989–1993 - Dean of the Faculty of Physics, Mathematics and Natural Sciences; in 1993–1998 - Rector of the Peoples' Friendship University of Russia; in 1998-2004 - Minister of General and Professional Education, Minister of Education of the Russian Federation; in 2004-2005 - Assistant to the Chairman of the Government of the Russian Federation (in the field of education and culture); from 2005 to May 2020- Rector of the Peoples' Friendship University of Russia, since May 2020 - President of the Peoples' Friendship University of Russia, since 2013 - Chairman of the Higher Attestation Commission of the Ministry of Science and Higher Education of the Russian Federation.

In 1980, he defended his PhD thesis in the V.A. Steklov Mathematical Institute of Academy of Sciences of the USSR on specialty 01.01.01 - mathematical analysis (supervisor - a corresponding member of the Academy of Sciences of the USSR, Professor L.D. Kudryavtsev), and in 1986 in the same Institute he defended his doctoral thesis "Quasi-classical solutions of inverse problems of the calculus of variations in non-Eulerian classes of functionals and function spaces". In 1987, he was awarded the academic title of a professor.

V.M. Filippov is an academician of the Russian Academy of Education; a foreign member of the Ukrainian Academy of Pedagogical Sciences; President of the UNESCO International Organizing Committee for the World Conference on Higher Education (2007-2009); Vice-President of the Eurasian Association of Universities; a member of the Presidium of the Rectors' Council of Moscow and Moscow Region Universities, of the Governing Board of the Institute of Information Technologies in Education (UNESCO), of the Supervisory Board of the European Higher Education Center of UNESCO (Bucharest, Romania),

Research interests: variational methods; non-potential operators; inverse problems of the calculus of variations; function spaces.

In his Ph.D thesis, V.M. Filippov solved a long standing problem of constructing an integral extremal variational principle for the heat equation. In his further research he developed a general theory of constructing extremal variational principles for broad classes of differential equations with non-potential (in classical understanding) operators. He showed that all previous attempts to construct variational principles for non-potential operators "failed" because mathematicians and mechanics from the time of L. Euler and J. Lagrange were limited in their research by functionals of the type Euler - Lagrange. Extending the classes of functionals, V.M. Filippov introduced a new scale of function spaces that generalize the Sobolev spaces, and thus significantly expanded the scope of the variational methods. In 1984, famous physicist, a Nobel Prize winner I.R. Prigogine presented the report of V.M. Filippov to the Royal Academy of Sciences of Belgium. Results of V.M. Filippov's variational principles for non-potential operators are quite fully represented in some of his and his colleagues' monographs.

Honors: Honorary Legion (France), "Commander" (Belgium), Crown of the King (Belgium); in Russia - orders "Friendship", "Honor", "For Service to the Fatherland" III and IV degrees; Prize of the President of the Russian Federation in the field of education; Prize of the Governement of the Russian Federation in the field of education; Gratitude of the President of the Russian Federation; "For Merits in the Social and Labor Sphere of the Russian Federation", "For Merits in the Development of the Olympic Movement in Russia", "For Strengthening the Combat Commonwealth; and a number of other medals, prizes and awards.

He is an author of more than 270 scientific and scientific-methodical works, including 32 monographs, 2 of which were translated and published in the United States by the American Mathematical Society.

V.M. Filippov meets his 70th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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REFINEMENT OF CONTINUOUS FORMS OF CLASSICAL INEQUALITIES

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Dedicated to the 80th anniversary of Professor Shoshana Abramovich

Key words: inequalities, Hölder-, Minkowski-, Popoviciu- and Bellman-type inequalities, continuous forms, measure spaces, related functionals

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Abstract. In this article we give refinements of the continuous forms of some classical inequalities i.e. of the inequalities which involve infinitely many functions instead of finitely many. We present new general results for such inequalities of Hölder-type and of Minkowski-type as well as for their reverses known as Popoviciu- and Bellman-type inequalities. Properties for related functionals are also established. As particular cases of these results we derive both well-known and new refinements of the corresponding classical inequalities for integrals and sums.

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1 Introduction

Let (X, μ) and (Y, ν) be two σ-finite measure spaces with non-negative measures. The classical Hölder and Minkowski inequalities can be generalized to hold for continuously many functions as we can see in the following theorem.

Theorem 1.1. *Let $f(x, y)$ be positive and measurable functions on $(X \times Y, \mu \times \nu)$ and let $u(x)$ and $v(y)$ be weight functions on X and on Y, respectively.*

(i) Then, for $p \geq 1$

$$\left(\int_Y \left(\int_X f(x,y)u(x)\,d\mu(x)\right)^p v(y)\,d\nu(y)\right)^{\frac{1}{p}} \leq \int_X \left(\int_Y f^p(x,y)v(y)\,d\nu(y)\right)^{\frac{1}{p}} u(x)\,d\mu(x). \tag{1.1}$$

(ii) Moreover, if $\int_X u(x)\,d\mu(x) = 1$, then

$$\int_Y \exp\left(\int_X \log f(x,y)u(x)\,d\mu(x)\right) v(y)\,d\nu(y) \leq \exp\left(\int_X \log\left(\int_Y f(x,y)v(y)\,d\nu(y)\right) u(x)\,d\mu(x)\right). \tag{1.2}$$

As usual, here and in the sequel, by a weight or a weight function $u(x)$ on X we mean a non-negative measurable function on X.

Remark 1. Inequality (1.1) is also known as the integral form of the Minkowski inequality (see e.g. [7, p. 41], [12]). The case $p < 1$ is described in details in paper [4] and under some additional assumptions the sign in inequality (1.1) is reversed. Inequality (1.2) is called a continuous form of the Hölder inequality (see e.g. [1], [6]). The reason why that inequality got its name is based on the following particular case: Namely, putting

$$f(x,y) := \begin{cases} f^p(y) & \text{for } x \in X_1 \\ g^q(y) & \text{for } x \in X_2, \end{cases}$$

where $X = X_1 \cup X_2, X_1 \cap X_2 = \emptyset$, $\int_{X_1} u(x)\, d\mu(x) = \frac{1}{p}$ and $\int_{X_2} u(x)\, d\mu(x) = \frac{1}{q}$, and hence $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then inequality (1.2) is reduced to the usual integral Hölder inequality for two functions

$$\|fg\|_1 \le \|f\|_p \cdot \|g\|_q, \tag{1.3}$$

where $\|F\|_p := \left(\displaystyle\int_Y |F(y)|^p v(y)\, d\nu(y) \right)^{1/p}$. This notation will be used through the whole of our paper.

Similarly, putting

$$f(x,y) := \begin{cases} \dfrac{f(y)}{\alpha_1} & \text{for } x \in X_1 \\ \dfrac{g(y)}{\alpha_2} & \text{for } x \in X_2, \end{cases}$$

where $X = X_1 \cup X_2, X_1 \cap X_2 = \emptyset$, $\int_{X_1} u(x)\, d\mu(x) = \alpha_1$ and $\int_{X_2} u(x)\, d\mu(x) = \alpha_2$, in (1.1) we get the usual integral Minkowski inequality for two functions:

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

A discussion about the reverse versions of these two inequalities is given in the third section.

Recently, in paper [2] the following result was proved:

Theorem 1.2. *Let $p, q > 1$ be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, f and g be real functions defined on $[u, v]$ such that $|f|^p$ and $|g|^q$ be integrable functions on $[u, v]$, $-\infty \le u < v \le \infty$ and α_1, α_2 be non-negative continuous functions on $[u, v]$ such that $\alpha_1(t) + \alpha_2(t) = 1$ for all $t \in [u, v]$. Then*

$$\int_u^v |f(y)g(y)|\, dy \quad \le \quad \sum_{i=1}^{2} \left[\int_u^v \alpha_i(y)|f(y)|^p\, dy \right]^{\frac{1}{p}} \cdot \left[\int_u^v \alpha_i(y)|g(y)|^q\, dy \right]^{\frac{1}{q}}. \tag{1.4}$$

A similar result for n functions α_i was also given in [2]. Moreover, in [5] it was proved that the right-hand side of inequality (1.4) is not greater than

$$\left(\int_u^v |f(y)|^p\, dy \right)^{\frac{1}{p}} \cdot \left(\int_u^v |g(y)|^q\, dy \right)^{\frac{1}{q}}.$$

So, together with inequality (1.4) we have a refinement of the Hölder inequality for integrals (1.3) with $Y = [u, v]$ and $v(y)\, d\nu(y) = dy$.

The aim of this paper is to discuss some similar refinements but for continuous forms of some classical inequalities. More exactly, in Section 2 we derive a new refinement of a continuous form of the Hölder inequality, which as special cases contain the above mentioned results in [2] and [5] (see Theorem 2.1). Moreover, also another new refinement of another continuous form of the Hölder inequality is proved and discussed (see Theorem 2.2.) In addition, the sharpness of these results are studied by investigating some functionals describing the "gaps" in these inequalties (see Theorem 2.3). The corresponding results concerning a continuous form of the Minkowski inequality are given in Section 3 (see Theorem 3.1 and Theorem 3.2.) Section 4 is used to state and prove the corresponding results related to classical Popoviciu and Bellman inequalities (see Theorems 4.2 and 4.5).

2 Refinements of some continuous forms of the Hölder inequality

The following theorem gives a continuous generalization of the result from [2] and [5].

Theorem 2.1. *Let $f(x,y)$ be positive and measurable functions on $(X \times Y, \mu \times \nu)$ and let $u(x)$ and $v(y)$ be weight functions on X such that $\int_X u(x)\,d\mu(x) = 1$. Moreover, let $(Z,\, dz)$ be a measure space and $\alpha(z,y)$ be a non-negative integrable function on $Z \times Y$ such that*

$$\int_Z \alpha(z,y)\,dz = 1, \qquad \text{for } y \in Y. \tag{2.1}$$

Then the following refinement of continuous form (1.2) of the Hölder inequality holds:

$$\begin{aligned}
&\int_Y \exp\left(\int_X \log f(x,y)u(x)\,d\mu(x)\right) v(y)\,d\nu(y)\\
&\leq \int_Z \left[\exp \int_X \log\left(\int_Y \alpha(z,y) f(x,z) v(y)\,d\nu(y)\right) u(x)\,d\mu(x)\right] dz\\
&\leq \exp\left[\int_X \log\left(\int_Y f(x,y)v(y)d\nu(y)\right) u(x)\,d\mu(x)\right].
\end{aligned} \tag{2.2}$$

Proof. By using condition (2.1) and the Fubini theorem we get

$$\begin{aligned}
&\int_Y \exp\left(\int_X \log f(x,y)u(x)\,d\mu(x)\right) v(y)\,d\nu(y)\\
=\;& \int_Y \left[\int_Z \alpha(z,y) \exp\left(\int_X \log f(x,y)u(x)\,d\mu(x)\right) dz\right] v(y)\,d\nu(y)\\
=\;& \int_Z \left[\int_Y \exp\left(\int_X \log f(x,y)u(x)\,d\mu(x)\right) \alpha(z,y)v(y)\,d\nu(y)\right] dz.
\end{aligned}$$

Now, by applying the continuous form of the Hölder inequality to the term in the square brackets and again the same inequality to the whole term we obtain that

$$\begin{aligned}
&\int_Z \left[\int_Y \exp\left(\int_X \log f(x,y)u(x)\,d\mu(x)\right) \alpha(z,y)v(y)\,d\nu(y)\right] dz\\
\leq\;& \int_Z \exp\left[\int_X \log\left(\int_Y f(x,y)\alpha(z,y)v(y)\,d\nu(y)\right) u(x)d\mu(x)\right] dz\\
\leq\;& \exp\left[\int_X \log\left(\int_Z \left(\int_Y f(x,y)\alpha(z,y)v(y)\,d\nu(y)\right) dz\right) u(x)d\mu(x)\right]\\
=\;& \exp\left[\int_X \log\left(\int_Y \left(\int_Z f(x,y)\alpha(z,y)dz\right) v(y)\,d\nu(y)\right) u(x)d\mu(x)\right]\\
=\;& \exp\left[\int_X \log\left(\int_Y f(x,y)v(y)\,d\nu(y)\right) u(x)d\mu(x)\right],
\end{aligned}$$

where the Fubini theorem and (2.1) are used in the last two equalities. □

Next we point out some consequences of the above-mentioned theorem and compare it with already known results.

Corollary 2.1. *a) Let the assumptions of Theorem* 2.1 *hold and let* $p(x)$ *be a measurable function on* X. *Then*

$$\begin{aligned}
&\int_Y \exp\left(\int_X p(x)\log f(x,y)\,u(x)\,d\mu(x)\right) v(y)\,d\nu(y)\\
&\le \int_Z \left[\exp\int_X \left(\int_Y \alpha(z,y) f(x,y)^{p(x)} v(y)\,d\nu(y)\right) u(x)\,d\mu(x)\right] dz\\
&\le \exp\left[\int_X \log\left(\int_Y f(x,y)^{p(y)} v(y)\,d\nu(y)\right) u(x)\,d\mu(x)\right]. \qquad (2.3)
\end{aligned}$$

b) Let $p,q>1$ *be such that* $\frac{1}{p}+\frac{1}{q}=1$. *If* $f(y), g(y), \alpha(z,y)$ *are non-negative functions such that* $fg\in L_1(Y), f, \alpha^{1/p}(z,.)f\in L_p(Y),\ g, \alpha^{1/q}(y,.)g\in L_q(Y),\ \int_Z \alpha(z,y)\,dz=1$ *for all* $y\in Y$, *then the following refinement of the Hölder inequality holds:*

$$\|fg\|_1 \le \int_Z \|\alpha^{1/p}(z,.)f(.)\|_p\cdot\|\alpha^{1/q}(z,.)g(.)\|_q\,dz \le \|f\|_p\|g\|_q. \qquad (2.4)$$

c) Let $p,q>1$ *be such that* $\frac{1}{p}+\frac{1}{q}=1$. *If* f,g,α *are non-negative functions on* Y *such that* $fg\in L_1(Y), f,\alpha^{1/p}f\in L_p(Y),\ g,\alpha^{1/q}g\in L_q(Y)$ *and* $\alpha(y)\le 1$ *for all* $y\in Y$, *then we find that also the following refinement of Hölder inequality holds:*

$$\|fg\|_1 \;\le\; \|\alpha^{\frac{1}{p}}f\|_p\cdot\|\alpha^{\frac{1}{q}}g\|_q+\|(1-\alpha)^{\frac{1}{p}}f\|_p\cdot\|(1-\alpha)^{\frac{1}{q}}g\|_q\le\|f\|_p\|g\|_q. \qquad (2.5)$$

Proof. a) This is a simple consequence of Theorem 2.1 applied with $f(x,y)^{p(x)}$ in place of $f(x,y)$.

b) By putting in the a) part of this corollary: $X=X_1\cup X_2, X_1\cap X_2=\emptyset$, such that $\int_{X_1} u(x)\,d\mu(x)=\frac{1}{p}$, $\int_{X_2} u(x)\,d\mu(x)=\frac{1}{q}$, and

$$f(x,y):=\begin{cases} f(y) & \text{for } x\in X_1\\ g(y) & \text{for } x\in X_2\end{cases}$$

$$p(x):=\begin{cases} p & \text{for } x\in X_1\\ q & \text{for } x\in X_2,\end{cases}$$

we get inequality (2.4).

c) Inequality (2.5) follows from inequality (2.4) by taking:

$$Z=[0,2],\quad Z_1=[0,1],\quad Z_2=[1,2],\quad \alpha(z,y):=\begin{cases}\alpha(y) & \text{for } z\in Z_1\\ 1-\alpha(y) & \text{for } z\in Z_2.\end{cases}$$

□

Our next remark shows in particular that Corollary 2.1 may be regarded as a genuine generalization of the results in [2] and [5].

Remark 2. a) The chain of inequalities from part c) of the above corollary for $Y = [a,b]$, $v(y)d\nu(y) = dy$ can be found in [5], while the first inequality was proved in [2]. Moreover, if $Y = [a,b]$, $v(y)d\nu(y) = dy$, and $\alpha(t) = b - t$ the chain of inequalities from part c) was proved in [2].

b) By using the same idea as in part b) we can derive a refinement of the Hölder inequality with n functions involved ($n = 2, 3, \ldots$).

Another nice inequality of Hölder type can be found in [1, VI.11.35]:

$$\exp\left[\int_X \log a(x)\, u(x)\, d\mu(x)\right] + \exp\left[\int_X \log b(x)\, u(x)\, d\mu(x)\right]$$
$$\leq \exp\left[\int_X \log\left(a(x) + b(x)\right)\, u(x)\, d\mu(x)\right] \tag{2.6}$$

(provided that all integrals exist and $a(.)$ and $b(.)$ are non-negative).

Next we will give a refinement also of this inequality.

Theorem 2.2. *Let (X,μ), (Y,ν) and (Z, dz) be σ-finite measure spaces. Let a and b be positive measurable functions on X, $u(x)$ be a weight on X, $v(y)$ be a weight on Y and $\alpha(z,y)$ be a non-negative function on $Z \times Y$ such that $\int_Z \alpha(z,y)\, dz = 1$ for all $y \in Y$. If Y has partition $Y = Y_1 \cup Y_2$, such that $\int_{Y_i} v(y)d\nu(y) = 1$, $i = 1,2$, and the integrals $A(z) := \int_{Y_1} \alpha(z,y)v(y)\, d\nu(y)$ and $B(z) := \int_{Y_2} \alpha(z,y)v(y)\, d\nu(y)$ exist, then*

$$\exp\left[\int_X \log a(x)u(x)\, d\mu(x)\right] + \exp\left[\int_X \log b(x)u(x)\, d\mu(x)\right]$$
$$\begin{aligned} &\leq \int_Z \left[\exp \int_X \log\Big(A(z)a(x) + B(z)b(x)\Big)u(x)\, d\mu(x)\right] dz \\ &\leq \exp\left[\int_X \log\Big(a(x) + b(x)\Big)u(x)\, d\mu(x)\right]. \end{aligned} \tag{2.7}$$

Proof. By putting in inequality (2.1)

$$f(x,y) := \begin{cases} a(x) & \text{for } y \in Y_1 \\ b(x) & \text{for } y \in Y_2 \end{cases}$$

after a straightforward calculation we get inequality (2.7). □

Remark 3. Let us take

$$Z = [0,2], \quad Z_1 = [0,1], \quad Z_2 = [1,2],$$

and let Y has partition $Y = Y_1 \cup Y_2$ such that $\int_{Y_i} v(y)d\nu(y) = 1$, $i = 1,2$.

Denote

$$\alpha(z,y) := \begin{cases} A & \text{for } z \in Z_1,\ y \in Y_1 \\ 1 - A & \text{for } z \in Z_1, y \in Y_2 \\ B & \text{for } z \in Z_2,\ y \in Y_1 \\ 1 - B & \text{for } z \in Z_2, y \in Y_2, \end{cases}$$

where $A, B \in [0,1]$.

The condition on $\alpha(z,y)$ is fulfilled.

Hence the middle term in (2.7) takes the form

$$\exp\left[\int_X \log\Big(Aa(x)+Bb(x)\Big)u(x)\,d\mu(x)\right]$$
$$+\exp\left[\int_X \log\Big((1-A)a(x)+(1-B)b(x)\Big)u(x)\,d\mu(x)\right],$$

where $A, B \in [0,1]$.

Next we will point out a technique which can give important complementary information on some of the refinements we have presented so far, but first we introduce some abbreviations and notations:

$$\begin{aligned}
L_H(v) &:= \int_Y \exp\left(\int_X \log f(x,y)u(x)\,d\mu(x)\right) v(y)\,d\nu(y),\\
M_H(v) &:= \int_Z \left[\exp\int_X \log\left(\int_Y \alpha(z,y)f(x,y)v(y)\,d\nu(y)\right) u(x)\,d\mu(x)\right] dz,\\
R_H(v) &:= \exp\left[\int_X \log\left(\int_Y f(x,y)v(y)d\nu(y)\right) u(x)\,d\mu(x)\right],\\
H_0(v) &:= R_H(v)-L_H(v),\\
H_1(v) &:= M_H(v)-L_H(v).
\end{aligned}$$

As we can see, the functional H_0 is the difference between the right-hand side and the left-hand side of the continuous Hölder inequality (1.2) while H_1 is the difference between the middle term in the refinement (2.2) and the left-hand side of (1.2). In the following theorem some superadditivity properties of the functionals H_0, H_1, R_H and M_H are given.

Theorem 2.3. *Let the assumptions of Theorem* 2.1 *hold and let $v(y)$ and $w(y)$ be weight functions on Y. Then L_H is linear and the functionals H_0, H_1, R_H and M_H satisfy*

$$\begin{aligned}
H_i(v+w) &\geq H_i(v)+H_i(w), \quad i=0,1,\\
M_H(v+w) &\geq M_H(v)+M_H(w),\\
R_H(v+w) &\geq R_H(v)+R_H(w).
\end{aligned}$$

Proof. By putting in (2.6)

$$a(x)=\int_Y \alpha(z,y)f(x,z)v(y)\,d\nu(y), \quad b(x)=\int_Y \alpha(z,y)f(x,z)w(y)\,d\nu(y)$$

we get that

$$\begin{aligned}
&\exp\left[\int_X \log\left(\int_Y \alpha(z,y)f(x,z)v(y)\,d\nu(y\right) u(x)\,d\mu(x)\right]\\
+\ &\exp\left[\int_X \log\left(\int_Y \alpha(z,y)f(x,z)w(y)\,d\nu(y\right) u(x)\,d\mu(x)\right]\\
\leq\ &\exp\left[\int_X \log\left(\int_Y \alpha(z,y)f(x,z)(v(y)+w(y))\,d\nu(y\right) u(x)\,d\mu(x)\right].
\end{aligned}$$

Now by integrating over Z we obtain that $M_H(v+w) \geq M_H(v)+M_H(w)$. The superadditivity of R_H can be proved in the similar manner. In the consideration for H_i we use the fact that

$L_H(v+w) = L_H(v) + L_H(w)$ and the just obtained properties for M_H and R_H. Particularly, for H_1 we have

$$\begin{aligned} & H_1(v+w) - H_1(v) - H_1(w) \\ = & \Big(M_H(v+w) - M_H(v) - M_H(w)\Big) - \Big(L_H(v+w) - L_H(v) - L_H(w)\Big) \geq 0. \end{aligned}$$

The proof is similar for H_0. □

Corollary 2.2. *The functionals H_0, H_1, R_H and M_H are non-decreasing.*

Proof. This is a corollary of the positivity and superadditivity of the considered functionals. For example, let us prove this fact for the functional H_0. If $v \geq w$, then $v - w \geq 0$ and from Theorem 2.1 we get that $H_0(v-w) \geq 0$. Hence, by using Theorem 2.3 we obtain that

$$H_0(v) = H_0(w + (v-w)) \geq H_0(w) + H_0(v-w) \geq H_0(w).$$

The proof is similar for the other functionals and therefore omitted. □

3 A refinement of the continuous form of the Minkowski inequality

Our first main result in this section reads:

Theorem 3.1. *Let $f(x,y)$ be a positive and measurable function on $(X \times Y, \mu \times \nu)$, let $u(x)$ and $v(y)$ be weight functions on X and Y, respectively. Moreover, let $\alpha(z,y)$ be a non-negative function such that*

$$\int_Z \alpha(z,y)\, dz = 1, \qquad \text{for } y \in Y.$$

If $p \geq 1$, then

$$\begin{aligned} & \int_Y \left(\int_X f(x,y) u(x)\, d\mu(x) \right)^p v(y)\, d\nu(y) \\ \leq & \int_Z \left[\int_X \left(\int_Y \alpha(z,y) f^p(x,y) v(y)\, d\nu(y) \right)^{1/p} u(x)\, d\mu(x) \right]^p dz \\ \leq & \left[\int_X \left(\int_Y f^p(x,y) v(y)\, d\nu(y) \right)^{1/p} u(x)\, d\mu(x) \right]^p. \end{aligned}$$

Proof. By using the condition on the function $\alpha(z,y)$ and the Fubini theorem we get

$$\begin{aligned} & \int_Y \left(\int_X f(x,y) u(x)\, d\mu(x) \right)^p v(y)\, d\nu(y) \\ = & \int_Y \left[\int_Z \alpha(z,y) \left(\int_X f(x,y) u(x)\, d\mu(x) \right)^p dz \right] v(y)\, d\nu(y) \\ = & \int_Z \left[\int_Y \alpha(z,y) \left(\int_X f(x,y) u(x)\, d\mu(x) \right)^p v(y)\, d\nu(y) \right] dz \end{aligned}$$

Moreover, by using the continuous Minkowski inequality on the term in the square brackets and, then, on the integrals over Z and X we obtain that

$$\begin{aligned}
&\int_Z \left[\int_Y \alpha(z,y)\left(\int_X f(x,y)u(x)\,d\mu(x)\right)^p v(y)\,d\nu(y)\right] dz \\
\leq\ & \int_Z \left[\int_X \left(\int_Y \alpha(z,y) f^p(x,y)v(y)\,d\nu(y)\right)^{1/p} u(x)\,d\mu(x)\right]^p dz \\
\leq\ & \left[\int_X \left[\int_Z \left(\int_Y \alpha(z,y) f^p(x,y)v(y)\,d\nu(y)\right) dz\right]^{1/p} u(x)\,d\mu(x)\right]^p \\
=\ & \left[\int_X \left(\int_Y \left(\int_Z \alpha(z,y) f^p(x,y) dz\right) v(y)\,d\nu(y)\right)^{1/p} u(x)\,d\mu(x)\right]^p \\
=\ & \left[\int_X \left(\int_Y f^p(x,y)v(y)\,d\nu(y)\right)^{1/p} u(x)\,d\mu(x)\right]^p,
\end{aligned}$$

where we also used the Fubini theorem and the condition on the function $\alpha(z,y)$. □

Example. By using the same substitutions as those described in Remark 1 for Minkowski inequality we get a refinement of usual Minkowski inequality with two functions involved. In particular, we have that

$$\begin{aligned}
\|f+g\|_p \ \leq\ & \int_Z \left[\left(\int_Y \alpha(z,y)|f(y)|^p\,dy\right)^{\frac{1}{p}} + \left(\int_Y \alpha(z,y)|g(y)|^p\,dy\right)^{\frac{1}{p}}\right] dz \\
=\ & \int_Z \left(\|\alpha^{\frac{1}{p}}(z,.)f(.)\|_p + \|\alpha^{\frac{1}{p}}(z,.)g(.)\|_p\right) dz \\
\leq\ & \|f\|_p + \|g\|_p,
\end{aligned}$$

which corresponds to the part b) of Corollary 2.1.

Moreover, as in part c) of Corollary 2.1 for α, $0 \leq \alpha(y) \leq 1$ on Y we get that

$$\begin{aligned}
\|f+g\|_p \ \leq\ & \|\alpha^{\frac{1}{p}} f\|_p + \|\alpha^{\frac{1}{p}} g\|_p + \|(1-\alpha)^{\frac{1}{p}} f\|_p + \|(1-\alpha)^{\frac{1}{p}} g\|_p \\
\leq\ & \|f\|_p + \|g\|_p.
\end{aligned}$$

These inequalities seem to be also new for this special case.

It is clear that in the same way we can derive the corresponding refinements of Minkowski inequality with n functions involved ($n = 2, 3, \ldots$).

Let us use the following abbreviations:

$$\begin{aligned}
L_M = L_M(v) \ =\ & \int_Y \left(\int_X f(x,y)u(x)\,d\mu(x)\right)^p v(y)\,d\nu(y), \\
M_M = M_M(v) \ =\ & \int_Z \left[\int_X \left(\int_Y \alpha(z,y) f^p(x,y)v(y)\,d\nu(y)\right)^{1/p} u(x)\,d\mu(x)\right]^p dz, \\
R_M = R_M(v) \ =\ & \left[\int_X \left(\int_Y f^p(x,y)v(y)\,d\nu(y)\right)^{1/p} u(x)\,d\mu(x)\right]^p.
\end{aligned}$$

Using the above abbreviations the refinement of the Minkowski inequality in Theorem 3.1 has the following form

$$L_M \le M_M \le R_M.$$

In our next theorem some properties of the functionals R_M, M_M, $M_0 := R_M - L_M$ and $M_1 := M_M - L_M$ are given.

Theorem 3.2. *Let the assumptions in Theorem 3.1 hold and let $v(y)$ and $w(y)$ be weight functions on Y. Then L_M is linear and the functionals M_0, M_1, R_M and M_M satisfy*

$$\begin{aligned} M_i(v+w) &\ge M_i(v) + M_i(w), \quad i = 0, 1, \\ M_M(v+w) &\ge M_M(v) + M_M(w), \\ R_M(v+w) &\ge R_M(v) + R_M(w). \end{aligned}$$

Proof. Let us first recall the following well known reversed form of integral Minkowski inequality for two functions and exponent $p \ge 1$ (see [10, p.114]):

$$\begin{aligned} &\left[\int_X \left(a^p(x) + b^p(x)\right)^{\frac{1}{p}} u(x)\, d\mu(x)\right]^p \\ &\ge \left[\int_X a(x)u(x)\, d\mu(x)\right]^p + \left[\int_X b(x)u(x)\, d\mu(x)\right]^p, \end{aligned} \tag{3.1}$$

where $a(x) \ge 0, b(x) \ge 0$ and $d\mu(x)$ is non-negative.

By using (3.1) with

$$a(x) = \left(\int_Y f^p(x,y)v(y)\, d\nu(y)\right)^{\frac{1}{p}} \text{ and } \quad b(x) = \left(\int_Y f^p(x,y)w(y)\, d\nu(y)\right)^{\frac{1}{p}}$$

we get that

$$\begin{aligned} &\left[\int_X \left(\int_Y f^p(x,y)v(y)\, d\nu(y) + \int_Y f^p(x,y)w(y)\, d\nu(y)\right)^{\frac{1}{p}} u(x)\, d\mu(x)\right]^p \\ &\ge \left[\int_X \left(\int_Y f^p(x,y)v(y)\, d\nu(y)\right)^{\frac{1}{p}} u(x)\, d\mu(x)\right]^p \\ &+ \left[\int_X \left(\int_Y f^p(x,y)w(y)\, d\nu(y)\right)^{\frac{1}{p}} u(x)\, d\mu(x)\right]^p, \end{aligned}$$

i.e.

$$R_M(v+w) \ge R_M(v) + R_M(w).$$

The proof of the other four inequalities can be done in a similar manner so we omit the details. □

Corollary 3.1. *The functionals M_0, M_1, R_M and M_M are non-decreasing.*

Proof. This is a consequence of Theorem 3.1 and Theorem 3.2. For instance for the proof about M_M we apply inequality (3.1) with

$$a(x) = \left(\int_Y f^p(x,y)\alpha(z,y)v(y)\, d\nu(y)\right)^{\frac{1}{p}} \text{ and } b(x) = \left(\int_Y f^p(x,y)\alpha(z,y)w(y)\, d\nu(y)\right)^{\frac{1}{p}}.$$

□

4 Refinements of the continuous forms of the Popoviciu and the Bellman inequalities

When discrete Hölder type inequalities for non-positive weights were researched a reversed type of the inequality was discovered. It is known as the Popoviciu inequality and has the following form:

$$w_0c_1c_2 - \sum_{i=1}^n w_if_ig_i \ \geq \ \left(w_0c_1^p - \sum_{i=1}^n w_if_i^p\right)^{\frac{1}{p}} \left(w_0c_2^q - \sum_{i=1}^n w_ig_i^q\right)^{\frac{1}{q}},$$

where $w_0, c_1, c_2 > 0$, $p, q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $w_i, f_i, g_i \geq 0, i = 1, 2, \ldots, n$, are real numbers such that $w_0c_1^p - \sum_{i=1}^n w_if_i^p > 0$, $w_0c_2^q - \sum_{i=1}^n w_ig_i^q > 0$.

More about this type of inequalities can be found in [10, p.125]. Very recently, we proved a continuous version of the Popoviciu inequality (see [9]). The continuous form of the Popoviciu inequality was given as the following result.

Theorem 4.1. *Let $u(x)$ and $v(y)$ be weight functions on the measure spaces (X, μ) and (Y, ν), respectively, $\int_X u(x)d\mu(x) = 1$. Let $f(x, y)$ be a positive measurable function on $X \times Y$, $v_0 \geq 0$, and assume that $f_0(x)$ is a function on X such that $v_0f_0(x) > \int_Y f(x, y)v(y)\, d\nu(y)$, for all $x \in X$. Then the following refinement of the continuous form of Popoviciu's inequality holds:*

$$\begin{aligned} &\exp\left(\int_X \log\left(v_0f_0(x)\right)u(x)\, d\mu(x)\right) \\ &- \int_Y \exp\left(\int_X \log f(x, y)u(x)\, d\mu(x)\right) v(y)\, d\nu(y) \\ &\geq \exp\left[\int_X \log\left(v_0f_0(x) - \int_Y f(x, y)v(y)\, d\nu(y)\right) u(x)\, d\mu(x)\right]. \end{aligned} \tag{4.1}$$

Example. Let $u(x) = v(y) = 1$, $v_0 = 1$, $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ with $\int_{X_1} d\mu(x) = \frac{1}{p}$, $\int_{X_2} d\mu(x) = \frac{1}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$; $f_0(x) = c_1^p$, $f(x, y) = f_i^p(y)$ for each $x \in X_1$ and $f_0(x) = c_2^q$, $f(x, y) = g_i^q(y)$ for each $x \in X_2$. Then we rediscover the unweighted Popoviciu inequality for integrals involving two functions:

$$c_1c_2 - \|fg\|_1 \geq (c_1^p - \|f^p\|_p)^{\frac{1}{p}} (c_2^q - \|g^q\|_q)^{\frac{1}{q}}. \tag{4.2}$$

Our first main result in this Section is the following generalization of Theorem 4.1.

Theorem 4.2. *Let the assumptions of Theorem* 4.1 *hold. Moreover, let $\alpha(z, y)$ be a non-negative integrable function on $Z \times Y$ such that $\int_Z \alpha(z, y)\, dz = 1$ for $y \in Y$, where (Z, dz) is a $\sigma-$finite measure space. Then the following refinement of the continuous form of the Popoviciu inequality* (4.1) *holds:*

$$\exp\left(\int_X \log\left(v_0f_0(x)\right)u(x)\, dx\right) - \int_Y \left(\exp\left(\int_X \log(f(x, y))u(x)\, dx\right)\right) v(y)\, d\nu(y)$$

$$\begin{aligned}
&\geq \exp\left(\int_X \log\Big(v_0 f_0(x)\Big) u(x)\,dx\right) \\
&\qquad - \int_Z \left[\exp \int_X \log\left(\int_Y f(x,y)\alpha(z,y)v(y)\,d\nu(y)\right) u(x)\,dx\right] dz \\
&\geq \exp\left[\int_X \log\left(v_0 f_0(x) - \int_Y f(x,y)v(y)\,d\nu(y)\right) u(x)\,dx\right] \geq 0. \qquad (4.3)
\end{aligned}$$

Proof. Let us denote

$$\begin{aligned}
C_H(v_0) &:= \exp\left(\int_X \log\Big(v_0 f_0(x)\Big) u(x)\,d\mu(x)\right), \\
K_H(v_0,v) &:= \exp\left[\int_X \log\left(v_0 f_0(x) - \int_Y f(x,y)v(y)\,d\nu(y)\right) u(x)\,d\mu(x)\right].
\end{aligned}$$

Using these abbreviations continuous Popoviciu inequality (4.1) has the form:

$$C_H(v_0) - L_H(v) \geq K_H(v_0,v).$$

In paper [9] the authors investigated the functionals

$$P_1(v_0,v) := C_H(v_0) - L_H(v) - K_H(v_0,v),$$

$$P_2(v_0,v) := C_H(v_0)) - R_H(v) - K_H(v_0,v),$$

where L_H and R_H are defined above in Theorem 2.3, and proved that, under the assumptions of Theorem 4.1, the above defined functionals have the following properties:

$$P_1(v_0,v) \geq P_2(v_0,v) \geq 0,$$

$$P_1(v_0+w_0, v+w) \leq P_1(v_0,v) + P_1(w_0,w), \qquad (4.4)$$

$$P_2(v_0+w_0, v+w) \leq P_2(v_0,v) + P_2(w_0,w). \qquad (4.5)$$

Let us define the functional P_3 by:

$$P_3 = P_3(v_0,v) := C_H(v_0) - K_H(v_0,v) - M_H(v).$$

Since the refinement of Hölder inequality (1.2) holds we have that

$$-L_H(v) \geq -M_H(v) \geq -R_H(v)$$

and, hence,

$$\begin{aligned}
C_H(v_0) - K_H(v_0,v) - L_H(v) &\geq C_H(v_0) - K_H(v_0,v) - M_H(v) \\
&\geq C_H(v_0) - K_H(v_0,v) - R_H(v),
\end{aligned}$$

so we can conclude that:

$$P_1(v_0,v) \geq P_3(v_0,v) \geq P_2(v_0,v) \geq 0.$$

Particularly, from the above inequalities we get that

$$C_H(v_0) - L_H(v) \geq C_H(v_0) - M_H(v) \geq K_H(v_0,v),$$

so (4.3) is proved. □

In the next remark we will point out the fact that in this case our result is new even for the case with only two functions involved.

Remark 4. The continuous version in the case of two functions f and g can be written shortly like

$$c_1c_2 - \|fg\|_1 \geq c_1c_2 - \int_Z \|\alpha^{\frac{1}{p}}(z,.)f(.)\|_p \|\alpha^{\frac{1}{q}}(z,.)g(.)\|_q \, dz$$

$$\geq (c_1^p - \|f\|_p^p)^{\frac{1}{p}} \cdot (c_2^q - \|g\|_q^q)^{\frac{1}{q}}$$

and we can see that this is a refinement of inequality (4.2).

Moreover, as in the part c) of Corollary 2.1 for α, $0 \leq \alpha(y) \leq 1$ on Y we get that

$$\begin{aligned} & c_1c_2 - \|fg\| \\ \geq \; & c_1c_2 - \left(\|\alpha^{\frac{1}{p}} f\|_p \cdot \|\alpha^{\frac{1}{q}} g\|_q + (\|(1-\alpha)^{\frac{1}{p}} f\|_p \cdot \|(1-\alpha)^{\frac{1}{q}} g\|_q\right) \\ \geq \; & \left(c_1^p - \|f\|_p^p\right)^{\frac{1}{p}} \cdot \left(c_2^q - \|g\|_q^q\right)^{\frac{1}{q}}, \end{aligned}$$

where $0 \leq \alpha(y) \leq 1$, $y \in Y$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $p, q > 1$.

Theorem 4.3. *If v_0, w_0 are positive real numbers and v, w weights such that the integrals in C_H, K_H and P_3 exist, then C_H is a linear function, and*

$$K_H(v_0 + w_0, v + w) \geq K_H(v_0, v) + K_H(w_0, w),$$

$$P_3(v_0 + w_0, v + w) \leq P_3(v_0, v) + P_3(w_0, w).$$

Proof. The statement about C_H is obvious so let us prove the inequality for the functional K_H. By using (2.6) with

$$a(x) := v_0 f_0^p(x) - \int_Y f^p(x,y)v(y)\, d\nu(y)$$

and

$$b(x) := w_0 f_0^p(x) - \int_Y f^p(x,y)w(y)\, d\nu(y)$$

the inequality for K_H follows from it. The inequality for P_3 follows easily from the already proved properties of C_H, M_H and K_H so we omit the details. $\square$

In the rest of this section we discuss a refinement of the continuous Bellman inequality.

The original (discrete) form of the Bellman inequality for two sequences reads: if $p \geq 1, c_1, c_2, w_0, w_i, a_i, b_i, i = 1, 2, \ldots, n$ are positive numbers and $w_0c_1^p - \sum_{i=1}^n w_i a_i^p > 0$ and $w_0 c_2^p - \sum_{i=1}^n w_i b_i^p > 0$, then

$$\left(w_0c_1^p - \sum_{i=1}^n w_i a_i^p\right)^{\frac{1}{p}} + \left(w_0c_2^p - \sum_{i=1}^n w_i b_i^p\right)^{\frac{1}{p}} \leq \left(w_0(c_1+c_2)^p - \sum_{i=1}^n w_i(a_i+b_i)^p\right)^{\frac{1}{p}}.$$

There exist also integral and functional forms of these inequalities (see [10, p.126]). A continuous version of the Bellman inequality was proved in [9] and it was given as the following theorem:

Theorem 4.4. *Let $f_0(x), v_0, u(x), v(y), X, Y, \mu, \nu$ be defined as in Theorem 4.1. Then, for $p \geq 1$,*

$$\left(\int_X \left[v_0 f_0^p(x) - \int_Y f^p(x,y)v(y)\,d\nu(y)\right]^{\frac{1}{p}} u(x)\,d\mu(x)\right)^p$$

$$\leq v_0 \left[\int_X f_0(x)u(x)\,d\mu(x)\right]^p - \int_Y \left[\int_X f(x,y)u(x)\,d\mu(x)\right]^p v(y)\,d\nu(y) \tag{4.6}$$

whenever $f_0(x) > \int_Y f(x,y)v(y)\,d\nu(y)$, for all $x \in X$.

In the same paper [10] the following functionals were considered:

$$\begin{aligned} B_1(v_0, v) &:= C_M(v_0) - L_M(v) - K_M(v_0, v) \\ B_2(v_0, v) &:= C_M(v_0) - R_M(v) - K_M(v_0, v) \end{aligned}$$

where we used the notations:

$$\begin{aligned} C_M(v_0) &:= v_0 \left[\int_X f_0(x)u(x)\,d\mu(x)\right]^p \\ K_M(v_0, v) &:= \left(\int_X \left[v_0 f_0^p(x) - \int_Y f^p(x,y)v(y)\,d\nu(y)\right]^{\frac{1}{p}} u(x)\,d\mu(x)\right)^p \end{aligned}$$

and the abbreviations $L_M(v)$ and $R_M(v)$ are defined in Section 3. The following properties of those functionals were proved in [9]:

$$B_1(v_0, v) \geq B_2(v_0, v) \geq 0,$$

$$B_1(v_0 + w_0, v + w) \leq B_1(v_0, v) + B_1(w_0, w),$$

$$B_2(v_0 + w_0, v + w) \leq B_2(v_0, v) + B_2(w_0, w).$$

We are now ready to present the following refinement of inequality (4.6):

Theorem 4.5. *Let the assumptions of Theorem 4.4 hold. Let $\alpha(z, y)$ be a non-negative integrable function on $Z \times Y$ such that $\int_Z \alpha(z, y)\,dz = 1$ for $y \in Y$ and where (Z, dz) is a $\sigma-$finite measure space. Then the following refinement of the continuous Bellman inequality holds for $p \geq 1$:*

$$\begin{aligned} & v_0 \left[\int_X f_0(x)u(x)\,d\mu(x)\right]^p - \int_Y \left(\int_X f(x,y)u(x)\,d\mu(x)\right)^p v(y)\,d\nu(y) \\ \geq\ & v_0 \left[\int_X f_0(x)u(x)u(x)\,d\mu(x)\right]^p \\ -\ & \int_Z \left[\int_X \left(\int_Y \alpha(z,y) f^p(x,y)v(y)\,d\nu(y)\right)^{1/p} u(x)\,d\mu(x)\right]^p dz \\ \geq\ & \left(\int_X \left[v_0 f_0^p(x) - \int_Y f^p(x,y)v(y)\,d\nu(y)\right]^{\frac{1}{p}} u(x)\,d\mu(x)\right)^p. \end{aligned} \tag{4.7}$$

Proof. Let us define the functional B_3 by

$$B_3 = B_3(v_0, v) := C_M(v_0) - M_M(v) - K_M(v_0, v).$$

Adding $C_M(v_0) - K_M(v_0, v)$ to the each term in the refinement of Minkowski inequality

$$-L_M(v) \geq -M_M(v) \geq -R_M(v)$$

we find that

$$B_1(v_0, v) \geq B_3(v_0, v) \geq B_2(v_0, v) \geq 0.$$

Particularly, from the above inequalities we obtain that

$$C_M(v_0) - L_M(v) \geq C_M(v_0) - M_M(v) \geq K_M(v_0, v),$$

i.e. that (4.7) holds. □

In our next remark we point out the fact that our results is new also in the classical case with only two functions involved.

Remark 5. In the case of two functions f and g we get the following "continuous" refinement of the unweighted Bellman inequality:

$$(c_1 + c_2)^p - \|f + g\|_p^p \geq (c_1 + c_2)^p - \int_Z \left[\|\alpha^{\frac{1}{p}}(z, .)f(.)\|_p + \|\alpha^{\frac{1}{p}}(z, .)g(.)\|_p\right]^p dz$$
$$\geq \left[(c_1^p - \|f\|_p^p)^{\frac{1}{p}} + (c_2^p - \|g\|_p^p)^{\frac{1}{p}}\right]^p.$$

Besides the continuous form, it is instructive to see how that inequality looks like in an integral form

In particular when instead of integral over Z we have only two summands this inequality reads:

$$(c_1 + c_2)^p - \|f + g\|_p^p$$
$$\geq (c_1 + c_2)^p - \left(\|\alpha^{\frac{1}{p}} f\|_p + \|\alpha^{\frac{1}{p}} g\|_p\right)^p + \left(\|(1 - \alpha)^{\frac{1}{p}} f\|_p + \|(1 - \alpha)^{\frac{1}{p}} g\|_p\right)^p$$
$$\geq \left[(c_1^p - \|f\|_p^p)^{\frac{1}{p}} + (c_2^p - \|g\|_p^p)^{\frac{1}{p}}\right]^p,$$

where $0 \leq \alpha(y) \leq 1$.

Theorem 4.6. *If v_0, w_0 are positive numbers and v, w weights such that the integrals in C_M, K_M and B_3 exist, then C_M is a linear function, and*

$$K_M(v_0 + w_0, v + w) \geq K_M(v_0, v) + K_M(w_0, w)$$
$$B_3(v_0 + w_0, v + w) \leq B_3(v_0, v) + B_3(w_0, w).$$

Proof. The proof of the linearity of C_M is obvious, so let us prove the stated inequality for the functional K_M. By inserting

$$a(x) := \left[v_0 f_0^p(x) - \int_Y f^p(x, y)v(y)\, d\nu(y)\right]^{\frac{1}{p}}$$
$$b(x) := \left[w_0 f_0^p(x) - \int_Y f^p(x, y)w(y)\, d\nu(y)\right]^{\frac{1}{p}}$$

into (3.1) we get that

$$a^p(x) + b^p(x) = (v_0 + w_0)f_0^p(x) - \int_Y f^p(x, y)(v(y) + w(y))\, d\nu(y)$$

and the inequality for K_M follows from (3.1). The stated inequality concerning B_3 follows from the above-mentioned properties of C_M and K_M and the superadditivity of M_M. □

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