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VLADIMIR MIKHAILOVICH FILIPPOV

(to the 70th birthday)

Vladimir Mikhailovich Filippov was born on 15 April 1951 in the city of Uryupinsk, Stalingrad Region of the USSR. In 1973 he graduated with honors from the Faculty of Physics and Mathematics and Natural Sciences of the Patrice Lumumba University of Peoples' Friendship in the specialty "Mathematics". In 1973-1975 he is a postgraduate student of the University; in 1976-1979 - Chairman of the Young Scientists' Council; in 1980-1987 - Head of the Research Department and the Scientific Department; in 1983-1984 - scientific work at the Free University of Brussels (Belgium); in 1985-2000 - Head of the Mathematical Analysis Department; from 2000 to the present - Head of the Comparative Educational Policy Department; in 1989–1993 - Dean of the Faculty of Physics, Mathematics and Natural Sciences; in 1993–1998 - Rector of the Peoples' Friendship University of Russia; in 1998-2004 - Minister of General and Professional Education, Minister of Education of the Russian Federation; in 2004-2005 - Assistant to the Chairman of the Government of the Russian Federation (in the field of education and culture); from 2005 to May 2020- Rector of the Peoples' Friendship University of Russia, since May 2020 - President of the Peoples' Friendship University of Russia, since 2013 - Chairman of the Higher Attestation Commission of the Ministry of Science and Higher Education of the Russian Federation.

In 1980, he defended his PhD thesis in the V.A. Steklov Mathematical Institute of Academy of Sciences of the USSR on specialty 01.01.01 - mathematical analysis (supervisor - a corresponding member of the Academy of Sciences of the USSR, Professor L.D. Kudryavtsev), and in 1986 in the same Institute he defended his doctoral thesis "Quasi-classical solutions of inverse problems of the calculus of variations in non-Eulerian classes of functionals and function spaces". In 1987, he was awarded the academic title of a professor.

V.M. Filippov is an academician of the Russian Academy of Education; a foreign member of the Ukrainian Academy of Pedagogical Sciences; President of the UNESCO International Organizing Committee for the World Conference on Higher Education (2007-2009); Vice-President of the Eurasian Association of Universities; a member of the Presidium of the Rectors' Council of Moscow and Moscow Region Universities, of the Governing Board of the Institute of Information Technologies in Education (UNESCO), of the Supervisory Board of the European Higher Education Center of UNESCO (Bucharest, Romania),

Research interests: variational methods; non-potential operators; inverse problems of the calculus of variations; function spaces.

In his Ph.D thesis, V.M. Filippov solved a long standing problem of constructing an integral extremal variational principle for the heat equation. In his further research he developed a general theory of constructing extremal variational principles for broad classes of differential equations with non-potential (in classical understanding) operators. He showed that all previous attempts to construct variational principles for non-potential operators "failed" because mathematicians and mechanics from the time of L. Euler and J. Lagrange were limited in their research by functionals of the type Euler - Lagrange. Extending the classes of functionals, V.M. Filippov introduced a new scale of function spaces that generalize the Sobolev spaces, and thus significantly expanded the scope of the variational methods. In 1984, famous physicist, a Nobel Prize winner I.R. Prigogine presented the report of V.M. Filippov to the Royal Academy of Sciences of Belgium. Results of V.M. Filippov's variational principles for non-potential operators are quite fully represented in some of his and his colleagues' monographs.

Honors: Honorary Legion (France), "Commander" (Belgium), Crown of the King (Belgium); in Russia - orders "Friendship", "Honor", "For Service to the Fatherland" III and IV degrees; Prize of the President of the Russian Federation in the field of education; Prize of the Governement of the Russian Federation in the field of education; Gratitude of the President of the Russian Federation; "For Merits in the Social and Labor Sphere of the Russian Federation", "For Merits in the Development of the Olympic Movement in Russia", "For Strengthening the Combat Commonwealth; and a number of other medals, prizes and awards.

He is an author of more than 270 scientific and scientific-methodical works, including 32 monographs, 2 of which were translated and published in the United States by the American Mathematical Society.

V.M. Filippov meets his 70th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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RANKS FOR FAMILIES OF ALL THEORIES OF GIVEN LANGUAGES

N.D. Markhabatov, S.V. Sudoplatov

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Abstract. For families of all theories of arbitrary given languages we describe ranks and degrees. In particular, we characterize (non-)totally transcendental families. We apply these characterizations for the families of all theories of given languages, with models of given finite or infinite cardinality.

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1 Introduction

The rank [9] for families of theories, similar to the Morley rank, can be considered as a measure of complexity for these families. Thus increasing the rank by extensions of families we produce more rich families obtaining families with the infinite rank that can be considered "rich enough". This measure of complexity is related to definability and interpretability [1], [3], [5], [6].

In the present paper, for families of all theories of an arbitrary given language, we describe ranks and degrees, partially answering the question in [9]. In particular, we characterize (non-)totally transcendental families. Thus, we describe rich families with respect to the rank. Besides, we apply these characterizations for the families of all theories of given languages, with models of a given finite or infinite cardinality.

Throughout we consider families $\mathcal{T}$ of complete first-order theories of a language $\Sigma = \Sigma(\mathcal{T})$. For a sentence φ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$.

Definition 1. [9]. Let $\mathcal{T}$ be a family of theories and T be a theory, $T \notin \mathcal{T}$. The theory T is called $\mathcal{T}$*-approximated*, or *approximated by* $\mathcal{T}$, or $\mathcal{T}$*-approximable*, or a *pseudo-*$\mathcal{T}$*-theory*, if for any sentence $\varphi \in T$ there is $T' \in \mathcal{T}$ such that $\varphi \in T'$.

If T is $\mathcal{T}$-approximated then $\mathcal{T}$ is called an *approximating family* for T, theories $T' \in \mathcal{T}$ are *approximations* for T, and T is an *accumulation point* for $\mathcal{T}$.

An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

It was shown in [11] that any e-minimal family $\mathcal{T}$ has unique accumulation point T with respect to neighbourhoods $\mathcal{T}_\varphi$, and $\mathcal{T} \cup \{T\}$ is also called *e-minimal*.

Following [9] we define the *rank* $\mathrm{RS}(\cdot)$ for the families of theories, similar to Morley rank [4], [8], and a hierarchy with respect to these ranks in the following way.

For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, for finite nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) = 0$, and $\mathrm{RS}(\mathcal{T}) \geq 1$ for infinite families $\mathcal{T}$.

For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

If α is a limit ordinal, then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.

We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.

If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

Proposition 1.1. [9]. *If an infinite family $\mathcal{T}$ does not have e-minimal subfamilies $\mathcal{T}_\varphi$ then $\mathcal{T}$ is not totally transcendental.*

If $\mathcal{T}$ is totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Recall the definition of the Cantor–Bendixson rank. It is defined on the elements of a topological space X by induction: $\mathrm{CB}_X(p) \geq 0$ for all $p \in X$; $\mathrm{CB}_X(p) \geq \alpha$ if and only if for any $\beta < \alpha$, p is an accumulation point of the points of CB_X-rank at least β. $\mathrm{CB}_X(p) = \alpha$ if and only if both $\mathrm{CB}_X(p) \geq \alpha$ and $\mathrm{CB}_X(p) \not\geq \alpha+1$ hold; if such an ordinal α does not exist then $\mathrm{CB}_X(p) = \infty$. Isolated points of X are precisely those having rank 0, points of rank 1 are those which are isolated in the subspace of all non-isolated points, and so on. For a non-empty $C \subseteq X$ we define $\mathrm{CB}_X(C) = \sup\{\mathrm{CB}_X(p) \mid p \in C\}$; in this way $\mathrm{CB}_X(X)$ is defined and $\mathrm{CB}_X(\{p\}) = \mathrm{CB}_X(p)$ holds. If X is compact and C is closed in X then the sup is achieved: $\mathrm{CB}_X(C)$ is the maximum value of $\mathrm{CB}_X(p)$ for $p \in C$; there are finitely many points of maximal rank in C and the number of such points is the CB_X-*degree* of C, denoted by $n_X(C)$.

If X is countable and compact then $\mathrm{CB}_X(X)$ is a countable ordinal and every closed subset has ordinal-valued rank and finite CB_X-degree $n_X(X) \in \omega \setminus \{0\}$.

For any ordinal α the set $\{p \in X \mid \mathrm{CB}_X(p) \geq \alpha\}$ is called the α-th CB-*derivative* X_α of X.

Elements $p \in X$ with $\mathrm{CB}_X(p) = \infty$ form the *perfect kernel* X_∞ of X.

Clearly, $X_\alpha \supseteq X_{\alpha+1}$, for any $\alpha \in \mathrm{Ord}$, where Ord is the class of all ordinals, and $X_\infty = \bigcap\limits_{\alpha\in\mathrm{Ord}} X_\alpha$.

It is noticed in [9] that any e-totally transcendental family $\mathcal{T}$ defines a superatomic Boolean algebra $\mathcal{B}(\mathcal{T})$ with $\mathrm{RS}(\mathcal{T}) = \mathrm{CB}_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$, $\mathrm{ds}(\mathcal{T}) = n_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$, i.e., the pair $(\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T}))$ consists of Cantor–Bendixson invariants for $\mathcal{B}(\mathcal{T})$ [2]. The algebra $\mathcal{B}(\mathcal{T})$ is the sentence algebra, i.e., the Lindenbaum–Tarski algebra, and the invariants $\mathrm{CB}_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$, $n_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$ can be obtained on a base of classification for sentence algebras [7].

By the definition for any e-totally transcendental family $\mathcal{T}$ each theory $T \in \mathcal{T}$ obtains the CB-rank $\mathrm{CB}_{\mathcal{T}}(T)$ starting with $\mathcal{T}$-isolated points T_0, of $\mathrm{CB}_{\mathcal{T}}(T_0) = 0$. We will denote the values $\mathrm{CB}_{\mathcal{T}}(T)$ by $\mathrm{RS}_{\mathcal{T}}(T)$ as the rank for the point T in the topological space on the E-closure $\mathrm{Cl}_E(\mathcal{T})$ [9] of $\mathcal{T}$ which is defined with respect to $\Sigma(\mathcal{T})$-sentences.

Definition 2. [9]. Let α be an ordinal. A family $\mathcal{T}$ of rank α is called α-*minimal* if for any sentence $\varphi \in \Sigma(T)$, $\mathrm{RS}(\mathcal{T}_\varphi) < \alpha$ or $\mathrm{RS}(\mathcal{T}_{\neg\varphi}) < \alpha$.

Proposition 1.2. [9]. (1) *A family $\mathcal{T}$ is* 0*-minimal if and only if $\mathcal{T}$ is a singleton.*

(2) *A family $\mathcal{T}$ is* 1*-minimal if and only if $\mathcal{T}$ is e-minimal.*

(3) *For any ordinal α a family $\mathcal{T}$ is α-minimal if and only if* $\mathrm{RS}(\mathcal{T}) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}) = 1$.

Proposition 1.3. [9]. *For any family $\mathcal{T}$,* $\mathrm{RS}(\mathcal{T}) = \alpha$, *with* $\mathrm{ds}(\mathcal{T}) = n$, *if and only if $\mathcal{T}$ is represented as a disjoint union of subfamilies $\mathcal{T}_{\varphi_1}, \ldots, \mathcal{T}_{\varphi_n}$, for some pairwise inconsistent sentences $\varphi_1, \ldots, \varphi_n$, such that each $\mathcal{T}_{\varphi_i}$ is α-minimal.*

2 Ranks for families of theories depending of given languages

Let Σ be a language. If Σ is relational, i.e., it does not contain functional symbols, then we denote by $\mathcal{T}_\Sigma$ the family of all theories of the language Σ. If Σ contains functional symbols f then $\mathcal{T}_\Sigma$ is the family of all theories of the language Σ', which is obtained by replacements of all n-ary symbols f with $(n+1)$-ary predicate symbols R_f interpreted by $R_f = \{(\bar{a}, b) \mid f(\bar{a}) = b\}$.

Theorem 2.1. *For any language Σ the family $\mathcal{T}_\Sigma$ is e-minimal if and only if $\Sigma = \emptyset$ or Σ consists of one constant symbol.*

Proof. If $\Sigma = \emptyset$ or Σ consists of one constant symbol then $\mathcal{T}_\Sigma$ is countable and consists of theories T_n with n-element models, $n \in \omega \setminus \{0\}$, and of the theory T_∞ with infinite models. The theories T_n are finitely axiomatizable by the sentences witnessing the cardinalities of models and T_∞ is a unique accumulation point for $\mathcal{T}_\Sigma$. Thus, $\mathcal{T}_\Sigma$ is e-minimal.

Now we assume that $\Sigma \neq \emptyset$ and it is not exhausted by one constant symbol. Below we consider all possible cases.

If Σ has a relational symbol P, then $\mathcal{T}_\Sigma$ is divided into infinite definable parts: with empty P and with nonempty P. Therefore, there is a sentence φ with infinite $(\mathcal{T}_\Sigma)_\varphi$ and infinite $(\mathcal{T}_\Sigma)_{\neg\varphi}$. Hence, $\mathcal{T}_\Sigma$ is not e-minimal.

If Σ has at least two constant symbols c_1 and c_2, then the family $\mathcal{T}_\Sigma$ is divided into two infinite parts: with $c_1 = c_2$ and with $c_1 \neq c_2$. It implies that again $\mathcal{T}_\Sigma$ is not e-minimal.

Finally, if Σ contains an n-ary functional symbol f, $n \geq 1$, then $\mathcal{T}_\Sigma$ is divided into two infinite parts: with identical f for each element a: $f(a, \ldots, a) = a$, and with $f(a, \ldots, a) \neq a$ for some a. It means that again $\mathcal{T}_\Sigma$ is not e-minimal. □

By Propositions 1.2 and 1.3 each theory T in e-minimal $\mathcal{T}_\Sigma$ has $\mathrm{RS}_{\mathcal{T}_\Sigma}(T) \leq 1$, with a unique theory having the RS-rank 1. Here, following Theorem 2.1, $\mathrm{RS}_{\mathcal{T}_\Sigma}(T) = 1$ if and only if T has infinite models.

Proposition 2.1. *If Σ is a language of 0-ary predicates, then either $\mathrm{RS}(\mathcal{T}_\Sigma) = 1$ with $\mathrm{ds}(\mathcal{T}_\Sigma) = 2^n$, if Σ consists of $n \in \omega$ symbols, or $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$, if Σ has infinitely many symbols.*

Proof. If Σ consists of $n \in \omega$ 0-ary predicates $P_1, \ldots, P_n$, then $\mathcal{T}_\Sigma$ has 2^n accumulation points T_i such that each T_i has infinite models and $(P_1, \ldots, P_n)$ has values $(\delta_1, \ldots, \delta_n) \in \{0, 1\}^n$.

If Σ consists of infinitely many 0-ary predicates P_i, then there is an infinite 2-tree [8] formed by independent values for P_i in $\{0, 1\}$, witnessing that there are no e-minimal subfamilies $\mathcal{T}_\varphi$ and producing $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$ by Proposition 1.1. □

By Proposition 2.1 a totally transcendental family $\mathcal{T}_\Sigma$, for a language Σ of n 0-ary predicates, has 2^n theories of RS-rank 1, each of which has infinite models.

Proposition 2.2. *If Σ is a language of 0-ary and unary predicates, with at least one unary symbol P, then either $\mathrm{RS}(\mathcal{T}_\Sigma) = 2^k$ with $\mathrm{ds}(\mathcal{T}_\Sigma) = 2^m$, if Σ consists of $k \in \omega$ unary symbols and $m \in \omega$ 0-ary predicates, or $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$, if Σ has infinitely many symbols.*

Proof. If Σ contains $k \in \omega$ unary symbols P_i then universes can be divided into 2^k parts by P_i such that cardinalities of these parts can vary from 0 to infinity. So varying finite cardinalities of the parts we obtain infinitely many pairwise inconsistent sentences allowing to vary cardinalities of other parts. Continuing the process for remaining parts we have 2^n steps forming $\mathrm{RS}(\mathcal{T}_\Sigma) = 2^k$. Having $m \in \omega$ 0-ary predicates Q_j, sentences witnessing $\mathrm{RS}(\mathcal{T}_\Sigma) = 2^k$ are implied by 2^m pairwise inconsistent sentences describing values for Q_j. Thus, $\mathrm{ds}(\mathcal{T}_\Sigma) = 2^m$.

If Σ contains infinitely many predicate symbols, 0-ary and unary, we construct an infinite 2-tree of sentences formed by independent values of predicates. Hence, $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$ using Proposition 1.1. □

In view of Proposition 2.2 there are 2^m theories T in $\mathcal{T}_\Sigma$ having the maximal $\mathrm{RS}_{\mathcal{T}_\Sigma}(T) = 2^k$. Each such T has only infinite parts with respect to the predicates P_i. Notice also that $\mathrm{RS}_{\mathcal{T}_\Sigma}(T) = s \leq 2^k$ if and only if T has models with exactly s infinite parts.

Proposition 2.3. *If Σ is a language of constant symbols then either $\mathrm{RS}(\mathcal{T}_\Sigma) = 1$ with $\mathrm{ds}(\mathcal{T}_\Sigma) = P(n)$, where $P(n)$ is the number for partitions of n-element sets, if Σ consists of $n \in \omega$ symbols, or $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$, if Σ has infinitely many symbols.*

Proof. If Σ consists of constant symbols $c_1, \ldots, c_n$ then we can write in sentences that these constants can arbitrarily coincide or not coincide. The sentences $(c_i \approx c_j)^\delta$, $\delta \in \{0, 1\}$, define partitions of the set $C = \{c_1, \ldots, c_n\}$. The number $P(n)$ of these partitions [10, Section 5.4] defines all possibilities for $\mathrm{ds}(\mathcal{T}_\Sigma)$. Since all Σ-sentences are reduced to the descriptions φ of these partitions as well as to the descriptions ψ of cardinalities of the sets $\overline{C} = M \setminus C$, where M are models of theories in $\mathcal{T}_\Sigma$, we have $\mathrm{RS}(\mathcal{T}_\Sigma) = 1$, witnessed by ψ, and $\mathrm{ds}(\mathcal{T}_\Sigma) = P(n)$, witnessed by φ.

If Σ contains infinitely many constant symbols, we construct an infinite 2-tree of sentences formed by independent (in)equalities of constants. Hence, $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$ using Proposition 1.1. $\square$

By Proposition 2.3, for $\mathrm{RS}(\mathcal{T}_\Sigma) = 1$ there are $P(n)$ theories T in $\mathcal{T}_\Sigma$ with $\mathrm{RS}_{\mathcal{T}_\Sigma}(T) = 1$. Each such T is characterized by existence of infinite models.

Proposition 2.4. *If Σ is a language of 0-ary and unary predicates, and constant symbols, then either $\mathrm{RS}(\mathcal{T}_\Sigma)$ is finite, if Σ consists of finitely many symbols, or $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$, if Σ has infinitely many symbols.*

Proof. If Σ is finite then we can increase $\mathrm{RS}(\mathcal{T}_\Sigma)$ till 2^k using unary predicates $P_1, \ldots, P_k$ repeating arguments for Proposition 2.2. The degree $\mathrm{ds}(\mathcal{T}_\Sigma)$ is bounded by finitely many possibilities for values of 0-ary predicates and for partitions of constants combining Propositions 2.2 and 2.3.

If Σ has infinitely many symbols then it has either infinitely many 0-ary predicates, or unary predicates, or constant symbols. Anyway it is possible to construct an infinite 2-tree, as for Propositions 2.2 and 2.3, guaranteeing that $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$. $\square$

As above, RS-ranks for theories T in a totally transcendental family $\mathcal{T}_\Sigma$ in Proposition 2.4 are characterized by the number of infinite P_i-parts in models of T.

Proposition 2.5. *If Σ is a language containing an m-ary predicate symbol, for $m \geq 2$, or an n-ary functional symbol, for $n \geq 1$, then $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$.*

Proof. Using the arguments for the propositions above it suffices to show that having a binary predicate symbol Q or a unary functional symbol f it is possible to define infinitely many independent definable subsets X_n, $n \in \omega$, of universes M for models of theories in $\mathcal{T}_\Sigma$. It is possible to code these sets X_n, even by acyclic directed graphs (i.e., by directed graphs without paths connecting a vertex with itself), by the existence of paths from some elements a without preimages to elements $b \in X_n$ such that the (a, b)-path has the length n. Coding the sets X_n we can form an infinite 2-tree for elements in $Y = \bigcup\limits_{n \in \omega} X_n$ such that some sentences divide Y into continuum many parts by (non)existence of paths having the lengths n. The existence of this 2-tree implies that $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$ using Proposition 1.1. $\square$

Remark 1. The arguments for Proposition 2.5 allow to restrict families $\mathcal{T}_\Sigma$ with binary relational symbols R to the families $\mathcal{T}_{\{R\},\mathrm{ag}}$ in graph languages $\{R\}$, of theories of acyclic graphs, and such that $\mathrm{RS}(\mathcal{T}_{\{R\},\mathrm{ag}}) = \infty$.

Summarizing arguments above we obtain the following theorem.

Theorem 2.2. *For any language Σ either* $\mathrm{RS}(\mathcal{T}_\Sigma)$ *is finite, if Σ consists of finitely many 0-ary and unary predicates, and finitely many constant symbols, or* $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$*, otherwise.*

Proof. If Σ consists of finitely many 0-ary and unary predicates, and constant symbols then $\mathrm{RS}(\mathcal{T}_\Sigma)$ is finite by Proposition 2.4. Otherwise, $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$ by Propositions 2.4 and 2.5. $\square$

3 Application for families of theories depending on cardinalities of models

The technique for counting of the ranks $\mathrm{RS}(\mathcal{T}_\Sigma)$ can be applied to families $\mathcal{T}_{\Sigma,n}$ of all theories in $\mathcal{T}_\Sigma$ having n-element models, $n \in \omega$, as well as for families $\mathcal{T}_{\Sigma,\infty}$ of all theories in $\mathcal{T}_\Sigma$ having infinite models.

Clearly, for any language Σ, $\mathcal{T}_\Sigma = \bigcup\limits_{n\in\omega} \mathcal{T}_{\Sigma,n} \cup \mathcal{T}_{\Sigma,\infty}$. Therefore, by the monotonicity of RS, we have for any $n \in \omega$:

$$\mathrm{RS}(\mathcal{T}_{\Sigma,n}) \leq \mathrm{RS}(\mathcal{T}_\Sigma), \tag{3.1}$$

$$\mathrm{RS}(\mathcal{T}_{\Sigma,\infty}) \leq \mathrm{RS}(\mathcal{T}_\Sigma). \tag{3.2}$$

Using (3.1) and (3.2), the following theorems and their arguments allow to count the ranks $\mathrm{RS}(\mathcal{T}_{\Sigma,n})$ and $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty})$ depending on Σ.

Theorem 3.1. *For any language Σ either* $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = 0$*, if Σ is finite or $n = 1$ and Σ has finitely many predicate symbols, or* $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = \infty$*, otherwise.*

Proof. If Σ is finite then $\mathcal{T}_{\Sigma,n}$ is finite for any $n \in \omega$, since there are finitely many isomorphism types for n-element structures in the language Σ. If $n = 1$ and Σ has finitely many predicate symbols then again there are finitely many isomorphism types for 1-element structures $\langle A; \Sigma\rangle$, since there are finitely many possibilities for distributions of empty predicates, all nonempty predicates are complete, all constants has same interpretations, and all functions are identical.

If Σ has infinitely many predicate symbols P_i, we can form an infinite 2-tree of sentences allowing P_i independently be empty or complete. If Σ has infinitely many constant symbols c_i, then, for $n \geq 2$ and $c_0 \neq c_1$, we again can form an infinite 2-tree of sentences allowing c_i independently be equal to c_0 or c_1. Finally, if Σ has infinitely many functional symbols f_i, then, for $n \geq 2$, we can form an infinite 2-tree of sentences allowing f_i be (non)identical. Each possibility above immediately implies $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = \infty$. $\square$

Recall that for a predicate $P \subseteq A^m$ and for an operation $f\colon A^n \to A$ the values m and n are the *arities* for P and f, respectively. These values are also *arities* for language symbols in Σ with interpretations P and f, respectively.

Theorem 3.2. *For any language Σ either* $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty})$ *is finite, if Σ is finite and without predicate symbols of arities $m \geq 2$ as well as without functional symbols of arities $n \geq 1$, or* $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty}) = \infty$*, otherwise.*

Proof. Let Σ be finite and without predicate symbols of arities $m \geq 2$ as well as without functional symbols of arities $n \geq 1$, i.e., Σ contains only finitely many 0-ary and unary predicate symbols as well as finitely many constant symbols. Then applying Propositions 2.1–2.4 and inequality (3.2) we have $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty}) < \omega$.

If Σ has predicate symbols of arities $m \geq 2$ or functional symbols of arities $n \geq 1$ then $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = \infty$ repeating arguments for Proposition 2.5 and constructing a 2-tree of sentences.

If Σ is infinite then by the previous case it suffices to consider languages with either infinitely many 0-ary predicates, or infinitely many unary predicates, or infinitely many constants. In these cases we repeat arguments for Propositions 2.1–2.4 and construct 2-trees of sentences guaranteeing $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = \infty$. $\square$

Notice that, similarly to the remark after Proposition 2.4, RS-ranks for theories T in a totally transcendental family $\mathcal{T}_{\Sigma,\infty}$ are characterized by numbers of infinite parts, in models of T, with respect to unary predicates.

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