ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2021, Volume 12, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

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VLADIMIR MIKHAILOVICH FILIPPOV

(to the 70th birthday)

Vladimir Mikhailovich Filippov was born on 15 April 1951 in the city of Uryupinsk, Stalingrad Region of the USSR. In 1973 he graduated with honors from the Faculty of Physics and Mathematics and Natural Sciences of the Patrice Lumumba University of Peoples' Friendship in the specialty "Mathematics". In 1973-1975 he is a postgraduate student of the University; in 1976-1979 - Chairman of the Young Scientists' Council; in 1980-1987 - Head of the Research Department and the Scientific Department; in 1983-1984 - scientific work at the Free University of Brussels (Belgium); in 1985-2000 - Head of the Mathematical Analysis Department; from 2000 to the present - Head of the Comparative Educational Policy Department; in 1989–1993 - Dean of the Faculty of Physics, Mathematics and Natural Sciences; in 1993–1998 - Rector of the Peoples' Friendship University of Russia; in 1998-2004 - Minister of General and Professional Education, Minister of Education of the Russian Federation; in 2004-2005 - Assistant to the Chairman of the Government of the Russian Federation (in the field of education and culture); from 2005 to May 2020- Rector of the Peoples' Friendship University of Russia, since May 2020 - President of the Peoples' Friendship University of Russia, since 2013 - Chairman of the Higher Attestation Commission of the Ministry of Science and Higher Education of the Russian Federation.

In 1980, he defended his PhD thesis in the V.A. Steklov Mathematical Institute of Academy of Sciences of the USSR on specialty 01.01.01 - mathematical analysis (supervisor - a corresponding member of the Academy of Sciences of the USSR, Professor L.D. Kudryavtsev), and in 1986 in the same Institute he defended his doctoral thesis "Quasi-classical solutions of inverse problems of the calculus of variations in non-Eulerian classes of functionals and function spaces". In 1987, he was awarded the academic title of a professor.

V.M. Filippov is an academician of the Russian Academy of Education; a foreign member of the Ukrainian Academy of Pedagogical Sciences; President of the UNESCO International Organizing Committee for the World Conference on Higher Education (2007-2009); Vice-President of the Eurasian Association of Universities; a member of the Presidium of the Rectors' Council of Moscow and Moscow Region Universities, of the Governing Board of the Institute of Information Technologies in Education (UNESCO), of the Supervisory Board of the European Higher Education Center of UNESCO (Bucharest, Romania),

Research interests: variational methods; non-potential operators; inverse problems of the calculus of variations; function spaces.

In his Ph.D thesis, V.M. Filippov solved a long standing problem of constructing an integral extremal variational principle for the heat equation. In his further research he developed a general theory of constructing extremal variational principles for broad classes of differential equations with non-potential (in classical understanding) operators. He showed that all previous attempts to construct variational principles for non-potential operators "failed" because mathematicians and mechanics from the time of L. Euler and J. Lagrange were limited in their research by functionals of the type Euler - Lagrange. Extending the classes of functionals, V.M. Filippov introduced a new scale of function spaces that generalize the Sobolev spaces, and thus significantly expanded the scope of the variational methods. In 1984, famous physicist, a Nobel Prize winner I.R. Prigogine presented the report of V.M. Filippov to the Royal Academy of Sciences of Belgium. Results of V.M. Filippov's variational principles for non-potential operators are quite fully represented in some of his and his colleagues' monographs.

Honors: Honorary Legion (France), "Commander" (Belgium), Crown of the King (Belgium); in Russia - orders "Friendship", "Honor", "For Service to the Fatherland" III and IV degrees; Prize of the President of the Russian Federation in the field of education; Prize of the Governement of the Russian Federation in the field of education; Gratitude of the President of the Russian Federation; "For Merits in the Social and Labor Sphere of the Russian Federation", "For Merits in the Development of the Olympic Movement in Russia", "For Strengthening the Combat Commonwealth; and a number of other medals, prizes and awards.

He is an author of more than 270 scientific and scientific-methodical works, including 32 monographs, 2 of which were translated and published in the United States by the American Mathematical Society.

V.M. Filippov meets his 70th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 12, Number 2 (2021), 39 – 51

APPROXIMATION BY MODIFIED LUPAŞ-STANCU OPERATORS BASED ON (p, q)-INTEGERS

A. Khan, Z. Abbas, M. Qasim, M. Mursaleen

Communicated by V.I. Burenkov

Key words: Lupaş operators, post quantum analogue, q analogue, Peetre's K-functional, Korovkin type theorem, convergence theorems.

AMS Mathematics Subject Classification: 41A10, 41A25, 41A36.

Abstract. The purpose of this paper is to construct a new class of Lupaş operators in the frame of post quantum setting. We obtain a Korovkin type approximation theorem, study the rate of convergence of these operators by using the concept of the K-functional and modulus of continuity, also give a convergence theorem for the Lipschitz continuous functions.

DOI: https://doi.org/10.32523/2077-9879-2021-12-2-39-51

1 Introduction

Approximation theory deals with approximation of functions by simpler functions or more easily calculated functions. Broadly it is divided into theoretical and constructive approximation. Inspired by the binomial probability distribution, in 1912 S.N. Bernstein [4] was the first to construct a sequence of positive linear operators to provide a constructive proof of the prominent Weierstrass approximation theorem [34] using the probabilistic approach.

In order to obtain more flexibility, Stancu [32] applied another technique for choosing nodes. He observed that the distance between two successive nodes and between the zero and first node and similarly between the last and first goes to zero as $m \to \infty$. After these observation Stancu introduced the following positive linear operators

$$\big(P_m^{(\gamma,\delta)} f\big)(u) = \sum_{k=0}^{m} \binom{m}{k} u^k (1-u)^{m-k} f\left(\frac{k+\gamma}{m+\delta}\right), \tag{1.1}$$

converging to a continuous function f uniformly on [0,1] for each real γ, δ such that $0 \leq \gamma \leq \delta$.

A. Lupaş [17] introduced the linear positive operators at the International Dortmund Meeting held in Witten (Germany, March, 1995) as follows:

$$L_m(f; u) = (1-a)^{mu} \sum_{\ell=0}^{\infty} \frac{(mu)_\ell a^\ell}{\ell!} f\left(\frac{\ell}{m}\right), \quad u \geq 0, \tag{1.2}$$

where $(mu)_\ell$ is the rising factorial defined as:

$$(mu)_0 = 1, \quad (mu)_\ell = mu(mu+1)(mu+2)\cdots(mu+\ell-1), \quad \ell \geq 0,$$

with $0 < a < 1$ and $f : [0, \infty) \to \mathbb{R}$. If $L_m(u) = u$, then one can prove that $a = \frac{1}{2}$. Thus operators (1.2) become

$$L_m(f;u) = 2^{-mu} \sum_{\ell=0}^{\infty} \frac{(mu)_\ell}{\ell! 2^\ell} f\left(\frac{\ell}{m}\right), \quad u \geq 0, \tag{1.3}$$

The q-analogue of Lupaş operators (1.3) is defined in [33] as:

$$L_m^{p,q}(f;u) = 2^{-[m]_q u} \sum_{\ell=0}^{\infty} \frac{([m]_q u)_\ell}{[\ell]_q! 2^\ell} f\left(\frac{[\ell]_q}{[m]_q}\right), \quad u \geq 0. \tag{1.4}$$

Where, for any fixed real number $q > 0$, the q-integer $[m]_q$, for $m \in \mathbb{N}$ (set of natural numbers) is defined as

$$[m]_q := \begin{cases} \frac{(1-q^m)}{(1-q)}, & q \neq 1 \\ m, & q = 1, \end{cases}$$

and $([m]_q u)_\ell$ is the rising factorial defined as:

$$\begin{aligned} ([m]_q u)_0 &= 1, \\ ([m]_q u)_\ell &= ([m]_q u)([m]_q u + 1)([m]_q u + 2) \cdots ([m]_q u + \ell - 1), \quad \ell \geq 0. \end{aligned}$$

Recently, Mursaleen et al. ([25], [26]) introduced (p,q)-calculus in approximation theory and constructed a post quantum analogue of Bernstein operators. On the other hand Khan and Lobiyal defined a (p,q)- analogue of Lupaş-Bernstein operators in [12] and showed its application to computer aided geometric design (CAGD) for the construction of Beizer curves and surfaces. For related literature, one can refer to papers [1]-[3], [5, 8, 9, 11, 13, 14, 18, 19], [20]-[24], [27]-[31] based on q and (p,q) integers in approximation theory and CAGD.

Motivated by the above mentioned work, in this article, we introduce positive linear Lupaş-Stancu operators based on (p,q)-integers as follows:

$$L_{m,p,q}^{\gamma,\delta}(f;u) = 2^{-[m]_{p,q} u} \sum_{\ell=0}^{\infty} \frac{([m]_{p,q} u)_\ell}{[\ell]_{p,q}! 2^\ell} f\left(\frac{p^{\ell-m}[\ell]_{p,q} + \gamma}{[m]_{p,q} + \delta}\right), \quad u \geq 0. \tag{1.5}$$

where $f : [0, \infty) \to \mathbb{R}$, $0 < q < p \leq 1$ with $0 \leq \gamma \leq \delta$ and for any $m \in \mathbb{N}$.

The sequence of (p,q)-Lupaş-Stancu operators constructed in (1.5) however does not preserve the test functions t and t^2. Hence one cannot guarantee approximation via these operators. Therefore, we construct the modified (p,q)- Lupaş-Stancu operators as follows.

Let $0 < q < p \leq 1$ with $0 \leq \gamma \leq \delta$ and $m \in \mathbb{N}$. For $f : [0, \infty) \to \mathbb{R}$, we define the (p,q)-analogue of Lupaş-Stancu operators as:

$$\mathcal{L}_{m,p,q}^{\gamma,\delta}(f;u) = 2^{-[m]_{p,q} u} \sum_{\ell=0}^{\infty} \frac{([m]_{p,q} u)_\ell}{[\ell]_{p,q}! a^\ell} f\left(\frac{[\ell]_{p,q} + \gamma}{[m]_{p,q} + \delta}\right), \quad u \geq 0. \tag{1.6}$$

Obviously, when $p = 1$ and $\gamma = \delta = 0$ operator (1.6) reduces to operator (1.4).

Before proceeding further, let us mention some basic definitions and notations of (p,q)-calculus. Let $p > 0$, $q > 0$. For any nonnegative integers ℓ and m, $m \geq \ell \geq 0$, the (p,q)-integer, (p,q)-binomial are defined, as

$$[j]_{p,q} = p^{j-1} + p^{j-2}q + p^{j-3}q^2 + ... + pq^{j-2} + q^{j-1} = \begin{cases} \frac{p^j - q^j}{p-q}, & \text{if} \quad p \neq q \neq 1, \\ j\ p^{j-1}, & \text{if } p = q \neq 1, \\ [j]_q, & \text{if } p = 1, \\ j, & \text{if } p = q = 1. \end{cases}$$

where $[j]_q$ denotes the q-integers and $m = 0, 1, 2, \cdots$.

The formula for (p,q)-binomial expansion is as follows:

$$(au + bv)^m_{p,q} := \sum_{\ell=0}^{m} p^{\frac{(m-\ell)(m-\ell-1)}{2}} q^{\frac{\ell(\ell-1)}{2}} \left[\begin{array}{c} m \\ \ell \end{array} \right]_{p,q} a^{m-\ell} b^{\ell} u^{m-\ell} v^{\ell},$$

where (p,q)-binomial coefficients are defined by

$$\left[\begin{array}{c} m \\ \ell \end{array} \right]_{p,q} = \frac{[m]_{p,q}!}{[\ell]_{p,q}![m-\ell]_{p,q}!}.$$

The goals of this paper are to construct a new class of Lupaş operators based on the post quantum analogue and also to estimate the rate of convergence of these new operators to the unit operator.

The rest of the paper is organized as follows. In Section 2, we calculate moments and central moments for the operators which are required for our main results. In Section 3, we prove a Korovkin type approximation theorem and estimate the rate of convergence of operator (1.6) to the unit operator. Finally in Section 4, we prove a weighted approximation theorem for operator (1.6).

2 Some auxiliary results

Lemma 2.1. *Let $0 < q < p \leq 1$, $0 \leq \gamma \leq \delta$ and $m \in \mathbb{N}$. We have*

(*i*) $\mathcal{L}^{\gamma,\delta}_{m,p,q}(1; u) = 1,$

(*ii*) $\mathcal{L}^{\gamma,\delta}_{m,p,q}(t; u) = \frac{[m]_{p,q}}{[m]_{p,q}+\delta} u + \frac{\gamma}{[m]_{p,q}+\delta},$

(*iii*) $\mathcal{L}^{\gamma,\delta}_{m,p,q}(t^2; u) = \frac{[m]_{p,q}}{([m]_{p,q}+\delta)^2 (2-p)^{([m]_{p,q}u+1)}} u + \frac{(2\gamma+q)[m]_{p,q}}{([m]_{p,q}+\delta)^2} u + \frac{[m]^2_{p,q}}{([m]^2_{p,q}+\delta)^2} qu^2 + \frac{\gamma^2}{([m]_{p,q}+\delta)^2}.$

Proof. We have

(i)

$$\mathcal{L}^{\gamma,\delta}_{m,p,q}(1; u) = 2^{-[m]_{p,q}u} \sum_{\ell=0}^{\infty} \frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}! 2^\ell} = 1.$$

(ii)

$$\begin{aligned}
\mathcal{L}_{m,p,q}^{\gamma,\delta}(t;u) &= 2^{-[m]_{p,q}u}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}!2^\ell}\frac{[\ell]_{p,q}+\gamma}{[m]_{p,q}+\delta}\\
&= 2^{-[m]_{p,q}u}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}!2^\ell}\frac{[\ell]_{p,q}}{[m]_{p,q}+\delta}\\
&+ 2^{-[m]_{p,q}u}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}!2^\ell}\frac{\gamma}{[m]_{p,q}+\delta}\\
&= 2^{-[m]_{p,q}u}\sum_{\ell=1}^{\infty}\frac{([m]_{p,q}u)([m]_{p,q}u+1)_{\ell-1}}{[\ell]_{p,q}[\ell-1]_{p,q}!2^\ell}\frac{[\ell]_{p,q}}{[m]_{p,q}+\delta}\\
&+ 2^{-[m]_{p,q}u}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}!2^\ell}\frac{\gamma}{[m]_{p,q}+\delta}\\
&= \frac{[m]_{p,q}}{[m]_{p,q}+\delta}u2^{-[m]_{p,q}u-1}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u+1)_\ell}{[\ell]_{p,q}!2^\ell}+\frac{\gamma}{[m]_{p,q}+\delta}\\
&= \frac{[m]_{p,q}}{[m]_{p,q}+\delta}u+\frac{\gamma}{[m]_{p,q}+\delta}.
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathcal{L}_{m,p,q}^{\gamma,\delta}(t^2;u) &= 2^{-[m]_{p,q}u}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}!2^\ell}\frac{([\ell]_{p,q}+\gamma)^2}{([m]_{p,q}+\delta)^2}\\
&= 2^{-[m]_{p,q}u}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}!2^\ell}\frac{[\ell]_{p,q}^2+\gamma^2+2\ell\gamma}{([m]_{p,q}+\delta)^2}\\
&= 2^{-[m]_{p,q}u}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}!2^\ell}\frac{[\ell]_{p,q}^2}{([m]_{p,q}+\delta)^2}\\
&+ 2^{-[m]_{p,q}u}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}!2^\ell}\frac{\gamma^2}{([m]_{p,q}+\delta)^2}\\
&+ 2^{-[m]_{p,q}u}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}!2^\ell}\frac{2\ell\gamma}{([m]_{p,q}+\delta)^2}\\
&\equiv I_1+I_2+I_3.
\end{aligned}$$

After calculating I_1, I_2 and I_3, we get

$$\begin{aligned}
&\mathcal{L}_{m,p,q}^{\gamma,\delta}(t^2;u)\\
= \;&\frac{[m]_{p,q}}{([m]_{p,q}+\delta)^2(2-p)^{([m]_{p,q}u+1)}}u+\frac{(2\gamma+q)[m]_{p,q}}{([m]_{p,q}+\delta)^2}u+\frac{[m]_{p,q}}{([m]_{p,q}^2+\delta)^2}qu^2+\frac{\gamma^2}{([m]_{p,q}+\delta)^2}.
\end{aligned}$$

□

Corollary 2.1. *Using Lemma 2.1, we get the following formulas for moments.*

$\mathcal{L}^{\gamma,\delta}_{m,p,q}(t-u;u)=0$

$\mathcal{L}^{\gamma,\delta}_{m,p,q}((t-u)^2;u)=\big(\frac{1}{(2-p)^{([m]_{p,q}u+1)}}+2\gamma+q\big)\frac{[m]_{p,q}}{([m]_{p,q}+\delta)^2}u+\frac{[m]^2_{p,q}}{([m]^2_{p,q}+\delta)^2}qu^2+\frac{\gamma^2}{([m]_{p,q}+\delta)^2}$
$-\frac{2[m]_{p,q}}{([m]_{p,q}+\delta)}u^2+\frac{2\gamma}{([m]_{p,q}+\delta)}u+u^2\equiv\sigma_m(u).$

3 Main results

Theorem 3.1. *Let $f\in C_B[0,\infty)$ and $q_m\in(0,1)$, $p_m\in(q_m,1]$ be such that $q_m\to 1$, $p_m\to 1$ as $m\to\infty$. Then for each $u\in[0,\infty)$ we have*

$$\lim_{n\to\infty}\mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(f;u)=f(u).$$

Proof. By Korovkin's theorem it is enough to show that

$$\lim_{m\to\infty}\mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(t^m;u)=u^m,\quad m=0,1,2.$$

By Lemma 2.1, it is clear that

$$\lim_{m\to\infty}\mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(1;u)=1$$

$$\lim_{m\to\infty}\mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(1;u)=u$$

and

$$\begin{aligned}\lim_{m\to\infty}\mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(t^2;u) &= \lim_{m\to\infty}\Bigg[\frac{[m]_{p,q}}{([m]_{p,q}+\delta)^2(2-p)^{([m]_{p,q}u+1)}}u\\ &+\frac{(2\gamma+q)[m]_{p,q}}{([m]_{p,q}+\delta)^2}u+\frac{[m]^2_{p,q}}{([m]^2_{p,q}+\delta)^2}qu^2+\frac{\gamma^2}{([m]_{p,q}+\delta)^2}\Bigg]=u^2.\end{aligned}$$

□

Next, in order to prove the convergence of $\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)$, we give the following definitions of the K-functional and modulus of smoothness. Let $C_B[0,\infty)$ be the space of all real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \|=\sup_{0\leq u<\infty}\mid f(x)\mid .$$

The K-functional is defined as

$$K_2(f,\delta)=\inf_{s\in W^2}\{\| f-s\|+\delta\| s^{''}\|\},$$

where $\sigma>0$ and $W^2=\{s\in C_B[0,\infty):s^{'},s^{''}\in C_B[0,\infty)\}$.

Then as in ([6], p. 177, Theorem 2.4), there exists an absolute constant $C>0$ such that

$$K_2(f,\sigma)\leq C\omega_2(f,\sqrt{\sigma}).\tag{3.1}$$

The second order modulus of smoothness of $f \in C_B[0,\infty)$ is as follows

$$\omega_2(f,\sqrt{\sigma}) = \sup_{0<h\leq\sqrt{\sigma}} \sup_{u\in[0,\infty)} \mid f(u+2h) - 2f(u+h) + f(u) \mid .$$

The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f,\sigma) = \sup_{0<h\leq\delta} \sup_{u\in[0,\infty)} \mid f(u+h) - f(u) \mid .$$

Theorem 3.2. *Let $f \in C_B[0,\infty)$, $p,q \in (0,1)$ be such that $0 < q < p \leq 1$. Then for every $u \in [0,\infty)$ we have*

$$\mid \mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u) - f(u) \mid \leq C\omega_2(f;\sigma_m(u)),$$

where $C > 0$ is an absolute constant and $\sigma_m(u)$ is defined in Corollary 2.1

Proof. Let $s \in \mathcal{W}^2$. Then from Taylor's expansion, we get

$$s(t) = s(u) + s'(u)(t-u) + \int_u^t (t-u)s''(u)\mathrm{d}u, \quad t \in [0,\mathcal{D}], \quad \mathcal{D} > 0.$$

Now by Corollary 2.1, we have

$$\mathcal{L}^{\gamma,\delta}_{m,p,q}(s;u) = s(u) + \mathcal{L}^{\gamma,\delta}_{m,p,q}\left(\int_u^t (t-u)s''(u)\mathrm{d}u;u\right).$$

$$\begin{aligned} |\mathcal{L}^{\gamma,\delta}_{m,p,q}(s;u) - s(u)| &\leq \mathcal{L}^{\gamma,\delta}_{m,p,q}\left(\left|\int_u^t \mid (t-u) \mid \mid s''(u) \mid \mathrm{d}u;u\right|\right) \\ &\leq \mathcal{L}^{\gamma,\delta}_{m,p,q}\left((t-u)^2;u\right) \parallel s'' \parallel, \end{aligned}$$

hence we get

$$\begin{aligned} &|\mathcal{L}^{\gamma,\delta}_{m,p,q}(s;u) - s(u)| \\ &\leq \parallel s'' \parallel \Bigg(\Big(\frac{1}{(2-p)^{([m]_{p,q}u+1)}} + 2\gamma + q\Big)\frac{[m]_{p,q}}{([m]_{p,q}+\delta)^2}u \\ &+ \frac{[m]^2_{p,q}}{([m]^2_{p,q}+\delta)^2}qu^2 + \frac{\gamma^2}{([m]_{p,q}+\delta)^2} - \frac{2[m]_{p,q}}{([m]_{p,q}+\delta)}u^2 \\ &+ \frac{2\gamma}{([m]_{p,q}+\delta)}u + u^2\Bigg). \end{aligned}$$

By (1.6), we have

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)| \leq 2^{-[m]_{p,q}u}\sum_{\ell=0}^{\infty}\frac{([m]_{p,q}u)_\ell}{[\ell]_{p,q}!2^\ell}\left|f\left(\frac{[\ell]_{p,q}+\gamma}{[m]_{p,q}+\delta}\right)\right| \leq \parallel f \parallel .$$

Thus, we have

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)| \leq \mid \mathcal{L}^{\gamma,\delta}_{m,p,q}((f-s);u) - (f-s)(u) \mid + |\mathcal{L}^{\gamma,\delta}_{m,p,q}(s;u) - s(u)|.$$

After substituting all values, we get

$$\begin{aligned}|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| &\leq \| f-s \| + \| s'' \| \Bigg(\Big(\frac{1}{(2-p)^{([m]_{p,q}u+1)}}+2\gamma+q\Big)\frac{[m]_{p,q}}{([m]_{p,q}+\delta)^2}u \\ &+ \frac{[m]^2_{p,q}}{([m]^2_{p,q}+\delta)^2}qu^2+\frac{\gamma^2}{([m]_{p,q}+\delta)^2}-\frac{2[m]_{p,q}}{([m]_{p,q}+\delta)}u^2 \\ &+ \frac{2\gamma}{([m]_{p,q}+\delta)}u+u^2\Bigg).\end{aligned}$$

By taking the infimum in the right hand side over all $s \in \mathcal{W}^2$, we get

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| \leq CK_2\left(f,\sigma^2_m(u)\right).$$

By using the property (3.1) of the K-functional, we have

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| \leq C\omega_2\left(f,\sigma_m(u)\right).$$

□

Theorem 3.3. *Let $0<\alpha\leq 1$ and E be any bounded subset of the interval $[0,\infty)$. If $f \in C_B[0,\infty)$ is locally Lip(α), i.e., the condition*

$$|f(v)-f(u)| \leq Ł|v-u|^{\alpha}, \qquad v\in E \quad and \qquad u \in [0,\infty) \tag{3.2}$$

holds, then, for each $u\in[0,\infty)$, we have

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| \leq Ł\left\{\sigma_m(u)^{\frac{\alpha}{2}}+2(d(u,E))^{\alpha}\right\}, \quad u\in[0,\infty)$$

where L is a constant depending on α and f and $d(u;E)$ is the distance between u and E defined by $d(u,E)=inf\,\{|t-u|; t\in E\}$ and $\sigma_m(u)$ is defined in Corollary 2.1

Proof. Let $\overline{E}$ be the closure of E in $[0,\infty)$. Then, there exists a point $t_0\in\overline{E}$ such that $d(u,E)=|u-t_0|$. Using the triangle inequality, we have

$$|f(t)-f(u)| \leq |f(t)-f(t_0)|+|f(t_0)-f(u)|.$$

By using (3.2) we get,

$$\begin{aligned}&|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| \\ \leq\ & \mathcal{L}^{\gamma,\delta}_{m,p,q}\left(|f(t)-f(t_0)|;u\right)+\mathcal{L}^{\gamma,\delta}_{m,p,q}\left(|f(u)-f(t_0)|;u\right) \\ \leq\ & Ł\left\{\mathcal{L}^{\gamma,\delta}_{m,p,q}\left(|t-t_0|^{\alpha};u\right)+\left(|u-t_0|^{\alpha};u\right)+|u-t_0|^{\alpha}\right\} \\ \leq\ & Ł\left\{\mathcal{L}^{\gamma,\delta}_{m,p,q}\left(|t-u|^{\alpha};u\right)+2|u-t_0|^{\alpha}\right\}.\end{aligned}$$

By choosing $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, we get $\frac{1}{p}+\frac{1}{q}=1$. Then by using Hölder's inequality we get

$$\begin{aligned}|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| &\leq Ł\left\{\mathcal{L}^{\gamma,\delta}_{m,p,q}\left(|t-u|^{\alpha p};u\right)^{\frac{1}{p}}\left[\mathcal{L}^{\gamma,\delta}_{m,p,q}(1^q;u)\right]^{\frac{1}{q}}+2(d(u,E))^{\alpha}\right\} \\ &\leq Ł\left\{\mathcal{L}^{\gamma,\delta}_{m,p,q}\left(((t-u)^2;u)\right)^{\frac{\alpha}{2}}+2(d(u,E))^{\alpha}\right\} \\ &\leq Ł\left\{\sigma_m(u)^{\frac{\alpha}{2}}+2(d(u,E))^{\alpha}\right\}.\end{aligned}$$

□

Now, let

$$\widetilde{\omega}_\alpha(f;u) = \sup_{t\neq u, t\in(0,\infty)} \frac{|f(t)-f(u)|}{|t-u|^\alpha},\ u\in[0,\infty) \text{ and } \alpha\in(0,1], \tag{3.3}$$

given by Lenze [16]. Then we get the following result.

Theorem 3.4. *Let $f\in C_B[0,\infty)$ and $\alpha\in(0,1]$. Then, for all $u\in[0,\infty)$, we have*

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| \leq \widetilde{\omega}_\alpha(f;u)\Big(\sigma_m(u)\Big)^{\frac{\alpha}{2}},$$

where $\sigma_m(u)$ is defined in Corollary 2.1.

Proof. We know that

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| \leq \mathcal{L}^{\gamma,\delta}_{m,p,q}(|f(t)-f(u)|;u).$$

From equation (3.3), we have

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| \leq \widetilde{\omega}_\alpha(f;u)\mathcal{L}^{\gamma,\delta}_{m,p,q}(|t-u|^\alpha;u).$$

From Hölder's inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, we have

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| \leq \widetilde{\omega}_\alpha(f;u)\big(\mathcal{L}^{\gamma,\delta}_{m,p,q}(|t-u|^2;u)\big)^{\frac{\alpha}{2}},$$

which proves the desired result. □

4 Weighted approximation by $\mathcal{L}^{\gamma,\delta}_{m,p,q}$

In this section we shall discuss weighted approximation theorems for the operators $\mathcal{L}^{\gamma,\delta}_{m,p,q}$ on the interval $[0,\infty)$. Let $\rho(u)=1+u^2$ be a weight function; $\mathcal{B}_\rho[0,\infty)$ be the weighted space defined by

$$\mathcal{B}_\rho[0,\infty)=\{f:[0,\infty)\to\mathbb{R}\big||f(u)|\leq\mathcal{K}_f\rho(u), u\geq 0\},$$

where $\mathcal{K}_f$ is a constant which depends only on f. $\mathcal{B}_\rho[0,\infty)$ is a normed linear space equipped with the norm

$$\parallel f\parallel_\rho = \sup_{u\in[0,\infty)}\frac{|f(u)|}{\rho(u)}.$$

Also, we define the following subspaces of $\mathcal{B}_\rho[0,\infty)$ as

$$\mathcal{C}_\rho[0,\infty)=\{f\in\mathcal{B}_\rho[0,\infty):\quad f \ \ \textit{is continuous on} \ \ [0,\infty)\},$$
$$\mathcal{C}^*_\rho[0,\infty)=\left\{f\in\mathcal{C}_\rho[0,\infty):\quad \lim_{u\to\infty}\frac{f(u)}{\rho(u)}=\mathcal{K}_f\right\},$$

where $\mathcal{K}_f$ is a constant depending on f and

$$U_\rho[0,\infty)=\{f\in\mathcal{C}_\rho[0,\infty):\quad \frac{f(u)}{\rho(u)} \ \ \text{is uniformly continuous on} \ \ [0,\infty)\}.$$

Obviously,

$$\mathcal{C}^*_\rho[0,\infty)\subset U_\rho[0,\infty)\subset\mathcal{C}_\rho[0,\infty)\subset\mathcal{B}_\rho[0,\infty).$$

Theorem 4.1 (cf. [15]). *Let $(\mathcal{A}_m)$ be a sequence of linear positive operators from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ satisfying*

$$\lim_{m\to\infty} \| \mathcal{A}_m k_i - k_i \|_\rho = 0, \qquad i = 0,1,2,$$

where $k_i = x^i, i = 0,1,2$ are the test functions.

Then for any function $f \in C^*_\rho[0,\infty)$

$$\lim_{m\to\infty} \| \mathcal{A}_m f - f \|_\rho = 0.$$

Theorem 4.2. *Let (q_m) and (p_m) be two sequences such that $0 < q_m < p_m \leq 1$ for all m which converge to 1. Then for each function $f \in C^*_\rho[0,\infty)$, we get*

$$\lim_{m\to\infty} \| \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m} f - f \|_\rho = 0.$$

Proof. By Theorem 4.1, it is enough to show that

$$\lim_{m\to\infty} \| \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m} k_i - k_i \|_\rho \;=\; 0, \qquad i = 0,1,2. \tag{4.1}$$

By Lemma 2.1 (i) and (ii), it is clear that

$$\| \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(1;u) - 1 \|_\rho \;=\; 0$$

$$\begin{aligned}
&\| \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(t;u) - u \|_\rho \\
=\; & \sup_{u\in[0,\infty)} \frac{\left(\frac{[m]_{p,q}}{[m]_{p,q}+\delta} - 1\right) u}{1+u^2} + \frac{\gamma}{[m]_{p,q}+\delta} \\
\leq\; & \left(\frac{[m]_{p,q}}{[m]_{p,q}+\delta} - 1\right) + \frac{\gamma}{[m]_{p,q}+\delta}.
\end{aligned}$$

and by Lemma 2.1 (iii), we have

$$\begin{aligned}
&\| \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(t^2;u) - u^2 \|_\rho \\
=\; & \sup_{u\in[0,\infty)} \frac{\left(\frac{[m]_{p,q}}{([m]_{p,q}+\delta)^2(2-p_m)^{([m]_{p,q}u+1)}} + \frac{(2\gamma+q_m)[m]_{p,q}}{([m]_{p,q}+\delta)^2}\right) u + \left(\frac{[m]^2_{p,q}}{([m]^2_{p,q}+\delta)^2} q_m - 1\right) u^2}{1+u^2} \\
+\; & \frac{\gamma^2}{([m]_{p,q}+\delta)^2} \\
\leq\; & \left(\frac{[m]_{p,q}}{([m]_{p,q}+\delta)^2(2-p_m)^{([m]_{p,q}u+1)}} + \frac{(2\gamma+q_m)[m]_{p,q}}{([m]_{p,q}+\delta)^2} + \frac{[m]^2_{p,q}}{([m]^2_{p,q}+\delta)^2} q_m - 1\right) \\
+\; & \frac{\gamma^2}{([m]_{p,q}+\delta)^2}.
\end{aligned}$$

The last inequality means that (4.1) holds for $i = 2$. By Theorem 4.1, the proof is completed. □

Theorem 4.3. *Let $q_m \in (0,1)$, $p_m \in (q,1]$ such that $q_m \to 1$, $p_m \to 1$ as $m \to \infty$. Let $f \in C^*_\rho[0,\infty)$, and $\omega_{d+1}(f;\sigma)$ be its modulus of continuity defined on finite interval $[0,d+1] \subset [0,\infty), d > 0$. Then, for $m \geq 1$ we have*

$$\|\mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(f;u) - f(u)\|_{C[0,d]} \leq 6M_f(1+d^2)\sigma_m(u) + 2\omega_{d+1}(f;\sqrt{\sigma_m(u)}),$$

where M_f is a constant which depends on f, $\sigma_m(u)$ is defined in Corollary 2.1.

Proof. From ([10], p. 378), for $u \in [0,d]$ and $t \leq d+1$, we have

$$|f(t)-f(u)| \leq 6M_f(1+d^2)(t-u)^2 + \left(1+\frac{|t-u|}{\sigma}\right)\omega_{d+1}(f;\sigma).$$

Applying $\mathcal{L}^{\gamma,\delta}_{m,p,q}$ to both sides, we have

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| \leq 6M_f(1+d^2)\mathcal{L}^{\gamma,\delta}_{m,p,q}((t-u)^2;u) + \left(1+\frac{\mathcal{L}^{\gamma,\delta}_{m,p,q}(|t-u|;u)}{\sigma}\right)\omega_{d+1}(f;\sigma).$$

Applying Cauchy-Schwartz inequality, for $u \in [0,d]$ we get

$$\begin{aligned} |\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| &\leq \mathcal{L}^{\gamma,\delta}_{m,p,q}\left(|(f;u)-f(u)|;u\right) \\ &\leq 6M_f(1+d^2)\mathcal{L}^{\gamma,\delta}_{m,p,q}((t-u)^2;u) \\ &+ \omega_{d+1}(f;\sigma)\left(1+\frac{1}{\sigma}\mathcal{L}^{\gamma,\delta}_{m,p,q}\left((t-u)^2;u\right)^{\frac{1}{2}}\right). \end{aligned}$$

Thus, from Corollary 2.1, for $u \in [0,d]$, we get

$$|\mathcal{L}^{\gamma,\delta}_{m,p,q}(f;u)-f(u)| \leq 6M_f(1+d^2)\sigma_m(u) + \omega_{d+1}(f;\delta)\left(1+\frac{\sqrt{\sigma_m(u)}}{\sigma}\right).$$

By choosing $\sigma = \sqrt{\sigma_m(u)}$, we get the required result. □

Now, we prove a theorem on approximation of all functions in $C^*_\rho[0,\infty)$. Results of such type are given in [7] for locally integrable functions.

Theorem 4.4. *Let $0 < q_m < p_m \leq 1$ such that $q_m \to 1$, $p_m \to 1$ as $m \to \infty$. Then for each function $f \in C^*_\rho[0,\infty)$, and $\alpha > 1$*

$$\lim_{m\to\infty} \sup_{u\in[0,\infty)} \frac{|\ \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(f;u)-f(u)\ |}{(1+u^2)^\alpha} = 0.$$

Proof. Let for any fixed $u_0 > 0$,

$$\begin{aligned} &\sup_{u\in[0,\infty)} \frac{|\ \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(f;u)-f(u)\ |}{(1+u^2)^\alpha} \\ \leq\ & \sup_{u\leq u_0} \frac{|\ \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(f;u)-f(u)\ |}{(1+u^2)^\alpha} + \sup_{u\geq u_0} \frac{|\ \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(f;u)-f(u)\ |}{(1+u^2)^\alpha} \\ \leq\ & \|\ \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(f)-f\ \|_{C[0,u_0]} + \|\ f\ \|_\rho \sup_{u\geq u_0} \frac{|\ \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(1+t^2;u)\ |}{(1+u^2)^\alpha} \\ +\ & \sup_{u\geq u_0} \frac{|\ f(u)\ |}{(1+u^2)^\alpha}. \end{aligned} \tag{4.2}$$

Since, $|\ f(u)\ | \leq M_f(1+u^2)$ we have,

$$\sup_{u\geq u_0} \frac{|\ f(u)\ |}{(1+u^2)^\alpha} \leq \sup_{u\geq u_0} \frac{M_f}{(1+u^2)^{\alpha-1}} \leq \frac{M_f}{(1+u^2)^{\alpha-1}}.$$

Let $\epsilon > 0$, and let us choose u_0 such that

$$\frac{M_f}{(1+{u_0}^2)^{\alpha-1}} < \frac{\epsilon}{3} \tag{4.3}$$

and in view of Lemma 2.1 we get

$$\begin{aligned} \| f \|_\rho &= \lim_{m\to\infty} \frac{| \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(1+t^2;u) |}{(1+u^2)^\alpha} \\ &= \| f \|_\rho \frac{1+u^2}{(1+u^2)^\alpha} \\ &\leq \frac{\| f \|_\rho}{(1+u^2)^{\alpha-1}} \leq \frac{\| f \|_\rho}{(1+{u_0}^2)^{\alpha-1}} \leq \frac{\epsilon}{3}. \end{aligned}$$

By using Theorem 4.3, the first term of inequality (4.2) becomes

$$\| \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(f) - f \|_{C[0,u_0]} < \frac{\epsilon}{3}. \tag{4.4}$$

Hence we get the required proof by combining (4.3)-(4.4)

$$\sup_{u\in[0,\infty)} \frac{| \mathcal{L}^{\gamma,\delta}_{m,p_m,q_m}(f;u) - f(u) |}{(1+u^2)^\alpha} = 0.$$

□

Acknowledgments

We are very thankful to the referee. The third author (M. Qasim) is also grateful to the Council of Scientific and Industrial Research (CSIR), India, for providing the Senior Research Fellowship(09/1172(0001)/2017-EMR-I).

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Received: 18.04.2019
Revised: 19.02.2020