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eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 562
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

VLADIMIR MIKHAILOVICH FILIPPOV

(to the 70th birthday)

Vladimir Mikhailovich Filippov was born on 15 April 1951 in the city of Uryupinsk, Stalingrad Region of the USSR. In 1973 he graduated with honors from the Faculty of Physics and Mathematics and Natural Sciences of the Patrice Lumumba University of Peoples' Friendship in the specialty "Mathematics". In 1973-1975 he is a postgraduate student of the University; in 1976-1979 - Chairman of the Young Scientists' Council; in 1980-1987 - Head of the Research Department and the Scientific Department; in 1983-1984 - scientific work at the Free University of Brussels (Belgium); in 1985-2000 - Head of the Mathematical Analysis Department; from 2000 to the present - Head of the Comparative Educational Policy Department; in 1989–1993 - Dean of the Faculty of Physics, Mathematics and Natural Sciences; in 1993–1998 - Rector of the Peoples' Friendship University of Russia; in 1998-2004 - Minister of General and Professional Education, Minister of Education of the Russian Federation; in 2004-2005 - Assistant to the Chairman of the Government of the Russian Federation (in the field of education and culture); from 2005 to May 2020- Rector of the Peoples' Friendship University of Russia, since May 2020 - President of the Peoples' Friendship University of Russia, since 2013 - Chairman of the Higher Attestation Commission of the Ministry of Science and Higher Education of the Russian Federation.

In 1980, he defended his PhD thesis in the V.A. Steklov Mathematical Institute of Academy of Sciences of the USSR on specialty 01.01.01 - mathematical analysis (supervisor - a corresponding member of the Academy of Sciences of the USSR, Professor L.D. Kudryavtsev), and in 1986 in the same Institute he defended his doctoral thesis "Quasi-classical solutions of inverse problems of the calculus of variations in non-Eulerian classes of functionals and function spaces". In 1987, he was awarded the academic title of a professor.

V.M. Filippov is an academician of the Russian Academy of Education; a foreign member of the Ukrainian Academy of Pedagogical Sciences; President of the UNESCO International Organizing Committee for the World Conference on Higher Education (2007-2009); Vice-President of the Eurasian Association of Universities; a member of the Presidium of the Rectors' Council of Moscow and Moscow Region Universities, of the Governing Board of the Institute of Information Technologies in Education (UNESCO), of the Supervisory Board of the European Higher Education Center of UNESCO (Bucharest, Romania),

Research interests: variational methods; non-potential operators; inverse problems of the calculus of variations; function spaces.

In his Ph.D thesis, V.M. Filippov solved a long standing problem of constructing an integral extremal variational principle for the heat equation. In his further research he developed a general theory of constructing extremal variational principles for broad classes of differential equations with non-potential (in classical understanding) operators. He showed that all previous attempts to construct variational principles for non-potential operators "failed" because mathematicians and mechanics from the time of L. Euler and J. Lagrange were limited in their research by functionals of the type Euler - Lagrange. Extending the classes of functionals, V.M. Filippov introduced a new scale of function spaces that generalize the Sobolev spaces, and thus significantly expanded the scope of the variational methods. In 1984, famous physicist, a Nobel Prize winner I.R. Prigogine presented the report of V.M. Filippov to the Royal Academy of Sciences of Belgium. Results of V.M. Filippov's variational principles for non-potential operators are quite fully represented in some of his and his colleagues' monographs.

Honors: Honorary Legion (France), "Commander" (Belgium), Crown of the King (Belgium); in Russia - orders "Friendship", "Honor", "For Service to the Fatherland" III and IV degrees; Prize of the President of the Russian Federation in the field of education; Prize of the Governement of the Russian Federation in the field of education; Gratitude of the President of the Russian Federation; "For Merits in the Social and Labor Sphere of the Russian Federation", "For Merits in the Development of the Olympic Movement in Russia", "For Strengthening the Combat Commonwealth; and a number of other medals, prizes and awards.

He is an author of more than 270 scientific and scientific-methodical works, including 32 monographs, 2 of which were translated and published in the United States by the American Mathematical Society.

V.M. Filippov meets his 70th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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ON AN INVERSE PROBLEM FOR A PARABOLIC EQUATION IN A DEGENERATE ANGULAR DOMAIN

M.T. Jenaliyev, M.I. Ramazanov, M.G. Yergaliyev

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Key words: coefficient inverse problem, heat equation, degenerate domain, angular domain, parabolic equation.

AMS Mathematics Subject Classification: 65M32, 35K05, 35K65, 35K10.

Abstract. We consider a coefficient inverse problem for a parabolic equation in a degenerate angular domain when the moving part of the boundary changes linearly. We show that the inverse problem for the homogeneous heat equation with homogeneous boundary conditions has a nontrivial solution up to a constant factor consistent with an additional condition. The boundedness of this solution and this additional condition is proved. Moreover, the solution of the considered inverse problem is found in an explicit form and it is proved that the required coefficient is determined uniquely. It is shown that the obtained nontrivial solution of the inverse problem has no singularities and the additional condition also has no singularities.

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1 Introduction

Despite the fact that the theory of inverse problems has been developed for not so long, inverse problems are used in many areas of science. Now a lot of attention is paid to the study of inverse problems arising in geophysics, to the study of inverse problems associated with the analysis of electrophysiological processes occurring in the human body, etc.

The inverse problems of the type that we consider in the article were investigated in papers [17]-[18] (see also literature therein). In those papers it was assumed that the movable boundaries move according to the Hölder condition, that the domain is not degenerate and the time interval is bounded. The uniqueness and existence of a solution of the inverse problem where the required coefficient is a continuous function were established and numerical solutions were obtained.

The peculiarity of our study is that we consider the inverse problem for the heat equation in a degenerate angular domain. For the sake of simplicity and for the purpose of showing the effect of degeneration of the domain, we consider the problem, where, firstly, the moving part of the boundary changes linearly; secondly, the boundary value problem is completely homogeneous; thirdly, the time interval is bounded. It is known that when a domain degenerates at some points, the methods of separation of variables and integral transformations are generally not applicable to this type of problems. In this paper, to prove the existence of a nontrivial solution for the original problem we use the methods and results of our earlier works [1]-[4], [8], [14], [15] where solutions are found with the help of the theory of thermal potentials and the Volterra integral equations of the second kind.

We also note works [7] and [9] devoted to the study of the heat conduction problems in degenerate domains. In paper [16] a theorem on the unique solvability of the non-homogeneous boundary value problem in weighted Hölder spaces was obtained. We also refer to publications [5], [10]-[13] of other authors that are close to the contents of this paper.

The work consists of 7 sections. The first section is Introduction. In Section 2, we formulate the original inverse problem. In Section 3, we present auxiliary problem and a solution to this auxiliary problem. In Section 4 we show the boundedness of the additional condition and of the solution of an equivalent form of the auxiliary problem. Section 5 is devoted to the asymptotic behaviour of the additional condition. In Section 6, we formulate the main result. Finally, conclusions are made in Section 7.

2 Statement of the original inverse problem

We consider an inverse problem of finding a coefficient $\lambda(t)$ and a function $u(x,t)$ in the domain $G_T = \{(x,t)|\ 0 < x < \sqrt{k}t,\ \ 0 < t < T\}$, $k > 0$, $T < +\infty$, for the following heat equation:

$$u_t(x,t) = u_{xx}(x,t) - \lambda(t)u(x,t),\ \ (x,t) \in G_T, \tag{2.1}$$

with the homogeneous boundary conditions

$$u(x,t)|_{x=0} = 0, \quad u(x,t)|_{x=\sqrt{k}t} = 0,\ 0 < t < T, \tag{2.2}$$

subject to the overspecification

$$u(\mu(t),t) = E(t), \quad E(t) \geq \delta > 0,\ 0 < t < T, \tag{2.3}$$

where $0 < \mu(t) < \sqrt{k}t$, $\mu(0) = 0$, $0 < \mu'(0) < \sqrt{k}$, $\mu(t) \in C^1(0,T)$ and $E(t) \in L_\infty(0,T)$ are given functions.

3 Auxiliary problems

3.1 First auxiliary problem

We transform original inverse problem (2.1)–(2.3) by replacing the independent variables and domain

$$\begin{cases} \bar{t} = k\,t,\ \bar{x} = \sqrt{k}\,x,\ G_{T_k} = \{(\bar{x},\bar{t})|0 < \bar{x} < \bar{t},\ 0 < \bar{t} < T_k = k\,T\}, \\ \bar{u}(\bar{x},\bar{t}) = u(x,t)|_{x=\bar{x}/\sqrt{k},\,t=\bar{t}/k},\ \bar{\lambda}(\bar{t}) = \frac{1}{k}\lambda(\bar{t}/k). \end{cases} \tag{3.1}$$

Further, for inverse problem (2.1)–(2.3) we obtain the following inverse problem of finding the coefficient $\bar{\lambda}(\bar{t})$ and the function $\bar{u}(\bar{x},\bar{t})$ in the domain G_{T_k}:

$$\bar{u}_{\bar{t}}(\bar{x},\bar{t}) = \bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}) - \bar{\lambda}(\bar{t})\bar{u}(\bar{x},\bar{t}),\ (\bar{x},\bar{t}) \in G_{T_k}, \tag{3.2}$$

with the homogeneous boundary conditions

$$\bar{u}(\bar{x},\bar{t})|_{\bar{x}=0} = 0,\ \bar{u}(\bar{x},\bar{t})|_{\bar{x}=\bar{t}} = 0, \tag{3.3}$$

subject to the overspecification

$$\bar{u}(\bar{\mu}(\bar{t}),\bar{t}) = \bar{E}(\bar{t}), \quad \bar{E}(\bar{t}) \geq \delta > 0,\ 0 < \bar{t} < T_k, \tag{3.4}$$

where $\bar{\mu}(\bar{t}) = \sqrt{k}\mu(t)|_{t=\bar{t}/k}$, $0 < \bar{\mu}(\bar{t}) < \bar{t}$ and $\bar{E}(\bar{t}) = E(t)|_{t=\bar{t}/k}$.

3.2 Second auxiliary problem

In accordance to problem (3.2)–(3.4) we will set an auxiliary inverse problem of finding the coefficient $\bar{\lambda}_1(\bar{t})$ and the function $v(\bar{x},\bar{t})$ in the domain $G_\infty = \{(\bar{x},\bar{t})|\ 0<\bar{x}<\bar{t},\ \ \bar{t}>0\}$:

$$v_{\bar{t}}(\bar{x},\bar{t}) = v_{\bar{x}\bar{x}}(\bar{x},\bar{t}) - \bar{\lambda}_1(\bar{t})v(\bar{x},\bar{t}),\ (\bar{x},\bar{t}) \in G_\infty, \tag{3.5}$$

with the homogeneous boundary conditions

$$v(\bar{x},\bar{t})|_{\bar{x}=0} = 0, \quad v(\bar{x},\bar{t})|_{\bar{x}=\bar{t}} = 0,\ \bar{t}>0, \tag{3.6}$$

subject to the overspecification

$$v(\tilde{\mu}(\bar{t}),\bar{t}) = \tilde{E}(\bar{t}), \quad \bar{t}>0, \tag{3.7}$$

$$\tilde{E}(\bar{t}) = \begin{cases} \bar{E}(\bar{t}), & 0<\bar{t}<T_k, \\ E_1(\bar{t}), & T_k \le \bar{t} < \infty, \end{cases} \tag{3.8}$$

where $E_1(\bar{t}) \ge \delta > 0$ is an arbitrary bounded function and

$$\tilde{\mu}(\bar{t}) = \begin{cases} \bar{\mu}(\bar{t}), & 0<\bar{t}<T_k, \\ \mu_1(\bar{t}), & T_k \le \bar{t} < \infty, \end{cases} \tag{3.9}$$

where $\mu_1(\bar{t})$ is an arbitrary continuous function satisfying the condition $0<\tilde{\mu}(\bar{t})<\bar{t},\ \bar{t}>0$.

Remark 1. By solving in G_∞ problem (3.5)–(3.9) and restricting its solution to the domain G_{T_k}, we can find a solution $\{\bar{u}(\bar{x},\bar{t}),\bar{\lambda}(\bar{t});\ (\bar{x},\bar{t})\in G_{T_k}\}$ of inverse problem (3.2)–(3.4).

3.3 Equivalent problem

In problem (3.5)–(3.9) we replace the required function $v(\bar{x},\bar{t})$ by the following function

$$w(\bar{x},\bar{t}) = \hat{\lambda}_1(\bar{t})v(\bar{x},\bar{t}), \text{ where } \ \hat{\lambda}_1(\bar{t}) = \exp\left\{\int_0^{\bar{t}} \bar{\lambda}_1(s)ds\right\}. \tag{3.10}$$

Then inverse problem (3.5)–(3.9) reduces to the following problem for the homogeneous heat equation:

$$w_{\bar{t}}(\bar{x},\bar{t}) = w_{\bar{x}\bar{x}}(\bar{x},\bar{t}), \quad (\bar{x},\bar{t}) \in G_\infty, \tag{3.11}$$

with the homogeneous boundary conditions

$$w(\bar{x},\bar{t})|_{\bar{x}=0} = 0, \quad w(\bar{x},\bar{t})|_{\bar{x}=\bar{t}} = 0,\ \bar{t}>0, \tag{3.12}$$

subject to the overspecification

$$w(\tilde{\mu}(\bar{t}),\bar{t}) = \hat{\lambda}_1(\bar{t})\tilde{E}(\bar{t}), \quad \tilde{E}(\bar{t}) \ge \delta > 0,\ \bar{t}>0. \tag{3.13}$$

3.4 On a nontrivial solution of homogeneous boundary value problem (3.11)–(3.12)

It follows from our previous results [1], [2], [3], [4], [8] that homogeneous boundary value problem (3.11)–(3.12) along with a trivial solution has a nontrivial solution up to a constant factor defined by the following formulas:

$$w(\bar{x},\bar{t}) = \frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}} \frac{\bar{x}}{(\bar{t}-\tau)^{3/2}} \exp\left\{-\frac{\bar{x}^2}{4(\bar{t}-\tau)}\right\}\nu(\tau)d\tau$$

$$+\frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}} \frac{\bar{x}-\tau}{(\bar{t}-\tau)^{3/2}} \exp\left\{-\frac{(\bar{x}-\tau)^2}{4(\bar{t}-\tau)}\right\}\varphi(\tau)d\tau, \tag{3.14}$$

$$\nu(\bar{t}) = \frac{1}{2\sqrt{\pi}}\int_0^{\bar{t}} \frac{\tau}{(\bar{t}-\tau)^{3/2}} \exp\left\{-\frac{\tau^2}{4(\bar{t}-\tau)}\right\}\varphi(\tau)d\tau, \tag{3.15}$$

where the function $\varphi(\bar{t})$ is defined according to the formulas:

$$\varphi(\bar{t}) = C\varphi_0(\bar{t}), \quad C = \text{const} \neq 0, \tag{3.16}$$

$$\varphi_0(\bar{t}) = \frac{1}{\sqrt{\bar{t}}}\exp\left\{-\frac{\bar{t}}{4}\right\} + \frac{\sqrt{\pi}}{2}\left[1+\text{erf}\left(\frac{\sqrt{\bar{t}}}{2}\right)\right], \tag{3.17}$$

where erf(x) is the error function. Moreover, the function $\varphi(\bar{t})$ belongs to the following class:

$$\theta(\bar{t})\varphi(\bar{t}) \in L_\infty(R_+), \text{ i.e. } \varphi(\bar{t}) \in L_\infty(R_+;\theta(\bar{t})), \tag{3.18}$$

where

$$\theta(\bar{t}) = \begin{cases} \sqrt{\bar{t}}\exp\{\bar{t}/4\}, & \text{if} \quad 0<\bar{t}\le T_1, \\ 1, & \text{if} \quad T_1<\bar{t}<+\infty, \end{cases} \tag{3.19}$$

and T_1 does not necessarily coincide with T.

Substituting $\nu(\bar{t})$, defined by (3.15) in (3.14), we obtain

$$w(\bar{x},\bar{t}) = w_+(\bar{x},\bar{t}) + w_-(\bar{x},\bar{t}),\ (\bar{x},\bar{t}) \in G_\infty, \tag{3.20}$$

where

$$w_+(\bar{x},\bar{t}) = \frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}} \frac{\bar{x}+\tau}{(\bar{t}-\tau)^{3/2}} \exp\left\{-\frac{(\bar{x}+\tau)^2}{4(\bar{t}-\tau)}\right\}\varphi(\tau)d\tau, \tag{3.21}$$

$$w_-(\bar{x},\bar{t}) = \frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}} \frac{\bar{x}-\tau}{(\bar{t}-\tau)^{3/2}} \exp\left\{-\frac{(\bar{x}-\tau)^2}{4(\bar{t}-\tau)}\right\}\varphi(\tau)d\tau. \tag{3.22}$$

3.5 Solution of inverse problem (3.2)–(3.4)

From (3.16) and (3.20)–(3.22) we obtain for the solution $w(\bar{x},\bar{t})$ of homogeneous boundary value problem (3.11)–(3.12) the following representation:

$$w(\bar{x},\bar{t}) = Cw_0(\bar{x},\bar{t}),$$

where

$$w_0(\bar{x},\bar{t}) = w_{0+}(\bar{x},\bar{t}) + w_{0-}(\bar{x},\bar{t}),\ (\bar{x},\bar{t}) \in G_\infty, \tag{3.23}$$

and

$$w_{0+}(\bar{x},\bar{t}) = \frac{1}{4\sqrt{\pi}} \int_0^{\bar{t}} \frac{\bar{x}+\tau}{(\bar{t}-\tau)^{3/2}} \exp\left\{-\frac{(\bar{x}+\tau)^2}{4(\bar{t}-\tau)}\right\} \varphi_0(\tau)d\tau, \tag{3.24}$$

$$w_{0-}(\bar{x},\bar{t}) = \frac{1}{4\sqrt{\pi}} \int_0^{\bar{t}} \frac{\bar{x}-\tau}{(\bar{t}-\tau)^{3/2}} \exp\left\{-\frac{(\bar{x}-\tau)^2}{4(\bar{t}-\tau)}\right\} \varphi_0(\tau)d\tau. \tag{3.25}$$

Further, using representation (3.23)–(3.25) for condition (3.13), we get:

$$w_0(\tilde{\mu}(\bar{t}),\bar{t}) = w_{0+}(\tilde{\mu}(\bar{t}),\bar{t}) + w_{0-}(\tilde{\mu}(\bar{t}),\bar{t})$$

$$= \frac{1}{4\sqrt{\pi}} \int_0^{\bar{t}} \frac{\tilde{\mu}(\bar{t})+\tau}{(\bar{t}-\tau)^{3/2}} \exp\left\{-\frac{(\tilde{\mu}(\bar{t})+\tau)^2}{4(\bar{t}-\tau)}\right\} \varphi_0(\tau)d\tau$$

$$+\frac{1}{4\sqrt{\pi}} \int_0^{\bar{t}} \frac{\tilde{\mu}(\bar{t})-\tau}{(\bar{t}-\tau)^{3/2}} \exp\left\{-\frac{(\tilde{\mu}(\bar{t})-\tau)^2}{4(\bar{t}-\tau)}\right\} \varphi_0(\tau)d\tau = \hat{\lambda}_{10}(\bar{t})\tilde{E}(\bar{t}),\ \bar{t}\in(0,\infty), \tag{3.26}$$

where $\hat{\lambda}_{10}(\bar{t}) = \hat{\lambda}_1(\bar{t})/C$.

From (3.10), (3.13), (3.26) and $w(\bar{x},\bar{t}) = Cw_0(\bar{x},\bar{t})$ we find the required coefficient

$$\bar{\lambda}_1(\bar{t}) = \frac{d\ln(\hat{\lambda}_1(\bar{t}))}{d\bar{t}} = \frac{(\hat{\lambda}_1(\bar{t}))'}{\hat{\lambda}_1(\bar{t})} = \frac{(C\hat{\lambda}_{10}(\bar{t}))'}{C\hat{\lambda}_{10}(\bar{t})} = \bar{\lambda}_{10}(\bar{t}), \tag{3.27}$$

where we have used the equality

$$\left(\frac{w(\tilde{\mu}(\bar{t}),\bar{t})}{\tilde{E}(\bar{t})}\right)' : \frac{w(\tilde{\mu}(\bar{t}),\bar{t})}{\tilde{E}(\bar{t})} = \left(\frac{w_0(\tilde{\mu}(\bar{t}),\bar{t})}{\tilde{E}(\bar{t})}\right)' : \frac{w_0(\tilde{\mu}(\bar{t}),\bar{t})}{\tilde{E}(\bar{t})},\ \bar{t}\in(0,\infty), \tag{3.28}$$

Thus, from (3.23)–(3.25), (3.26)–(3.28) we obtain the following theorem.

Theorem 3.1. *Inverse problem* (3.2)–(3.4) *has the following solution* $\{\bar{u}(\bar{x},\bar{t}),\ \bar{\lambda}(\bar{t})\}$*: the coefficient* $\bar{\lambda}(\bar{t}) = \bar{\lambda}_0(\bar{t})$ *is determined uniquely by formula* (3.27)–(3.28) *by restricting it to the finite interval* $(0,T_k)$ *and the solution* $\bar{u}(\bar{x},\bar{t})$ *is found by means of restricting the function:*

$$v(\bar{x},\bar{t}) = Cv_0(\bar{x},\bar{t}),\ \text{where}\ v_0(\bar{x},\bar{t}) = [\hat{\lambda}_{10}(\bar{t})]^{-1} w_0(\bar{x},\bar{t}),\ (\bar{x},\bar{t})\in G_\infty,\ C=\text{const}, \tag{3.29}$$

to the bounded triangle G_{T_k} *and* $w_0(\bar{x},\bar{t})$ *is defined by formula* (3.23).

Remark 2. According to formulas (3.23)–(3.25) the solution $w_0(\bar{x},\bar{t})$ is a nonnegative function. It should be noted that the function $\tilde{E}(\bar{t})$ in (3.13) is also a nonnegative function, since the left-hand side of equality (3.26) is nonnegative and the coefficient $\hat{\lambda}_{10}(\bar{t})$ is a nonnegative function.

In the following section we will show the boundedness of solution (3.23)–(3.25) of boundary value problem (3.11)–(3.12) and of condition (3.26) taking into account that the function $\varphi_0(\bar{t})$ defined by (3.17) belongs to class (3.18)–(3.19).

4 Estimates

4.1 Estimate of solution (3.23)–(3.25)

Let the function $\varphi_0(\bar{t})$ defined by (3.17) belong to class (3.18)–(3.19). The following statement is true.

Theorem 4.1. *The solution of problem* (3.11)–(3.12) *is bounded on* G_∞.

The proof of Theorem 4.1 will follow from Lemmas 4.1–4.2.

Lemma 4.1. *Let* $0<\bar t<T_1$ *and* $C=\|\theta(\bar t)\varphi_0(\bar t)\|_{L_\infty(0,T_1)}$. *Then the following estimate holds*

$$w_0(\bar x,\bar t)=w_{0+}(\bar x,\bar t)+w_{0-}(\bar x,\bar t)=\frac{1}{4\sqrt{\pi}}\int_0^{\bar t}\frac{\bar x+\tau}{(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar x+\tau)^2}{4(\bar t-\tau)}\right\}\varphi_0(\tau)d\tau$$

$$+\frac{1}{4\sqrt{\pi}}\int_0^{\bar t}\frac{\bar x-\tau}{(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar x-\tau)^2}{4(\bar t-\tau)}\right\}\varphi_0(\tau)d\tau\le C\frac{\sqrt{\pi}}{4}. \tag{4.1}$$

Proof.

$$w_{0+}(\bar x,\bar t)=\frac{1}{4\sqrt{\pi}}\int_0^{\bar t}\frac{\bar x+\tau}{(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar x+\tau)^2}{4(\bar t-\tau)}\right\}\varphi_0(\tau)d\tau$$

$$\le\frac{C}{4\sqrt{\pi}}\int_0^{\bar t}\frac{\bar x+\tau}{\sqrt{\tau}(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar x+\tau)^2}{4(\bar t-\tau)}-\frac{\tau}{4}\right\}d\tau\equiv CI_{1+}(\bar x,\bar t),$$

$$w_{0-}(\bar x,\bar t)=\frac{1}{4\sqrt{\pi}}\int_0^{\bar t}\frac{\bar x-\tau}{(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar x-\tau)^2}{4(\bar t-\tau)}\right\}\varphi_0(\tau)d\tau$$

$$\le\frac{C}{4\sqrt{\pi}}\int_0^{\bar t}\frac{\bar x-\tau}{\sqrt{\tau}(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar x-\tau)^2}{4(\bar t-\tau)}-\frac{\tau}{4}\right\}d\tau\equiv CI_{1-}(\bar x,\bar t).$$

We transform the kernels in integrals $I_{1+}(\bar x,\bar t)$ and $I_{1-}(\bar x,\bar t)$. We have

$$\frac{\bar x+\tau}{\sqrt{\tau}(\bar t-\tau)^{3/2}}=\frac{\bar x+\bar t-(\bar t-\tau)}{\sqrt{\tau}(\bar t-\tau)^{3/2}}=\frac{\bar x+\bar t}{\sqrt{\tau}(\bar t-\tau)^{3/2}}-\frac{1}{\sqrt{\tau(\bar t-\tau)}},$$

$$\frac{\bar x-\tau}{\sqrt{\tau}(\bar t-\tau)^{3/2}}=\frac{\bar x-\bar t+(\bar t-\tau)}{\sqrt{\tau}(\bar t-\tau)^{3/2}}=\frac{\bar x-\bar t}{\sqrt{\tau}(\bar t-\tau)^{3/2}}+\frac{1}{\sqrt{\tau(\bar t-\tau)}},$$

$$-\frac{(\bar x\pm\tau)^2}{4(\bar t-\tau)}=-\frac{[\bar x\pm\bar t\mp(\bar t-\tau)]^2}{4(\bar t-\tau)}=-\frac{(\bar x\pm\bar t)^2}{4(\bar t-\tau)}+\frac{\pm2\bar x+\bar t}{4}+\frac{\tau}{4},$$

therefore

$$I_{1+}(\bar x,\bar t)\le\frac{1}{4\sqrt{\pi}}\exp\left\{\frac{2\bar x+\bar t}{4}\right\}\int_0^{\bar t}\frac{\bar x+\bar t}{\sqrt{\tau}(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar x+\bar t)^2}{4(\bar t-\tau)}\right\}d\tau$$

$$+\frac{1}{4\sqrt{\pi}}\int_0^{\bar t}\frac{1}{\sqrt{\tau(\bar t-\tau)}}\exp\left\{-\frac{(\bar x+\tau)^2}{4(\bar t-\tau)}-\frac{\tau}{4}\right\}d\tau\equiv I^1_{1+}(\bar x,\bar t)+I^2_{1+}(\bar x,\bar t), \tag{4.2}$$

$$I_{1-}(\bar x,\bar t)\le\frac{1}{4\sqrt{\pi}}\exp\left\{\frac{-2\bar x+\bar t}{4}\right\}\int_0^{\bar t}\frac{\bar x-\bar t}{\sqrt{\tau}(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar x-\bar t)^2}{4(\bar t-\tau)}\right\}d\tau$$

$$+\frac{1}{4\sqrt{\pi}}\int_0^{\bar t}\frac{1}{\sqrt{\tau(\bar t-\tau)}}\exp\left\{-\frac{(\bar x-\tau)^2}{4(\bar t-\tau)}-\frac{\tau}{4}\right\}d\tau\equiv I^1_{1-}(\bar x,\bar t)+I^2_{1-}(\bar x,\bar t). \tag{4.3}$$

First, we calculate integrals $I^1_{1+}(\bar x,\bar t)$ and $I^1_{1-}(\bar x,\bar t)$. In order to do this we introduce the following substitutions $2z_+=(\bar x+\bar t)(\bar t-\tau)^{-1/2}$, $z^2_{1+}=z^2_+-(\bar x+\bar t)^2(4\bar t)^{-1}$ and $2z_-=(\bar t-\bar x)(\bar t-\tau)^{-1/2}$, $z^2_{1-}=z^2_--(\bar t-\bar x)^2(4\bar t)^{-1}$. Then we obtain

$$I^1_{1+}(\bar x,\bar t)=\frac{2\exp\{-\frac{\bar x^2}{4\bar t}\}}{\sqrt{\pi\bar t}}\int_0^\infty\exp\{-z^2_{1+}\}dz_{1+}=\frac{\exp\{-\frac{\bar x^2}{4\bar t}\}}{\sqrt{\bar t}},$$

$$I^1_{1-}(\bar{x},\bar{t}) = -\frac{2\exp\{-\frac{\bar{x}^2}{4\bar{t}}\}}{\sqrt{\pi\bar{t}}}\int_0^\infty \exp\{-z^2_{1-}\}dz_{1-} = -\frac{\exp\{-\frac{\bar{x}^2}{4\bar{t}}\}}{\sqrt{\bar{t}}}.$$

From these relations by (3.23)–(3.25) we have

$$I^1_{1+}(\bar{x},\bar{t}) + I^1_{1-}(\bar{x},\bar{t}) = 0. \tag{4.4}$$

For the second integrals $I^2_{1+}(\bar{x},\bar{t})$ and $I^2_{1-}(\bar{x},\bar{t})$ in formulas (4.2) and (4.3) we have:

$$I^2_{1\pm}(\bar{x},\bar{t}) = \frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}} \frac{1}{\sqrt{\tau(\bar{t}-\tau)}}\exp\left\{-\frac{(\bar{x}\pm\tau)^2}{4(\bar{t}-\tau)} - \frac{\tau}{4}\right\}d\tau$$

$$\leq \frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}} \frac{1}{\sqrt{\tau(\bar{t}-\tau)}}d\tau = \frac{\sqrt{\pi}}{4}.$$

□

Lemma 4.2. *Let $T_1 < \bar{t} < \infty$ and $C = \|\theta(\bar{t})\varphi_0(\bar{t})\|_{L_\infty(T_1,\infty)}$. Then the following estimate holds*

$$w_{0\pm}(\bar{x},\bar{t}) = \frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}} \frac{\bar{x}\pm\tau}{(\bar{t}-\tau)^{3/2}}\exp\left\{-\frac{(\bar{x}\pm\tau)^2}{4(\bar{t}-\tau)}\right\}\varphi_0(\tau)d\tau \leq C. \tag{4.5}$$

Proof. As in the proof of Lemma 4.1, using similar transformations of the independent variables, we obtain

$$w_{0\pm}(\bar{x},\bar{t}) \leq \frac{C}{4\sqrt{\pi}}\int_0^{\bar{t}} \frac{|\bar{x}\pm\tau|}{(\bar{t}-\tau)^{3/2}}\exp\left\{-\frac{(\bar{x}\pm\tau)^2}{4(\bar{t}-\tau)}\right\}d\tau$$

$$\leq \frac{C}{4\sqrt{\pi}}\exp\left\{\frac{\bar{x}\pm\bar{t}}{2}\right\}\int_0^{\bar{t}} \frac{|\bar{x}\pm\bar{t}|}{(\bar{t}-\tau)^{3/2}}\exp\left\{-\frac{(\bar{x}\pm\bar{t})^2}{4(\bar{t}-\tau)} - \frac{\bar{t}-\tau}{4}\right\}d\tau$$

$$+\frac{C}{4\sqrt{\pi}}\exp\left\{\frac{\bar{x}\pm\bar{t}}{2}\right\}\int_0^{\bar{t}} \frac{1}{\sqrt{\bar{t}-\tau}}\exp\left\{-\frac{(\bar{x}\pm\bar{t})^2}{4(\bar{t}-\tau)} - \frac{\bar{t}-\tau}{4}\right\}d\tau$$

$$\equiv C\left[I^1_{2\pm}(\bar{x},\bar{t}) + I^2_{2\pm}(\bar{x},\bar{t})\right]. \tag{4.6}$$

Using the substitution $2z_\pm = |\bar{x}\pm\bar{t}|(\bar{t}-\tau)^{-1/2}$, for the first integral we get:

$$I^1_{2\pm}(\bar{x},\bar{t}) = \frac{1}{\sqrt{\pi}}\exp\left\{\frac{\bar{x}\pm\bar{t}}{2}\right\}\int_{\frac{|\bar{x}\pm\bar{t}|}{\sqrt{\bar{t}}}}^\infty \exp\left\{-z^2 - \frac{(\bar{x}\pm\bar{t})^2}{16z^2}\right\}dz$$

$$\leq \frac{1}{\sqrt{\pi}}\exp\left\{\frac{\bar{x}\pm\bar{t}}{2}\right\}\int_0^\infty \exp\left\{-z^2 - \frac{(\bar{x}\pm\bar{t})^2}{16z^2}\right\}dz = \frac{1}{2}. \tag{4.7}$$

Here we used the well-known equality ([6], formula 3.325)

$$\int_0^\infty \exp\left\{-\mu x^2 - \frac{\eta}{x^2}\right\}dx = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\mu}}\exp\left\{-2\sqrt{\mu\eta}\right\}. \tag{4.8}$$

For the second integral $I^2_{2\pm}(\bar{x},\bar{t})$ we have

$$I^2_{2\pm}(\bar{x},\bar{t}) = \frac{1}{4\sqrt{\pi}}\exp\left\{\frac{\bar{x}\pm\bar{t}}{2}\right\}\int_0^{\bar{t}} \frac{1}{\sqrt{\bar{t}-\tau}}\exp\left\{-\frac{(\bar{x}\pm\bar{t})^2}{4(\bar{t}-\tau)} - \frac{\bar{t}-\tau}{4}\right\}d\tau$$

$$= \left(z_\pm = \frac{2\sqrt{\bar{t}-\tau}}{|\bar{x}\pm\bar{t}|}\right)$$

$$= \frac{|\bar{x}\pm\bar{t}|}{4\sqrt{\pi}}\exp\left\{\frac{\bar{x}\pm\bar{t}}{2}\right\}\int_0^{\frac{2\sqrt{\bar{t}}}{|\bar{x}\pm\bar{t}|}}\exp\left\{-\frac{(\bar{x}\pm\bar{t})^2}{16}z_\pm^2-\frac{1}{z_\pm^2}\right\}dz_\pm$$
$$\le \frac{|\bar{x}\pm\bar{t}|}{4\sqrt{\pi}}\exp\left\{\frac{\bar{x}\pm\bar{t}}{2}\right\}\int_0^{\infty}\exp\left\{-\frac{(\bar{x}\pm\bar{t})^2}{16}z_\pm^2-\frac{1}{z_\pm^2}\right\}dz_\pm=\frac{1}{2}, \tag{4.9}$$

where in (4.7) equality (4.8) was used. □

From estimates (4.1) and (4.5) established in Lemmas 4.1–4.2 we obtain the assertion of Theorem 4.1.

4.2 Estimate of integrals in (3.26)

In this subsection, we will show that the integrals in the left-hand side of formula (3.26) are bounded, taking into account that function $\varphi_0(\bar{t})$ defined by (3.17) belongs to class (3.18)–(3.19).

Theorem 4.2. *The integrals in* (3.26) *are bounded functions on the semi-axis* R_+.

The proof of Theorem 4.2 will follow from Lemmas 4.3–4.4.

Lemma 4.3. *Let* $0<\bar{t}<T_1$ *and* $C=\|\theta(t)\varphi_0(\bar{t})\|_{L_\infty(0,T_1)}$. *Then the following estimate holds*

$$w_{0\pm}(\tilde{\mu}(\bar{t}),\bar{t})=\frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}}\frac{\tilde{\mu}(\bar{t})\pm\tau}{(\bar{t}-\tau)^{3/2}}\exp\left\{-\frac{(\tilde{\mu}(\bar{t})\pm\tau)^2}{4(\bar{t}-\tau)}\right\}\varphi_0(\tau)d\tau\le C\frac{\sqrt{\pi}}{4}. \tag{4.10}$$

Lemma 4.4. *Let* $T_1<\bar{t}<\infty$ *and* $C=\|\theta(\bar{t})\varphi_0(\bar{t})\|_{L_\infty(T_1,\infty)}$. *Then the following estimate holds*

$$w_{0\pm}(\tilde{\mu}(\bar{t}),\bar{t})=\frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}}\frac{\tilde{\mu}(\bar{t})\pm\tau}{(\bar{t}-\tau)^{3/2}}\exp\left\{-\frac{(\tilde{\mu}(\bar{t})\pm\tau)^2}{4(\bar{t}-\tau)}\right\}\varphi_0(\tau)d\tau\le C. \tag{4.11}$$

Proofs of Lemmas 4.3–4.4 can be obtained similarly to the proofs of Lemmas 4.1–4.2 by replacing $\bar{x}$ with $\mu(\bar{t})$.

5 Asymptotics of integrals in the left-hand side of formula (3.26) as $\bar{t}\to 0^+$

In Theorem 4.2 we have established the boundedness of the integrals in the left-hand side of formula (3.26) for $\bar{t}\in R_+$. In this section, we want to give an answer to the question: what is the asymptotic behaviour of the integrals as $\bar{t}\to 0^+$? This is important for determining the classes of functions to which belong the solutions $\{\bar{u}(\bar{x},\bar{t}),\ (\bar{x},\bar{t})\in G_{T_k};\ \bar{\lambda}(\bar{t}),\ \bar{t}\in(0,T_k)\}$ of inverse problem (3.2)–(3.4).

Lemma 5.1. *In the case, in which the asymptotics of the function* $\tilde{\mu}(\bar{t})$ *is comparable with the function* $\bar{t}/2$, *the integrals in formula* (3.26) *tend to constants as* $\bar{t}\to 0^+$.

Proof. For this purpose, we will split each integral in (3.26) into three integrals taking into account formula (3.17):

$$w_{0+}^1(\bar{t}/2,\bar{t})=\frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}}\frac{\bar{t}/2+\tau}{\sqrt{\tau}(\bar{t}-\tau)^{3/2}}\exp\left\{-\frac{(\bar{t}/2+\tau)^2}{4(\bar{t}-\tau)}-\frac{\tau}{4}\right\}d\tau, \tag{5.1}$$

$$w_{0+}^2(\bar{t}/2,\bar{t})=\frac{1}{8}\int_0^{\bar{t}}\frac{\bar{t}/2+\tau}{(\bar{t}-\tau)^{3/2}}\exp\left\{-\frac{(\bar{t}/2+\tau)^2}{4(\bar{t}-\tau)}\right\}d\tau, \tag{5.2}$$

$$w_{0+}^3(\bar{t}/2,\bar{t})=\frac{1}{4\sqrt{\pi}}\int_0^{\bar{t}}\frac{\bar{t}/2+\tau}{(\bar{t}-\tau)^{3/2}}\exp\left\{-\frac{(\bar{t}/2+\tau)^2}{4(\bar{t}-\tau)}\right\}\mathrm{erf}\left(\frac{\sqrt{\tau}}{2}\right)d\tau, \tag{5.3}$$

$$w^1_{0-}(\bar t/2,\bar t)=\frac{1}{4\sqrt{\pi}}\int_0^{\bar t}\frac{\bar t/2-\tau}{\sqrt{\tau}(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar t/2-\tau)^2}{4(\bar t-\tau)}-\frac{\tau}{4}\right\}d\tau, \tag{5.4}$$

$$w^2_{0-}(\bar t/2,\bar t)=\frac{1}{8}\int_0^{\bar t}\frac{\bar t/2-\tau}{(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar t/2-\tau)^2}{4(\bar t-\tau)}\right\}d\tau, \tag{5.5}$$

$$w^3_{0-}(\bar t/2,\bar t)=\frac{1}{4\sqrt{\pi}}\int_0^{\bar t}\frac{\bar t/2-\tau}{(\bar t-\tau)^{3/2}}\exp\left\{-\frac{(\bar t/2-\tau)^2}{4(\bar t-\tau)}\right\}\operatorname{erf}\left(\frac{\sqrt{\tau}}{2}\right)d\tau. \tag{5.6}$$

In the integrals $w^1_{0+}(\bar t/2,\bar t)$ and $w^1_{0-}(\bar t/2,\bar t)$, making the transformations

$$\frac{\bar t}{2}+\bar t-(\bar t-\tau)=\frac{3}{2}\bar t-(\bar t-\tau), \tag{5.7}$$

$$\frac{\bar t}{2}-\bar t+(\bar t-\tau)=-\frac{1}{2}\bar t+(\bar t-\tau), \tag{5.8}$$

we obtain

$$w^1_{0+}(\bar t/2,\bar t)=\frac{3\bar t\exp\left\{\frac{\bar t}{2}\right\}}{8\sqrt{\pi}}\int\limits_0^{\bar t}\frac{1}{\sqrt{\tau}(\bar t-\tau)^{3/2}}\exp\left\{-\frac{9\bar t^2}{16(\bar t-\tau)}\right\}$$

$$-\frac{1}{4\sqrt{\pi}}\int_0^{\bar t}\frac{1}{\sqrt{\tau(\bar t-\tau)}}\exp\left\{-\frac{(\bar t/2+\tau)^2}{4(\bar t-\tau)}-\frac{\tau}{4}\right\}d\tau=w^{11}_{0+}(\bar t/2,\bar t)-w^{12}_{0+}(\bar t/2,\bar t),$$

$$w^1_{0-}(\bar t/2,\bar t)=-\frac{\bar t}{8\sqrt{\pi}}\int\limits_0^{\bar t}\frac{1}{\sqrt{\tau}(\bar t-\tau)^{3/2}}\exp\left\{-\frac{\bar t^2}{16(\bar t-\tau)}\right\}$$

$$+\frac{1}{4\sqrt{\pi}}\int_0^{\bar t}\frac{1}{\sqrt{\tau(\bar t-\tau)}}\exp\left\{-\frac{(\bar t/2-\tau)^2}{4(\bar t-\tau)}-\frac{\tau}{4}\right\}d\tau=-w^{11}_{0-}(\bar t/2,\bar t)+w^{12}_{0-}(\bar t/2,\bar t). \tag{5.9}$$

In the integral $w^{11}_{0+}(\bar t/2,\bar t)$ using the substitutions

$$z=\frac{3\bar t}{4\sqrt{\bar t-\tau}},\qquad z_1=\sqrt{z^2-\frac{9\bar t}{16}}$$

we obtain:

$$w^{11}_{0+}(\bar t/2,\bar t)=\frac{\exp\left\{-\frac{\bar t}{16}\right\}}{2\sqrt{\bar t}}. \tag{5.10}$$

In the integral $w^{11}_{0-}(\bar t/2,\bar t)$ using the following substitutions

$$z=\frac{\bar t}{4\sqrt{\bar t-\tau}},\qquad z_1=\sqrt{z^2-\frac{\bar t}{16}},$$

we get:

$$w^{11}_{0-}(\bar t/2,\bar t)=\frac{\exp\left\{-\frac{\bar t}{16}\right\}}{2\sqrt{\bar t}}. \tag{5.11}$$

By relations (5.10) and (5.11) we have

$$w^{11}_{0+}(\bar t/2,\bar t)-w^{11}_{0-}(\bar t/2,\bar t)=0.$$

Further

$$-w^{12}_{0+}(\bar t/2,\bar t)+w^{12}_{0-}(\bar t/2,\bar t)$$

$$= \frac{1}{4\sqrt{\pi}} \int_0^{\bar t} \frac{1}{\sqrt{\tau(\bar t-\tau)}} \exp\left\{-\frac{\tau}{4}\right\} \exp\left\{-\frac{(\bar t-2\tau)^2}{16(\bar t-\tau)}\right\} \left(1-\exp\left\{-\frac{\bar t\tau}{2(\bar t-\tau)}\right\}\right) d\tau.$$

It is easy to notice that when $\bar t$ tends to zero, the last expression also tends to zero.

For integrals $w^2_{0+}(\bar t/2,\bar t), w^3_{0+}(\bar t/2,\bar t), w^2_{0-}(\bar t/2,\bar t), w^3_{0-}(\bar t/2,\bar t)$ it is enough to consider integrals $w^2_{0+}(\bar t/2,\bar t)$ and $w^2_{0-}(\bar t/2,\bar t)$, since singularities of integrals $w^3_{0+}(\bar t/2,\bar t)$ and $w^3_{0-}(\bar t/2,\bar t)$ will not exceed singularities of integrals $w^2_{0+}(\bar t/2,\bar t)$ and $w^2_{0-}(\bar t/2,\bar t)$.

In integral (5.2) by using transformation (5.7) we obtain:

$$w^2_{0+}(\bar t/2,\bar t) = \frac{3\bar t\exp\left\{\frac{\bar t}{2}\right\}}{16} \int_0^{\bar t} \frac{1}{(\bar t-\tau)^{3/2}} \exp\left\{-\frac{9\bar t^2}{16(\bar t-\tau)}+\frac{\tau}{4}\right\}$$

$$-\frac{\exp\left\{\frac{\bar t}{2}\right\}}{8} \int_0^{\bar t} \frac{1}{\sqrt{\bar t-\tau}} \exp\left\{-\frac{9\bar t^2}{16(\bar t-\tau)}+\frac{\tau}{4}\right\} d\tau = \left(z=\frac{3\bar t}{4\sqrt{\bar t-\tau}}\right)$$

$$= \frac{\exp\left\{\frac{3\bar t}{4}\right\}}{2} \int_{\frac{3\sqrt{\bar t}}{4}}^{\infty} \exp\left\{-z^2-\frac{9\bar t^2}{64z^2}\right\} dz$$

$$-\frac{3\bar t\exp\left\{\frac{3\bar t}{4}\right\}}{16} \int_{\frac{3\sqrt{\bar t}}{4}}^{\infty} \frac{1}{z^2} \exp\left\{-z^2-\frac{9\bar t^2}{64z^2}\right\} dz = w^{21}_{0+}(\bar t/2,\bar t) - w^{22}_{0+}(\bar t/2,\bar t). \tag{5.12}$$

In the second integral in formula (5.12) replacing $z_1 = z^{-1}$, we get:

$$w^{21}_{0+}(\bar t/2,\bar t) - w^{22}_{0+}(\bar t/2,\bar t) = \frac{\exp\left\{\frac{3\bar t}{4}\right\}}{2} \int_{\frac{3\sqrt{\bar t}}{4}}^{\infty} \exp\left\{-z^2-\frac{9\bar t^2}{64z^2}\right\} dz$$

$$-\frac{3\bar t\exp\left\{\frac{3\bar t}{4}\right\}}{16} \int_0^{\frac{4}{3\sqrt{\bar t}}} \exp\left\{-\frac{9\bar t^2}{64}z^2-\frac{1}{z^2}\right\} dz.$$

Then

$$w^2_{0+}(\bar t/2,\bar t) = w^{21}_{0+}(\bar t/2,\bar t) - w^{22}_{0+}(\bar t/2,\bar t) = \frac{\exp\left\{\frac{3\bar t}{2}\right\}\sqrt{\pi}}{4}\left(1-\operatorname{erf}\left(\frac{5\sqrt{\bar t}}{4}\right)\right). \tag{5.13}$$

In integral (5.5) by using transformation (5.8) we obtain:

$$w^2_{0-}(\bar t/2,\bar t) = -\frac{\bar t}{16} \int_0^{\bar t} \frac{1}{(\bar t-\tau)^{3/2}} \exp\left\{-\frac{\bar t^2}{16(\bar t-\tau)}+\frac{\tau}{4}\right\}$$

$$+\frac{1}{8} \int_0^{\bar t} \frac{1}{\sqrt{\bar t-\tau}} \exp\left\{-\frac{\bar t^2}{16(\bar t-\tau)}+\frac{\tau}{4}\right\} d\tau = \left(z=\frac{\bar t}{4\sqrt{\bar t-\tau}}\right)$$

$$= -\frac{\exp\left\{\frac{\bar t}{4}\right\}}{2} \int_{\frac{\sqrt{\bar t}}{4}}^{\infty} \exp\left\{-z^2-\frac{\bar t^2}{64z^2}\right\} dz$$

$$+\frac{\bar{t}\exp\left\{\frac{\bar{t}}{4}\right\}}{16}\int\limits_{\frac{\sqrt{\bar{t}}}{4}}^{\infty}\frac{1}{z^2}\exp\left\{-z^2-\frac{\bar{t}^2}{64z^2}\right\}dz=-w_{0-}^{21}(\bar{t}/2,\bar{t})+w_{0-}^{22}(\bar{t}/2,\bar{t}). \quad (5.14)$$

In the second integral in formula (5.14) replacing $z_1=z^{-1}$, we get:

$$-w_{0-}^{21}(\bar{t}/2,\bar{t})+w_{0-}^{22}(\bar{t}/2,\bar{t})=-\frac{\exp\left\{\frac{\bar{t}}{4}\right\}}{2}\int\limits_{\frac{\sqrt{\bar{t}}}{4}}^{\infty}\exp\left\{-z^2-\frac{\bar{t}^2}{64z^2}\right\}dz$$

$$-\frac{\bar{t}\exp\left\{\frac{\bar{t}}{4}\right\}}{16}\int\limits_{0}^{\frac{4}{\sqrt{\bar{t}}}}\exp\left\{-\frac{\bar{t}^2}{64}z^2-\frac{1}{z^2}\right\}dz.$$

Then

$$w_{0-}^{2}(\bar{t}/2,\bar{t})=-w_{0-}^{21}(\bar{t}/2,\bar{t})+w_{0-}^{22}(\bar{t}/2,\bar{t})$$
$$=\frac{\exp\left\{\frac{\bar{t}}{2}\right\}\sqrt{\pi}}{4}\left(\exp\{\bar{t}\}-1-\exp\{\bar{t}\}\operatorname{erf}\left(\frac{5\sqrt{\bar{t}}}{4}\right)+\operatorname{erf}\left(\frac{3\sqrt{\bar{t}}}{4}\right)\right). \quad (5.15)$$

Thus, taking into account the asymptotics of (5.13) and (5.15) we see, that the integrals in (3.26) tend to constants. □

Theorem 5.1. *The solution* $\{\bar{u}(\bar{x},\bar{t}),\lambda(\bar{t})\}$ *of inverse problem* (3.2)–(3.4) *has no singularity as* $\bar{t}\to 0^+$.

Proof. According to the properties of the given function $\tilde{E}(\bar{t})$ in overspecification (3.8) and also from the statement of Lemma 5.1 and equality (3.26) we obtain that in case when the asymptotics of the function $\tilde{\mu}(\bar{t})$ is comparable to the asymptotics of the function $\bar{t}/2$, then asymptotic behaviour of the function $\hat{\lambda}_1(\bar{t})$ for small values of the variable $\bar{t}$ does not depend of the variable $\bar{t}$. Hence the coefficient $\bar{\lambda}_1(\bar{t})$ has no singularity as $\bar{t}\to 0^+$. Since

$$v(\bar{x},\bar{t})=\frac{w(\bar{x},\bar{t})}{\hat{\lambda}_1(\bar{t})}$$

the solution of boundary value problem (3.2)–(3.4) also has no singularity. Indeed, it follows from the statement of Theorem 4.1 that the solution $w(\bar{x},\bar{t})$ of boundary value problem (3.11)–(3.12) has no singularity. □

6 Solution of original inverse problem (2.1)–(2.3)

For the solution $\{\bar{u}(\bar{x},\bar{t}),\ \bar{\lambda}(\bar{t})\}$ of auxiliary inverse problem (3.2)–(3.4) which was found by using Theorem 3.1 and by applying transformation (3.1) we get the solution $\{u(x,t),\ \lambda(t)\}$ of original inverse problem (2.1)–(2.3).

7 Conclusion

In this work we consider an inverse problem for the heat equation in a degenerate angular domain when the moving part of the boundary changes linearly. We have shown that the inverse problem for the homogeneous heat equation with completely homogeneous boundary conditions has a nontrivial solution $\{u(x,t),\ \lambda(t)\}$ consistent with the additional condition. Moreover, the solution of the considered inverse problem was found in an explicit form and it was proved that the required coefficient is determined uniquely. It has also been shown that the obtained nontrivial solution $\{u(x,t),\lambda(t)\}$ of inverse problem (2.1)–(2.3) has no singularity as $t\to 0^+$.

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Muvasharkhan Tanabaevich Jenaliyev
Institute of Mathematics and Mathematical Modeling
125 Pushkin St,
050010 Almaty, Kazakhstan
E-mail: muvasharkhan@gmail.com

Murat Ibraevich Ramazanov
E.A. Buketov Karaganda State University
28 Universitetskaya St,
100028 Karaganda, Kazakhstan
E-mail: ramamur@mail.ru

Madi Gabidenovich Yergaliyev
Al-Farabi Kazakh National University,
Institute of Mathematics and Mathematical Modeling
71 al-Farabi Ave, 125 Pushkin St,
050040 Almaty, Kazakhstan, 050010 Almaty, Kazakhstan
E-mail: ergaliev.madi.g@gmail.com

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