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E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 562
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

VLADIMIR MIKHAILOVICH FILIPPOV

(to the 70th birthday)

Vladimir Mikhailovich Filippov was born on 15 April 1951 in the city of Uryupinsk, Stalingrad Region of the USSR. In 1973 he graduated with honors from the Faculty of Physics and Mathematics and Natural Sciences of the Patrice Lumumba University of Peoples' Friendship in the specialty "Mathematics". In 1973-1975 he is a postgraduate student of the University; in 1976-1979 - Chairman of the Young Scientists' Council; in 1980-1987 - Head of the Research Department and the Scientific Department; in 1983-1984 - scientific work at the Free University of Brussels (Belgium); in 1985-2000 - Head of the Mathematical Analysis Department; from 2000 to the present - Head of the Comparative Educational Policy Department; in 1989–1993 - Dean of the Faculty of Physics, Mathematics and Natural Sciences; in 1993–1998 - Rector of the Peoples' Friendship University of Russia; in 1998-2004 - Minister of General and Professional Education, Minister of Education of the Russian Federation; in 2004-2005 - Assistant to the Chairman of the Government of the Russian Federation (in the field of education and culture); from 2005 to May 2020- Rector of the Peoples' Friendship University of Russia, since May 2020 - President of the Peoples' Friendship University of Russia, since 2013 - Chairman of the Higher Attestation Commission of the Ministry of Science and Higher Education of the Russian Federation.

In 1980, he defended his PhD thesis in the V.A. Steklov Mathematical Institute of Academy of Sciences of the USSR on specialty 01.01.01 - mathematical analysis (supervisor - a corresponding member of the Academy of Sciences of the USSR, Professor L.D. Kudryavtsev), and in 1986 in the same Institute he defended his doctoral thesis "Quasi-classical solutions of inverse problems of the calculus of variations in non-Eulerian classes of functionals and function spaces". In 1987, he was awarded the academic title of a professor.

V.M. Filippov is an academician of the Russian Academy of Education; a foreign member of the Ukrainian Academy of Pedagogical Sciences; President of the UNESCO International Organizing Committee for the World Conference on Higher Education (2007-2009); Vice-President of the Eurasian Association of Universities; a member of the Presidium of the Rectors' Council of Moscow and Moscow Region Universities, of the Governing Board of the Institute of Information Technologies in Education (UNESCO), of the Supervisory Board of the European Higher Education Center of UNESCO (Bucharest, Romania),

Research interests: variational methods; non-potential operators; inverse problems of the calculus of variations; function spaces.

In his Ph.D thesis, V.M. Filippov solved a long standing problem of constructing an integral extremal variational principle for the heat equation. In his further research he developed a general theory of constructing extremal variational principles for broad classes of differential equations with non-potential (in classical understanding) operators. He showed that all previous attempts to construct variational principles for non-potential operators "failed" because mathematicians and mechanics from the time of L. Euler and J. Lagrange were limited in their research by functionals of the type Euler - Lagrange. Extending the classes of functionals, V.M. Filippov introduced a new scale of function spaces that generalize the Sobolev spaces, and thus significantly expanded the scope of the variational methods. In 1984, famous physicist, a Nobel Prize winner I.R. Prigogine presented the report of V.M. Filippov to the Royal Academy of Sciences of Belgium. Results of V.M. Filippov's variational principles for non-potential operators are quite fully represented in some of his and his colleagues' monographs.

Honors: Honorary Legion (France), "Commander" (Belgium), Crown of the King (Belgium); in Russia - orders "Friendship", "Honor", "For Service to the Fatherland" III and IV degrees; Prize of the President of the Russian Federation in the field of education; Prize of the Governement of the Russian Federation in the field of education; Gratitude of the President of the Russian Federation; "For Merits in the Social and Labor Sphere of the Russian Federation", "For Merits in the Development of the Olympic Movement in Russia", "For Strengthening the Combat Commonwealth; and a number of other medals, prizes and awards.

He is an author of more than 270 scientific and scientific-methodical works, including 32 monographs, 2 of which were translated and published in the United States by the American Mathematical Society.

V.M. Filippov meets his 70th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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GENERALIZED CAUCHY PRODUCT AND RELATED OPERATORS ON $\ell^p(\beta)$

Y. Estaremi

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Key words: Cauchy product, extended eigenvalue, multiplication operators.

AMS Mathematics Subject Classification: 47B37.

Abstract. In this paper first we give some necessary and sufficient conditions for the boundedness of the multiplication operator $D_f = M_{\circledast,f}$ with respect to the generalized Cauchy product $\circledast$, on $\ell^p(\beta)$. Also, under certain conditions, we give the characterization of the extended eigenvalues and extended eigenvectors of the multiplication operator $M_{\circledast,z}$ on $\ell^p(\beta)$. Finally we describe the commutants of $M_{\circledast,z}$ and consequently the collection of all hyperinvariant subspaces of $M_{\circledast,z}$.

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1 Introduction

Let X be a separable Banach space. We denote by $B(X)$ the set of all bounded linear operators on X. A complex number λ is called an extended eigenvalue of $T \in B(X)$ if there exists a non-zero operator $A \in B(X)$ such that $\lambda AT = TA$ (see [2]). We know that when $\lambda = 1$, the equation $\lambda AT = TA$ can be used to obtain information about the operator T based on the properties of the operator A. For $A \in B(X)$ a (closed, linear) subspace of X is a nontrivial invariant subspace (n.i.s.) for A if it is neither X nor $\{0\}$ and is invariant under A. This space is hyperinvariant for A if it is invariant for every operator in $\{A\}'$, the commutant of A. More generally, a subspace is defined to be invariant for a set of operators if it is invariant for each member of that set. For more information on the extended eigenvalue, commutant and invariant subspaces one can see [1, 2, 3, 4, 5, 6, 7]. In this paper we will characterize an extended eigenvalue of a special operator on a Banach space of formal power series.

Let $\{\beta(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 \le p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}$ such that

$$\|f\|_{\beta}^{p} = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

We shall use the formal notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ independently of whether or not the series converges for any complex values z. Such series are called formal power series. Throughout this paper, we consider the space $\ell^p(\beta)$ to be defined as

$$\ell^p(\beta) = \{f : f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n, \ \|f\|_{\beta} < \infty\}.$$

For more details about $\ell^p(\beta)$ one can see [8, 9]. Let $\{\delta_n\}_{n=0}^\infty$ be a sequence of positive numbers with $\delta_0 = 1$. Given arbitrary two functions $f(z) = \sum_{n=0}^\infty \hat{f}(n)z^n$ and $g(z) = \sum_{n=0}^\infty \hat{g}(n)z^n$ belonging to the space $\ell^p(\beta)$, we define the following generalized Cauchy product series

$$(f \circledast g)(z) = \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{\delta_{n+m}}{\delta_n \delta_m} \hat{f}(n)\hat{g}(m)z^{n+m}. \tag{1.1}$$

In this paper first we show that $\ell^p(\beta)$ is a unital commutative Banach algebra with respect to $\circledast$. Then under certain conditions, we describe the extended eigenvalues and extended eigenvectors of the multiplication operator $M_{\circledast,z}$ on $\ell^p(\beta)$. Finally we describe the commutants of $M_{\circledast,z}$ and consequently the collection of all hyperinvariant subspaces of $M_{\circledast,z}$.

2 Main results

Let X be a separable Banach space. An operator $T \in B(X)$ is called a well splitting operator in X if for every $x \in X$ there exists a bounded linear operator B_x such that $T^n x = B_x y_n$ for every $n \in \mathbb{N} \cup \{0\}$ and for some complete system $\{y_n\}_{n\geq 0}$ of the space X.

Let $1 < p < \infty$ and let q be the conjugate exponent to p. We define

$$C_o := \sup_{n\geq 0} \sum_{k=0}^n \left(\frac{\delta_n \beta(n)}{\delta_k \delta_{n-k} \beta(k)\beta(n-k)} \right)^q. \tag{2.1}$$

Throughout this section we assume that $1 < p < \infty$ and $C_o < \infty$.

Here first we compute the norm of T^N for each $N \in \mathbb{N} \cup \{0\}$, and then we show that T is a well splitting operator on $\ell^p(\beta)$ where T is a weighted shift operator defined on $\ell^p(\beta)$ as $T(f) = \sum_{n=0}^\infty \frac{\delta_{n+1}}{\delta_n} \hat{f}(n)z^{n+1}$. Also we determine the extended eigenvalues and extended eigenvectors of T.

Theorem 2.1. *The weighted shift operator T defined on $\ell^p(\beta)$ as $T(f) = \sum_{n=0}^\infty \frac{\delta_{n+1}}{\delta_n} \hat{f}(n)z^{n+1}$ is bounded and*

$$\| T^N \| = \sup_{n\geq 0} \frac{\beta(n+N)\delta_{n+N}}{\beta(n)\delta_n}, \qquad N \in \mathbb{N} \cup \{0\}.$$

Proof. Suppose that $f(z) = \sum_{n=0}^\infty \hat{f}(n)z^n$ is an arbitrary element of $\ell^p(\beta)$. Then

$$T^N(f) = \sum_{n=0}^\infty \frac{\delta_{n+N}}{\delta_n} \hat{f}(n)z^{n+N}.$$

Take $C = \sup_{n\geq 0} \frac{\beta(n+N)\delta_{n+N}}{\beta(n)\delta_n}$. Since $C_0 < \infty$, then $C < \infty$. It follows that

$$\|T^N f\|_\beta^p = \sum_{n=0}^\infty (\frac{\delta_{n+N}}{\delta_n})^p |\hat{f}(n)|^p \beta(n+N)^p$$

$$= \sum_{n=0}^\infty |\hat{f}(n)|^p \beta(n)^p (\frac{\delta_{n+N}\beta(n+N)}{\delta_n \beta(n)})^p \leq C^p \|f\|_\beta^p.$$

Hence $\|T^N\| \leq C$. To obtain the reverse inequality, put $f_n(z) = z^n$. Then $\|f_n\|_\beta = \beta(n)$ and $T^N(f_n) = \frac{\delta_{n+N}}{\delta_n} z^{n+N}$. Therefore we have

$$\frac{\beta(n+N)\delta_{n+N}}{\delta_n} = \|T^N(f_n)\|_\beta \leq \|T^N\| \|f_n\|_\beta = \|T^N\|\beta(n).$$

This implies that $C \leq \|T^N\|$ and so $\|T^N\| = C$. □

Let $\ell^0(\beta)$ be the set of all formal power series. For each $f \in \ell^p(\beta)$, let $D_f : \ell^p(\beta) \to \ell^0(\beta)$ defined by $D_f(g) = f \circledast g$ be its corresponding $\circledast$-multiplication linear operator. When $f(z) = z$, we take $D_z = M_{\circledast,z}$. It is easy to see that $M_{\circledast,z}(f) = \sum_{n=0}^{\infty} \frac{\delta_{n+1}}{\delta_n \delta_1} \hat{f}(n) z^{n+1}$ and $M^N_{\circledast,z}(f) = \frac{\delta_N}{\delta_1^N} z^N \circledast f$, for all $N \in \mathbb{N} \cup \{0\}$ and $f \in \ell^p(\beta)$. By Theorem 2.1, $M_{\circledast,z}$ is bounded and

$$\|M^N_{\circledast,z}\| = \sup_{n \in \mathbb{N}\cup\{0\}} \frac{\beta(n+N)\delta_{n+N}}{\delta_1^N \delta_n \beta(n)}, \qquad N \in \mathbb{N} \cup \{0\}.$$

In the next theorem we give certain necessary and sufficient conditions for the boundedness of D_f on $\ell^p(\beta)$. Indeed, the sufficient condition for boundedness of all operators of the form D_f, is a sufficient condition for $\ell^p(\beta)$ to be a Banach algebra with respect to the generalized Cauchy product.

Theorem 2.2. *Let D_f be the above mentioned operator. Then we have the following statements.*

(a) For each $f \in \ell^p(\beta)$, D_f is bounded on $\ell^p(\beta)$; that is $D_f(\ell^p(\beta)) \subseteq \ell^p(\beta)$.
(b) If D_f is a bounded operator on $\ell^p(\beta)$, then

$$\sup_{m \geq 0} \frac{(\sum_{k=0}^{\infty} (\frac{\delta_{m+k}}{\delta_k})^p |\hat{f}(k)|^p \beta(m+k)^p)^{\frac{1}{p}}}{\beta(m)\delta_m} < \infty$$

Proof. (a) Let $f, g \in \ell^p(\beta)$. Using (1.1), it is easy to see that

$$\widehat{(f \circledast g)}(n) = \sum_{k=0}^{n} \frac{\delta_n}{\delta_k \delta_{n-k}} \hat{f}(k)\hat{g}(n-k).$$

Let q be the conjugate exponent to p. By using Hölder inequality and (1.1) we have

$$\|D_f(g)\|_\beta^p = \sum_{n=0}^{\infty} |\widehat{(f \circledast g)}(n)|^p \beta(n)^p = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \frac{\delta_n}{\delta_k \delta_{n-k}} \hat{f}(k)\hat{g}(n-k) \right|^p \beta(n)^p$$

$$\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{\delta_n \beta(n)}{\delta_k \delta_{n-k} \beta(k)\beta(n-k)} |\hat{f}(k)||\hat{g}(n-k)|\beta(k)\beta(n-k) \right)^p$$

$$\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(|\hat{f}(k)|\beta(k)|\hat{g}(n-k)|\beta(n-k) \right)^p \right)^{\frac{p}{p}} \left(\sum_{k=0}^{n} \left(\frac{\delta_n \beta(n)}{\delta_k \delta_{n-k} \beta(k)\beta(n-k)} \right)^q \right)^{\frac{p}{q}}$$

$$\leq C_o^{\frac{p}{q}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} |\hat{f}(k)|^p \beta(k)^p |\hat{g}(n-k)|^p \beta(n-k)^p$$

$$= C_o^{\frac{p}{q}} \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p \right) \left(\sum_{n=0}^{\infty} |\hat{g}(n)|^p \beta(n)^p \right) = C_o^{\frac{p}{q}} \|f\|_\beta^p \ \|g\|_\beta^p.$$

Consequently, we get that

$$\|D_f(g)\|_\beta = \|f \circledast g\|_\beta \le C_o^{\frac{1}{q}}\|f\|_\beta\|g\|_\beta,$$

and so $\|D_f\| \le C_o^{\frac{1}{q}}\|f\|_\beta$.
(b) Let D_f be bounded. Then for every $m \in \mathbb{N}\cup\{0\}$ we have

$$\sup_m \frac{\|D_f(z^m)\|_\beta}{\|z^m\|_\beta} \le \|D_f\| < \infty.$$

Hence we get the result.

□

Here we show that the weighted shift operator T, as defined before, is well splitting on $\ell^p(\beta)$. Also, we give a characterization of the extended eigenvalues of T.

Theorem 2.3. *(a) The weighted shift operator T is a well splitting operator on $\ell^p(\beta)$.*
(b) Let λ be a nonzero complex number with $|\lambda| \le 1$, and let $A \in B(\ell^p(\beta))$ be a nonzero operator. Then $\lambda AT = TA$; i.e., λ is an extended eigenvalue of T, if and only if $A\Lambda_{\{\lambda\}} = D_{A(1)}$, where $\Lambda_{\{\lambda\}}(z^n) := \lambda^n z^n$.
(c) If $|\lambda| > 1$, then $\lambda AT = TA$ if and only if $A = D_{A1}\Lambda_{\frac{1}{\lambda}}$.

Proof. (a) For $f \in \ell^p(\beta)$, since $f(z) = \sum_{n=0}^{\infty}\hat{f}(n)z^n$, then we have

$$T^N(f) = f \circledast \delta_N z^N = D_f(\delta_N z^N).$$

Using Theorem 2.2 and the fact that D_f is bounded and defined on the whole of $\ell^p(\beta)$, we deduce that T is a well splitting operator on $\ell^p(\beta)$.

(b) Let $\lambda AT = TA$. Then $\lambda^n AT^n = T^n A$, for all $n \in \mathbb{N}\cup\{0\}$. In particular, $A\lambda^n T^n(1) = T^n A(1)$. By using (2.1), we obtain that

$$T^n(1) = 1 \circledast \delta_n z^n = \delta_n z^n \Rightarrow \lambda^n A(\delta_n z^n) = A(1) \circledast \delta_n z^n = \delta_n z^n \circledast A(1).$$

Thus $\delta_n A(\lambda^n z^n) = \delta_n(z^n \circledast A(1))$ and so $A\Lambda_{\{\lambda\}}(z^n) = z^n \circledast A(1)$. This implies that $A\Lambda_{\{\lambda\}}(P) = P \circledast A(1)$, for all polynomials P. Since polynomials are dense in $\ell^p(\beta)$, $A\Lambda_{\{\lambda\}}(f) = D_{A(1)}(f)$ for all $f \in \ell^p(\beta)$, which yields $A\Lambda_{\{\lambda\}} = D_{A(1)}$.

Conversely, suppose $A\Lambda_{\{\lambda\}} = D_{A(1)}$. Then we have

$$TA(z^n) = TA\Lambda_{\{\lambda\}}\Lambda_{\{\frac{1}{\lambda}\}}(z^n) = TD_{A(1)}\Lambda_{\{\frac{1}{\lambda}\}}(z^n)$$

$$= TD_{A(1)}(\frac{z^n}{\lambda^n}) = \delta_1 z \circledast D_{A(1)}(\frac{z^n}{\lambda^n}) = D_{\delta_1 z}D_{A(1)}(\frac{z^n}{\lambda^n})$$

$$= D_{A(1)}D_{\delta_1 z}(\frac{z^n}{\lambda^n}) = D_{A(1)}(\delta_1 z \circledast \frac{z^n}{\lambda^n}) = \lambda A\Lambda_{\{\lambda\}}(\delta_1 z \circledast \frac{z^n}{\lambda^{n+1}})$$

$$= \lambda A\Lambda_{\{\lambda\}}(\lambda_n \frac{z^{n+1}}{\lambda^{n+1}}) = \lambda A\Lambda_{\{\lambda\}}\Lambda_{\{\frac{1}{\lambda}\}}(\lambda_n z^{n+1}) = \lambda A(T(z^n)).$$

Thus for any polynomial P, $TA(P) = \lambda AT(P)$ and so $TA(f) = \lambda AT(f)$ for all $f \in \ell^p(\beta)$.
(c) Suppose that $|\lambda| > 1$ and $\lambda AT = TA$. Hence $AT = \frac{1}{\lambda}TA$ and so $AT^n = \frac{1}{\lambda^n}TA$ for all $n \ge 0$. By the same method of (b) we get that $A(z^n\delta_n) = \frac{1}{\lambda^n}TA(1)$ and so

$$A(z^n) = \frac{1}{\delta_n\lambda^n}TA(1) = A(1) \circledast \frac{z^n}{\lambda^n} = D_{A1}\Lambda_{\frac{1}{\lambda}}(z^n).$$

This implies that $Af = D_{A1}\Lambda_{\frac{1}{\lambda}}(f)$ for all $f \in \ell^p(\beta)$. □

Finally we get the following corollary, that gives us the description of commutants of the special weighted shift operator T. Recall that the commutant of T is defined as $\{T\}' = \{A \in B(\ell^p(\beta)) : AT = TA\}$.

Corollary 2.1. $\{T\}' = \{D_f : f \in \ell^p(\beta)\}$.

Let $\ell_0^p(\beta) = \ell^p(\beta)$, $\ell_\infty^p(\beta) = \{0\}$ and let for $i \in \mathbb{N} \cup \{0\}$, $\ell_i^p(\beta) = \{\Sigma_{n\geq i}\, c_n z^n \in \ell^p(\beta)\}$. Given two functions $f(z) = \Sigma_{n=0}^{\infty}\, \hat{f}(n) z^n \in \ell^p(\beta)$ and $g(z) = \Sigma_{n=i}^{\infty}\, \hat{g}(n) z^n \in \ell_i^p(\beta)$ we easily get that $f \circledast g \in \ell_i^p(\beta)$. This means that the closed subspace $\ell_i^p(\beta)$ of $\ell^p(\beta)$ is invariant under D_f for all $f \in \ell^p(\beta)$. Therefore the Corollary 2.1 enable us to get the following corollary.

Corollary 2.2. *The set* $\{\ell_i^p(\beta)\}_{i\in\mathbb{N}\cup\{0\}}$ *is the collection of all hyperinvariant subspace of the weighted shift operator* T *and equivalently of the multiplication operator* $D_z = M_{\circledast,z}$ *on* $\ell^p(\beta)$.

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Yousef Estaremi
Department of Mathematics and Computer Sciences
Golestan University
Gorgan, Iran
E-mail: estaremi@gmail.com

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