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VLADIMIR MIKHAILOVICH FILIPPOV

(to the 70th birthday)

Vladimir Mikhailovich Filippov was born on 15 April 1951 in the city of Uryupinsk, Stalingrad Region of the USSR. In 1973 he graduated with honors from the Faculty of Physics and Mathematics and Natural Sciences of the Patrice Lumumba University of Peoples' Friendship in the specialty "Mathematics". In 1973-1975 he is a postgraduate student of the University; in 1976-1979 - Chairman of the Young Scientists' Council; in 1980-1987 - Head of the Research Department and the Scientific Department; in 1983-1984 - scientific work at the Free University of Brussels (Belgium); in 1985-2000 - Head of the Mathematical Analysis Department; from 2000 to the present - Head of the Comparative Educational Policy Department; in 1989–1993 - Dean of the Faculty of Physics, Mathematics and Natural Sciences; in 1993–1998 - Rector of the Peoples' Friendship University of Russia; in 1998-2004 - Minister of General and Professional Education, Minister of Education of the Russian Federation; in 2004-2005 - Assistant to the Chairman of the Government of the Russian Federation (in the field of education and culture); from 2005 to May 2020- Rector of the Peoples' Friendship University of Russia, since May 2020 - President of the Peoples' Friendship University of Russia, since 2013 - Chairman of the Higher Attestation Commission of the Ministry of Science and Higher Education of the Russian Federation.

In 1980, he defended his PhD thesis in the V.A. Steklov Mathematical Institute of Academy of Sciences of the USSR on specialty 01.01.01 - mathematical analysis (supervisor - a corresponding member of the Academy of Sciences of the USSR, Professor L.D. Kudryavtsev), and in 1986 in the same Institute he defended his doctoral thesis "Quasi-classical solutions of inverse problems of the calculus of variations in non-Eulerian classes of functionals and function spaces". In 1987, he was awarded the academic title of a professor.

V.M. Filippov is an academician of the Russian Academy of Education; a foreign member of the Ukrainian Academy of Pedagogical Sciences; President of the UNESCO International Organizing Committee for the World Conference on Higher Education (2007-2009); Vice-President of the Eurasian Association of Universities; a member of the Presidium of the Rectors' Council of Moscow and Moscow Region Universities, of the Governing Board of the Institute of Information Technologies in Education (UNESCO), of the Supervisory Board of the European Higher Education Center of UNESCO (Bucharest, Romania),

Research interests: variational methods; non-potential operators; inverse problems of the calculus of variations; function spaces.

In his Ph.D thesis, V.M. Filippov solved a long standing problem of constructing an integral extremal variational principle for the heat equation. In his further research he developed a general theory of constructing extremal variational principles for broad classes of differential equations with non-potential (in classical understanding) operators. He showed that all previous attempts to construct variational principles for non-potential operators "failed" because mathematicians and mechanics from the time of L. Euler and J. Lagrange were limited in their research by functionals of the type Euler - Lagrange. Extending the classes of functionals, V.M. Filippov introduced a new scale of function spaces that generalize the Sobolev spaces, and thus significantly expanded the scope of the variational methods. In 1984, famous physicist, a Nobel Prize winner I.R. Prigogine presented the report of V.M. Filippov to the Royal Academy of Sciences of Belgium. Results of V.M. Filippov's variational principles for non-potential operators are quite fully represented in some of his and his colleagues' monographs.

Honors: Honorary Legion (France), "Commander" (Belgium), Crown of the King (Belgium); in Russia - orders "Friendship", "Honor", "For Service to the Fatherland" III and IV degrees; Prize of the President of the Russian Federation in the field of education; Prize of the Governement of the Russian Federation in the field of education; Gratitude of the President of the Russian Federation; "For Merits in the Social and Labor Sphere of the Russian Federation", "For Merits in the Development of the Olympic Movement in Russia", "For Strengthening the Combat Commonwealth; and a number of other medals, prizes and awards.

He is an author of more than 270 scientific and scientific-methodical works, including 32 monographs, 2 of which were translated and published in the United States by the American Mathematical Society.

V.M. Filippov meets his 70th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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MODULUS OF CONTINUITY FOR BESSEL TYPE POTENIIAL OVER LORENTZ SPACE

N.H. Alkhalil

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Key words: the generalized Bessel potential, the modulus of continuity of a potential, Lorentz space, rearrangement invariant space.

AMS Mathematics Subject Classification: 46A30, 42A16.

Abstract. The generalized Bessel potentials are constructed using convolutions of the generalized Bessel–McDonald kernels with functions belonging to a basic rearrangement invariant space. Under assumptions ensuring the embedding of potentials into the space of bounded continuous functions, differential properties of potentials are described by using the k-th order modulus of continuity in the uniform norm. In the paper, estimates are given for the k-th order modulus of continuity in the uniform norm in the case of the generalized Bessel potentials constructed over the basic weighted Lorentz space.

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1 Introduction

The paper is organized as follows. In Section 1 we give basic definitions of the potential theory. Main properties of kernels are considered and basic spaces for potentials are described. Section 2 contains some auxiliary results. Estimates for $\|u\|_C$ are presented for potentials, and properties of moduli of continuity are discussed. The main results of the paper are presented in Section 3. In Theorem 3.1 we establish estimates of the modulus of continuity in the uniform norm $\omega_C^k(u;\tau)$ in the case of the basic weighted Lorentz space, where $k \in \mathbb{N}$, $\omega_C^k(u;\tau)$ is the modulus of continuity for generalized Bessel potentials u.

2 Basic definitions

Let $v > 0$ be a measurable function on $\mathbb{R}_+$. The Lorentz space $\Lambda^p(v)$ is the space of all measurable functions on $\mathbb{R}^n$ with finite (quasi) norms (see [1])

$$\|f\|_{\Lambda^p(v)} = \begin{cases} \left(\int_0^\infty f^*(t)^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}; & 0 < p < \infty; \\ \operatorname*{essup}_{t\in\mathbb{R}_+} \left\{ f^*(t)v(t) \right\}; & p = \infty. \end{cases} \tag{2.1}$$

Here $f^*\colon \mathbb{R}_+ \to [0,\infty]$ is the decreasing rearrangement of a function $f\colon \mathbb{R}^n \to \mathbb{R}$, i.e f^* is a nonnegative decreasing right-continuous function on $\mathbb{R}_+ = (0,\infty)$ which is equimeasurable with f :

$$\mu_n \{x \in \mathbb{R}^n :\ |f(x)| > y\} = \mu_1 \{t \in \mathbb{R}_+ :\ f^*(t) > y\}, \quad y \in \mathbb{R}_+, \tag{2.2}$$

where μ_n is the n-dimensional Lebesgue measure. We assume that $0 < V(t) := \int_0^t v(\tau)d(\tau) < \infty$, $t \in \mathbb{R}_+$, and

$$\sup_{t\in\mathbb{R}_+}\left[\frac{V(2t)}{V(t)}\right] < \infty, \tag{2.3}$$

the so-called Δ_2-condition. Under these assumptions $E(\mathbb{R}^n) = \Lambda^p(v)$ is a (quasi) Banach space, which gives an important example of a rearrangement invariant space (shortly: RIS), because of the property:

$$g^* \le f^*, f \in E(\mathbb{R}^n) \Rightarrow g \in E(\mathbb{R}^n), \|g\|_E \le \|f\|_E.$$

(see C. Bennett and R. Sharpley [1]). Moreover, $E' = E'(\mathbb{R}^n)$ is the associated RIS for $E(\mathbb{R}^n)$, i.e. E' is RIS with the norm:

$$\|g\|_{E'} = \sup\Big\{\int\limits_{\mathbb{R}^n} |fg|\,\mathrm{d}\mu_n : \ f \in E, \|f\|_E \le 1\Big\}. \tag{2.4}$$

For $1 < p < \infty$ the description of the associated space for $E(\mathbb{R}^n) = \Lambda^p(v)$ was obtained by E. Sawyer [7]. Namely,

$$\|g\|_{E'} = \sup_{0\le h\downarrow}\frac{\int_0^\infty g^*(\tau)h(\tau)\,\mathrm{d}\tau}{\left(\int_0^\infty h(\tau)^p v(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}}} \approx \left(\int_0^\infty \left(\left(\int_0^\xi g^*(\tau)\mathrm{d}\tau\right)^{p'} \frac{v(\xi)d\xi}{V(\xi)^{p'}}\right)^{\frac{1}{p'}}\right). \tag{2.5}$$

We will use this description in Section 4. Here the symbol $\approx$ means that the ratio of left- and right-hand sides is between positive constants depending only on p (and not on v or g).

Remark 1. Note that, for $E(\mathbb{R}^n) = \Lambda^p(v)$,

$$E'(\mathbb{R}^n) \ne \{0\} \Leftrightarrow \exists\, T > 0 : \int_0^T \frac{t^{p'}v(t)dt}{V(t)^{p'}} < \infty. \tag{2.6}$$

Indeed, for $D \subset \mathbb{R}^n, \mu_n(D) = T$, we have $g(x) = \chi_D(x) \in E'(\mathbb{R}^n)$, because $g^*(\tau) = \chi_{(0,T)}(\tau)$ and

$$\int_0^\xi g^*(\tau)d\tau = \begin{cases} \xi, & 0 < \xi \le T, \\ T, & \xi > T; \end{cases}$$

$$(\|g\|_{E'})^{p'} \le c_1 \int_0^\infty \left(\int_0^\xi g^*(\tau)\mathrm{d}\tau\right)^{p'} \frac{v(\xi)d\xi}{V(\xi)^{p'}} = c_1\int_0^T \frac{\xi^{p'}v(\xi)d\xi}{V(\xi)^{p'}} + c_1 T^{p'} \int_T^\infty \frac{v(\xi)d\xi}{V(\xi)^{p'}} < \infty.$$

Here,

$$\int_T^\infty \frac{v(\xi)d\xi}{V(\xi)^{p'}} = \frac{V(\xi)^{1-p'}}{1-p'}\bigg|_{\xi=T}^{\infty} \le \frac{V(T)^{1-p'}}{p'-1} < \infty.$$

On the other hand, if $\exists\; g \in E'(\mathbb{R}^n),\ g \ne 0$ then there exists $c > 0$ and $T \in \mathbb{R}_+$ such that $g^*(\tau) \ge c,\ \tau \in (0,T)$. Then

$$\infty > (\|g\|_{E'})^{p'} \ge c_2\int_0^T \left(\int_0^\xi g^*(\tau)\mathrm{d}\tau\right)^{p'} \frac{v(\xi)d\xi}{V(\xi)^{p'}} \ge c_2 c^{p'}\int_0^T \frac{\xi^{p'}v(\xi)d\xi}{V(\xi)^{p'}}.$$

Everywhere in this paper we assume that condition (2.6) is satisfied.

The potential space $H_E^G \equiv H_E^G(\mathbb{R}^n)$ for $E(\mathbb{R}^n) = \Lambda^p(v)$ is defined as the set of convolutions of the potential kernel G with all functions belonging to the basic RIS $E(\mathbb{R}^n)$:

$$H_E^G(\mathbb{R}^n) = \big\{u = G * f : \ f \in E(\mathbb{R}^n)\big\}. \tag{2.7}$$

We define

$$\|u\|_{H_E^G} = \inf\big\{\|f\|_E : \ f \in E(\mathbb{R}^n), \ G * f = u\big\}. \tag{2.8}$$

We assume that the kernel G in representation (2.8) is admissible, i.e

$$G \in L_1(\mathbb{R}^n) + E'(\mathbb{R}^n).$$

Here the convolution $G * f$ is defined as the integral

$$(G * f)(x) = \int\limits_{\mathbb{R}^n} G(x-y) f(y)\, \mathrm{d}(y).$$

For $R \in \mathbb{R}_+$ we introduce the class of monotone functions $\mathfrak{I}_n(R)$ as follows. A function Φ: $(0,R) \to \mathbb{R}_+$ belongs to $\mathfrak{I}_n(R)$ if Φ satisfies the following conditions:

1. Φ is decreasing and continuous on $(0,R)$,

2. there is a constant $c \in \mathbb{R}_+$, such that

$$\int\limits_0^r \Phi(\rho)\rho^{n-1}\, \mathrm{d}\rho < \infty, \quad r \in (0,R). \tag{2.9}$$

The properties of kernels are discussed in Definitions 1–3 below.

Remark 2. Let $A(x), B(x)$ be positive functions on the set $D \subset \mathbb{R}^n$. We write $A(x) \cong B(x), x \in D$ if there exists a constant $c \geq 1$ such that $c^{-1} \leq \dfrac{A(x)}{B(x)} \leq c, \forall x \in D$.

Definition 1. Let $\Phi \in \mathfrak{I}_n(R)$. We write that $G \in S_R(\Phi)$, if

$$G(x) \cong \Phi(|x|), x \in B_R = \{x \in \mathbb{R}^n : \ |x| < R\}, \quad R \in \mathbb{R}_+. \tag{2.10}$$

Definition 2. Let $\Phi \in \mathfrak{I}_n(R)$, $X(\mathbb{R}^n)$ be a RIS. We write that $G \in S_R(\Phi; X)$, if

$$G(x) = G_R^0(x) + G_R^1(x);$$

$$G_R^0(x) = G(x)\chi_{B_R}(x); \quad G_R^1(x) = G(x)\chi_{B_R^c}(x), \tag{2.11}$$

$$G_R^0(x) \cong \Phi(|x|), x \in B_R; \quad G_R^1(x) \in X(\mathbb{R}^n). \tag{2.12}$$

Definition 3. Potentials $u \in H_E^G(\mathbb{R}^n)$ with $E(\mathbb{R}^n) = \Lambda^p(v)$ are called the generalized Bessel potentials, if for some $R \in \mathbb{R}_+$

$$\Phi \in \mathfrak{I}_n(R), \quad G \in S_R(\Phi; \ L_1 \cap E'), \quad \int\limits_{\mathbb{R}^n} G \mathrm{d}\mu_n \neq 0. \tag{2.13}$$

Remark 3. Note that the classical Bessel–McDonald the kernels have the form

$$G_\alpha(x) = c(\alpha, n)\rho^{-\gamma}K_\gamma(\rho), \quad \rho = |x| \in \mathbb{R}_+, \quad \alpha \in (0, n); \quad \gamma = \frac{n-\alpha}{2},$$

where K_γ is the McDonald function, see [6]. The well-known properties of these kernels state that

$$G_\alpha(x) \cong \Phi(|x|), \quad 0 < |x| < R; \quad \Phi(\rho) = \rho^{\alpha-n} \in \mathfrak{I}_n(R); \quad G_\alpha(x) \cong |x|^{-\gamma-\frac{1}{2}}e^{-|x|}, |x| > R.$$

In view of the embedding $L_1 \cap L_\infty \subset L_1 \cap E'$, where $E(\mathbb{R}^n) = \Lambda^p(v)$, our scheme includes the Bessel potentials.

Definition 4. Let $C(\mathbb{R}^n)$ be the space of all bounded and uniformly continuous functions with the norm

$$\|u\|_C = \sup_{x\in\mathbb{R}^n} |u(x)|. \tag{2.14}$$

For $u \in C(\mathbb{R}^n)$ the modulus of continuity of order $k \in \mathbb{N}$ is defined as:

$$\omega_C^k(u;\tau) = \sup\Big\{\big\|\Delta_h^k u\big\|_C \,:\, |h| \le \tau\Big\}, \quad \tau \in \mathbb{R}_+. \tag{2.15}$$

Here $\Delta_h^k u(x)$ is the k-th difference with the step $h \in \mathbb{R}^n$ at the point $x \in \mathbb{R}^n$. Let us note that for $u \in C(\mathbb{R}^n)$,

$$\omega_C^k(u;\tau) \to 0 \quad (\tau \to +0). \tag{2.16}$$

3 Auxiliary statements

The following results were proved in [3].

Theorem 3.1. *Let $G \in L_1(\mathbb{R}^n)$, $G \ne 0$, $\varphi(\tau) = G^*(\tau)$, $\tau \in \mathbb{R}_+$, and a function $f : \mathbb{R}^n \to \mathbb{R}$ be such that for some $T \in \mathbb{R}_+$.*

$$\int_0^T \varphi(\tau) f^*(\tau)\,d\tau < \infty. \tag{3.1}$$

1. *For the convolution*

$$u(x) = \int_{\mathbb{R}^n} G(x-y)f(y)\,dy, \quad x \in \mathbb{R}^n, \tag{3.2}$$

the following estimate holds

$$\sup_{x\in\mathbb{R}^n}\big|u(x)\big| \le c_0 \int_0^T \varphi(\tau) f^*(\tau)\,d\tau, \tag{3.3}$$

where

$$c_0 = 1 + \Big(\int_T^\infty \varphi(\tau)\,d\tau\Big)\Big(\int_0^T \varphi(\tau)\,d\tau\Big)^{-1}. \tag{3.4}$$

2. *Let, in addition $G \in C^k(\mathbb{R}^n \backslash 0)$, $k \in \mathbb{N}$, and for*

$$G_k(x) := \sum_{|\alpha|=k} \big|D^\alpha G(x)\big|, \quad x \in \mathbb{R}^n, \tag{3.5}$$

for some c_1*, the following estimate takes place*

$$|G_k(x)| \le c_1 \Psi_k(|x|), \quad x \in \mathbb{R}^n, \tag{3.6}$$

where the function

$$0 \le \varphi_k(\tau) := \Psi_k\left(\left(\frac{\tau}{V_n}\right)^{\frac{1}{n}}\right) \downarrow \quad \text{on } \mathbb{R}_+, \tag{3.7}$$

(V_n *is the volume of unit ball in* $\mathbb{R}^n$) *is such that the following relations hold*

$$\varphi_k(\tau) \le \tau^{-k/n}\varphi(\tau), \quad \tau \in (0,T] \tag{3.8}$$

$$\int_T^\infty \varphi_k(\tau)\,\mathrm{d}\tau < \infty. \tag{3.9}$$

Then convolution (3.2) *is continuous on* $\mathbb{R}_+$ *and for* $t \in (0,T]$

$$\omega_C^k\left(u; t^{\frac{1}{n}}\right) \le c_2 \int_0^T \left[\frac{\tau^{-\frac{k}{n}}}{\tau^{-\frac{k}{n}} + t^{-\frac{k}{n}}}\right] \varphi(\tau) f^*(\tau)\,\mathrm{d}\tau. \tag{3.10}$$

Here $c_2 = c_1 \tilde{c} d$*, where*

$$d = 1 + \frac{2}{T\varphi_k(T)}\left(\int_T^\infty \varphi_k(\tau)\,\mathrm{d}\tau\right), \tag{3.11}$$

c_1 *is the constant from condition* (3.6)*, and* $\tilde{c} = \tilde{c}(k,n) \in \mathbb{R}_+$ *depends only on* k *and* n*.*

Remark 4. Under the assumptions of Theorem 3.1 let $G \in L_1(\mathbb{R}^n) \cap E'(\mathbb{R}^n)$ for $E(\mathbb{R}^n) = \Lambda^p(v)$, see (2.1)–(2.6), and $f \in E(\mathbb{R}^n)$. Then, inequality (3.3) shows that convolution (3.2) is uniformly bounded. Moreover, formula (3.10) shows that $\omega_C^k\left(u; t^{\frac{1}{n}}\right) \to 0$ as $t \to +0$. which implies that $u \in C(\mathbb{R}^n)$.

Lemma 3.1. *Let the following inequality be valid:*

$$\int_t^T \tau^{-\frac{k}{n}} \varphi(\tau)\,\mathrm{d}\tau \le B_0\, t^{1-\frac{k}{n}} \varphi(t), \quad t \in (0,T), \tag{3.12}$$

where $B_0 \in \mathbb{R}_+$ *is independent of* t*. In addition, let the assumptions of Theorem 3.1 be fulfilled. Then*

$$\omega_C^k\left(u; t^{\frac{1}{n}}\right) \le c_3 \int_0^t \varphi(\tau) f^*(\tau)\,\mathrm{d}\tau, \quad t \in (0,T], \tag{3.13}$$

where $c_3 = (1 + B_0)c_2$*, and* c_2 *is the constant from (3.10).*

4 Main result

We preserve the notations of Sections 2 and 3.

Theorem 4.1. *Let $E(\mathbb{R}^n) = \Lambda^p(v)$ be the Lorentz space, see (2.1)–(2.6), and $H^G_E(\mathbb{R}^n)$ be the space of generalized Bessel potentials, see (2.13),*

$$\varphi(\tau) = \Phi((\frac{\tau}{V_n})^{\frac{1}{n}}), \quad \tau \in (0,T); \quad T = V_n R^n; \quad \varphi(\tau) = G^*(\tau); \quad \tau > T, \tag{4.1}$$

Moreover, we assume that estimate (3.12) holds, the kernel G satisfies the assumptions of Theorem 3.1, and

$$\sup_{t\in(0,T]} \left\{ \frac{1}{V(t)^{\frac{1}{p}}} \int_0^t \varphi(\tau) d\tau \right\} < \infty, \quad 0 < p \le 1; \tag{4.2}$$

$$\left(\int_0^T \left(\int_0^t \varphi(\tau) \mathrm{d}\tau \right)^{p'} \frac{v(t)dt}{V(t)^{p'}} \right)^{\frac{1}{p'}} < \infty, \quad 1 < p < \infty, \quad p' = \frac{p}{p-1}. \tag{4.3}$$

Then the following statements hold:

1) $u \in C(\mathbb{R}^n)$;

2) If $0 < p \le 1$, then there exists $c_3 \in \mathbb{R}_+$ such that for $0 < t < T$

$$\omega^k_C\Big(u; t^{\frac{1}{n}}\Big) \le c_3 w\Big(t^{\frac{1}{n}}\Big) \, \|u\|_{H^G_{\Lambda^p(v)}}, \tag{4.4}$$

where

$$w\Big(t^{\frac{1}{n}}\Big) = \sup_{\xi\in(0,t)} \left\{ \frac{1}{V(\xi)^{\frac{1}{p}}} \int_0^\xi \varphi(\tau) d\tau \right\}. \tag{4.5}$$

3) If $1 < p < \infty$, then there exists $c_4 \in \mathbb{R}_+$ such that for $0 < t \le T$

$$\omega^k_C\Big(u; t^{\frac{1}{n}}\Big) \le c_4 A(t) \|u\|_{H^G_{\Lambda^p(v)}}, \tag{4.6}$$

where

$$A(t) = \left[\int_0^t \left(\int_0^\xi \varphi(\tau) d\tau \right)^{p'} \frac{v(\xi)d\xi}{V(\xi)^{p'}} + \left(\int_0^t \varphi(\tau) d\tau \right)^{p'} V(t)^{-\frac{p'}{p}} \right]^{\frac{1}{p'}}. \tag{4.7}$$

Proof. 1) We have by Theorem 3.1 (see Remark 4) the inclusion

$$H^G_E(\mathbb{R}^n) \subset C(\mathbb{R}^n).$$

Under the assumptions of Theorem 4.1 we have equivalence (2.10), so that

$$G^*(\tau) \cong \varphi(\tau), \quad 0 < \tau \le T = V_n R^n.$$

Together with equality $\varphi(\tau) = G^*(\tau), \quad \tau > T$, we obtain

$$\varphi(\tau) \cong G^*(\tau), \quad \tau \in \mathbb{R}_+. \tag{4.8}$$

Thus, application of Theorem 3.1 and Remark 4 to a function $u \in H^G_E(\mathbb{R}^n)$, $E(\mathbb{R}^n) = \Lambda^p(v)$, gives: $u \in C(\mathbb{R}^n)$ and formulas (3.2)–(3.13) hold. Here $u = G * f$, $f \in \Lambda^p(v)$. Now, we use the equality

$$\int_0^t \varphi(\tau) f^*(\tau) \, \mathrm{d}\tau = \left\{ \frac{\int_0^t \varphi(\tau) f^*(\tau) \, \mathrm{d}\tau}{\|f\|_{\Lambda^p(v)}} \right\} \|f\|_{\Lambda^p(v)}.$$

Then, by (3.13), for $0 < t < T$

$$\omega^k_C\Big(u; t^{\frac{1}{n}}\Big) \le c_3 \sup_{\rho\in\Lambda^p(v)} \left\{ \frac{\int_0^t \varphi(\tau) \rho^*(\tau) \, \mathrm{d}\tau}{\|\rho\|_{\Lambda^p(v)}} \right\} \|f\|_{\Lambda^p(v)}. \tag{4.9}$$

We denote

$$\rho^* = h \in L_P(v); \quad \|\rho\|_{\Lambda^p(v)} = \Big(\int_0^\infty h^p v \mathrm{d}\tau\Big)^{\frac{1}{p}}; \quad 0 \le h \downarrow;$$

and obtain

$$F_t \equiv \sup_{\rho\in\Lambda^p(v)} \left\{ \frac{\int_0^t \varphi(\tau)\rho^*(\tau)\,\mathrm{d}\tau}{\|\rho\|_{\Lambda^p(v)}} \right\} = \sup_{0\le h\downarrow} \left\{ \frac{\int_0^t \varphi(\tau)h(\tau)\,\mathrm{d}\tau}{(\int_0^\infty h^p v \mathrm{d}\tau)^{\frac{1}{p}}} \right\}. \tag{4.10}$$

Here the numerator does not depend on the values $h(\tau)$ for $\tau \in (t,\infty)$. Therefore, "sup" is realized on functions h, such that $h(\tau) = 0$ for $\tau \in (t,\infty)$. It means that

$$\sup_{\rho\in\Lambda^p(v)} \left\{ \frac{\int_0^t \varphi(\tau)\rho^*(\tau)\,\mathrm{d}\tau}{\|\rho\|_{\Lambda^p v}} \right\} = \sup_{0\le h\downarrow} \frac{\int_0^t \varphi(\tau)h(\tau)\,\mathrm{d}\tau}{(\int_0^t h^p v \mathrm{d}\tau)^{\frac{1}{p}}} \equiv B(t). \tag{4.11}$$

2) Let $0 < p \le 1$. To calculate $B(t)$ we apply the result of [2]. Namely,

$$B(t) \equiv \sup_{\xi\in(0,t)} \left\{ \frac{\int_0^t \varphi(\tau)\mathrm{d}\tau}{(\int_0^\xi v(\tau)\mathrm{d}\tau)^{\frac{1}{p}}} \right\} = w\Big(t^{\frac{1}{n}}\Big).$$

From (4.9)–(4.11) we obtain for $u \in H_E^G(\mathbb{R}^n)$, $t \in (0,T]$,

$$\omega_C^k\Big(u; t^{\frac{1}{n}}\Big) \le c_3 w\Big(t^{\frac{1}{n}}\Big)\ \|f\|_{\Lambda^p(v)}, \quad 0 < p \le 1.$$

Here $f \in \Lambda^p(v)$ is any function such that $G * f = u$ for a given $u \in H_E^G(\mathbb{R}^n)$. Now, we take "inf" over the set of such functions $f \in \Lambda^p(v)$ for a given $u \in H_E^G(\mathbb{R}^n)$, and obtain

$$\omega_C^k\Big(u; t^{\frac{1}{n}}\Big) \le c_3 w\Big(t^{\frac{1}{n}}\Big)\ \|u\|_{H_E^G}, \quad E = \Lambda^p(v). \tag{4.12}$$

This is estimate (4.4) with constant c_3 from (3.13).

3) Let $1 < p < \infty$. We put in equality (4.10)

$$\varphi_t(\tau) = \varphi(\tau)\chi_{(0,t)}(\tau); \quad 0 \le \varphi_t(\tau) \downarrow \ \Rightarrow \quad \varphi_t^*(\tau) = \varphi_t(\tau).$$

Then

$$F_t = \sup_{0\le h\downarrow} \left\{ \frac{\int_0^\infty \varphi_t(\tau)h(\tau)\,\mathrm{d}\tau}{(\int_0^\infty h^p v \mathrm{d}\tau)^{\frac{1}{p}}} \right\} = \sup_{0\le h\downarrow} \left\{ \frac{\int_0^t \varphi_t^*(\tau)h(\tau)\,\mathrm{d}\tau}{(\int_0^\infty h^p v \mathrm{d}\tau)^{\frac{1}{p}}} \right\}.$$

Now, we apply formula (2.5) with $g(\tau) = \varphi_t(\tau)$, $\tau \in \mathbb{R}_+$, and obtain

$$F_t = \|\varphi_t\|_{E'} \cong \left(\int_0^\infty \Big(\int_0^\xi \varphi_t(\tau)\mathrm{d}\tau \Big)^{p'} \frac{v(\xi)d\xi}{V(\xi)^{p'}} \right)^{\frac{1}{p'}}. \tag{4.13}$$

We note that

$$\int_0^\xi \varphi_t(\tau)d\tau = \int_0^\xi \varphi(\tau)d\tau, \quad \xi \in (0,t); \qquad \int_0^\xi \varphi_t(\tau)d\tau = \int_0^t \varphi(\tau)d\tau, \quad \xi \ge t.$$

We put these equalities in (4.13) and see that $F_t \cong A(t)$, $0 < t \le T$. Therefore, from (4.9) and (4.10) it follows

$$\omega_C^k\Big(u; t^{\frac{1}{n}}\Big) \le c_3 F_t\ \|f\|_{\Lambda^p(v)} \le c_4 A(t)\ \|f\|_{\Lambda^p(v)}, \quad t \in (0,T].$$

Analogously to (4.12) we obtain from here that for $t \in (0,T]$.

$$\omega_C^k\Big(u; t^{\frac{1}{n}}\Big) \le c_4 A(t)\ \|u\|_{H_E^G}, \quad E = \Lambda^p(v). \tag{4.14}$$

This completes the proof of Theorem 4.1.

□

Corollary 4.1. *Let us concretize the answers in classical cases: for the classical space of potentials we put*

$$\varphi(t) = t^{\frac{\alpha}{n}-1}, \quad 0 < \alpha < n, \tag{4.15}$$

see Remark 3*; for the classical Lorentz space we have* $E(\mathbb{R}^n) = \Lambda^p(v)$ *with* $v(t) = t^\beta$*. In these case the condition of nontriviality* $E'(\mathbb{R}^n) \neq \{0\}$*, see* (2.6) *will be as follows:* $-1 < \beta \le p-1$ *for* $0 < p \le 1; \quad -1 < \beta < p-1$ *for* $p > 1$*.*
In all formulas

$$\int_0^t \varphi(\tau)d\tau = \frac{n}{\alpha}.t^{\frac{\alpha}{n}}, \quad t \in (0,T); \tag{4.16}$$

and condition (3.12) *will be as follows:* $0 < \alpha < k$ *As the answer we obtain*

$$w\Big(t^{\frac{1}{n}}\Big) = \frac{(\beta+1)n}{\alpha}.t^{\frac{\alpha}{n}-\beta-1}, \quad \frac{\alpha}{n} \ge \beta+1, \quad 0 < p \le 1; \tag{4.17}$$

$A(t) \cong t^{\frac{\alpha}{n}-\frac{\beta+1}{p}}; \quad \frac{\alpha}{n} \ge \frac{\beta+1}{p}, \quad 1 < p < \infty$.

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