ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2021, Volume 12, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 562
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

SHAVKAT ARIFJANOVICH ALIMOV

(to the 75th birthday)

Shavkat Arifjanovich Alimov was born on March 2, 1945 in the city of Nukus, Uzbekistan. In 1968, he graduated from the Department of Mathematics of Physical Faculty of the M.V. Lomonosov Moscow State University (MSU), receiving a diploma with honors. From 1968 to 1970, he was a post-graduate student in the same department under the supervision of Professor V.A. Il'in. He defended his PhD thesis in 1970. In May 1973, at the age of 28, he defended his doctoral thesis devoted to equations of mathematical physics. In 1973, for research on the spectral theory, he was awarded the highest youth prize of the USSR.

From 1974 to 1984, he worked as a professor in the Department of General Mathematics at the Faculty of Computational Mathematics and Cybernetics. In 1984, Sh.A. Alimov joined the Tashkent State University (TSU) as a professor. From 1985 to 1987 he worked as the Rector of the Samarkand State University, from 1987 to 1990 - the Rector of the TSU, from 1990 to 1992 - the Minister of Higher and Secondary Special Education of the Republic of Uzbekistan. From 1992 to 1994, he headed the Department of Mathematical Physics of the TSU.

After some years of diplomatic work, he continued his academic career as a professor of the Department of Mathematical Physics at the National University of Uzbekistan (NUU). From the first days of the opening of the Tashkent branch of the MSU in 2006, he worked as a professor in the Department of Applied Mathematics. From 2012 to 2017, he headed the Laboratory of Mathematical Modeling of the Malaysian Institute of Microelectronic Systems in Kuala Lumpur. From 2017 to 2019, he worked as a professor at the Department of Differential Equations and Mathematical Physics of the NUU. From 2019 to the present, Sh.A. Alimov is a Scientific Consultant at the Center for Intelligent Software Systems, and an adviser to the Rector of the NUU.

The main scientific activity of Sh.A. Alimov is connected with the spectral theory of partial differential equations and the theory of boundary value problems for equations of mathematical physics. He obtained series of remarkable results in these fields. They cover many important problems of the theory of Schrodinger equations with singular potentials, the theory of boundary control of the heat transfer process, the mathematical problems of peridynamics related to the theory of hypersingular integrals.

In 1984, Sh.A. Alimov was elected a corresponding member and in 2000 an academician of the Academy of Sciences of Uzbekistan. He was awarded several prestigious state prizes.

Sh.A. Alimov has over 150 published scientific and a large number of educational works. Among his pupils there are 10 doctors of sciences and more than 20 candidates of sciences (PhD) working at universities of Uzbekistan, Russia, USA, Finland, and Malaysia.

For about thirty years, Sh.A. Alimov has been actively involved in the reform of mathematical school education.

Sh.A. Alimov meets his 75th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 12, Number 1 (2021), 68 – 81

ON MULTIPERIODIC SOLUTIONS OF PERTURBED NONLINEAR AUTONOMOUS SYSTEMS WITH THE DIFFERENTIATION OPERATOR ON A VECTOR FIELD

B.Zh. Omarova, Zh.A. Sartabanov

Communicated by Ya.T. Sultanaev

Key words: multiperiodic solutions, autonomous system, differentiation operator, Lyapunov's vector field, perturbation.

AMS Mathematics Subject Classification: 34C46, 35B10, 35C15, 35F35, 35F50.

Abstract. A quasilinear system with the differentiation operator with respect to the directions of vector fields specified by Lyapunov's system with respect to space independent variables and a multiperiodic system with respect to time variables is considered. We study the problem of the existence and uniqueness of a multiperiodic solution of a quasilinear system and we use methods of the theory of multiperiodic solutions of linear systems. The research partially reflects the multiperiodic structure of a solution of the initial problem for quasilinear systems. Conditions for the existence and uniqueness of a multiperiodic solution, an existence theorem of a solution of the initial problem, and the problem of multiperiodic solutions are given. They are proved by the method of contraction mappings defined on spaces of smooth functions.

DOI: https://doi.org/10.32523/2077-9879-2021-12-1-68-81

1 Introduction

Many oscillatory phenomena, such as sound, light, electromagnetic, gas, and hydromechanical phenomena are described by systems of partial differential equations. In this regard, the research on solutions of such systems with oscillatory properties belongs to applied aspects of the theory of ordinary and partial differential equations. Existence conditions of periodic solutions of partial differential equations of the first order with the differentiation operator in the directions of a vector field on a torus were established in [1]. Note that the differential operator of work [1] is similar to the differentiation operator with respect to time variables considered in this paper. We note some similarity of the research methods used in [3] with the methods of this paper. It is explained by the fact that the Poincare-Lyapunov and Hamilton-Jacobi methods are the general basis of the methods for studying such problems.

Comparing the investigations in papers [1, 3] with this work, one should note the common general direction associated with partial differential equations of the first order describing oscillatory processes and some generality of used methods and operators, but the problems investigated are completely different.

The foundations of methods which are used in this paper were laid in [4, 9, 11, 17]. Some results on multiperiodic solutions of systems of equations obtained by the development of the methods of those works are given in [2, 5–14], which were further developed in [15, 16]. In these works, the forces causing systems oscillations were multiperiodic with known periods. Consequently, the authors were looking for solutions with pre-known periods. In the case of autonomous systems, we do not have such a possibility, but it is possible to reveal multiperiodic structures of solutions of problems accompanying the considered system (e.g. the characteristic equations in time and space variables, corresponding linear systems, etc.), which allows one to determine multiperiodic solutions of the original system.

In this paper, we consider a quasilinear system of equations with the differentiation operator in the directions of vector fields, where the characteristic systems of the differentiation operators with respect to time and space variables are independent. Moreover, the differentiation with respect to space variables is carried out in the directions determined by Lyapunov's system. We studied the problem of the existence of multiperiodic solutions of an analogous autonomous system in our previous works. In this paper, some input data are affected by perturbations depending on time variables. Obviously, multiperiodic solutions of the system are compositions of periodic components with rationally incommensurable periods. Along with multiperiodic components, non-periodic components can also enter a solution of the initial problem. The structure of a solution with selected periodic components is called its multiperiodic structure. The determination of the multiperiodic structure of a solution of the considered system fully reveals the oscillatory nature of the phenomenon described by the system of the initial problem.

In the case of a non-autonomous system, the frequencies of the required multiperiodic oscillations are mainly determined by the system itself. Therefore, the frequencies and their numbers are known in advance.

In the autonomous case, the main difficulty of the problem under consideration is the uncertainty of the frequencies of periodic oscillations, which are components of multiperiodic oscillations determined by a given system. This difficulty was overcome by using the fact that the characteristic vector field satisfies the conditions of Lyapunov's system.

2 Integral representation of multiperiodic solutions of a linear system. Statement of the main problem

Let, for $r > 0$ and $s \in \mathbb{N}$, B_r^s denote an open ball in $\mathbb{R}^s$ centered at the origin and let a vector-function $x = (x_1, \dots, x_n)$ depending on (τ, t)-time and $\zeta = (\zeta_1, \dots, \zeta_l)$-space variables with the components $\zeta_j = (\xi_j, \eta_j)\,, j = \overline{1, l}$ characterize the oscillatory process described by the system of equations

$$Dx = Ax + f(\tau, t, \zeta, x) \tag{2.1}$$

with the differentiation operator

$$D = \frac{\partial}{\partial \tau} + \left\langle a, \frac{\partial}{\partial t} \right\rangle + \left\langle \nu I \zeta + g, \frac{\partial}{\partial \zeta} \right\rangle. \tag{2.2}$$

Here $\tau \in \mathbb{R}$, $t = (t_1, \dots, t_m) \in \mathbb{R}^m$, $\zeta_j = (\xi_j, \eta_j) \in B_\delta^2$, $j = \overline{1, l}$, $\zeta = (\zeta_1, \dots, \zeta_l) \in B_\delta^{2l}$, $\delta > 0$; $\dfrac{\partial}{\partial t} = \left(\dfrac{\partial}{\partial t_1}, \dots, \dfrac{\partial}{\partial t_m}\right)$, $\dfrac{\partial}{\partial \zeta} = \left(\dfrac{\partial}{\partial \zeta_1}, \dots, \dfrac{\partial}{\partial \zeta_l}\right)$, $\dfrac{\partial}{\partial \zeta_j} = \left(\dfrac{\partial}{\partial \xi_j}, \dfrac{\partial}{\partial \eta_j}\right)$ are vector differentiation operators; $\nu = (\nu_1, \dots, \nu_l)$ is a constant vector, $\nu I = diag(\nu_1 I_2, \dots, \nu_l I_2)$ is the matrix with the symplectic unit I_2 of the second order, $a = (a_1(\tau, t), \dots, a_m(\tau, t)) = a(\tau, t)$ and $g = (g_1(\tau), \dots, g_l(\tau)) = g(\tau)$ are vector-functions, $\langle \, , \, \rangle$ denotes the scalar product of vectors, A is a constant n-matrix, $f = f(\tau, t, \zeta, x)$ is an n-vector-function of variables $(\tau, t, \zeta, x) \in \mathbb{R} \times \mathbb{R}^m \times \overline{B}_\delta^{2l} \times \overline{B}_\Delta^n$.

Assume that the following conditions are satisfied.

$a)$ The vector-function $a(\tau, t)$ has the property of (θ, ω)-periodicity and smoothness with respect to $(\tau, t) \in \mathbb{R} \times \mathbb{R}^m$ of order $(0, e) = (0, 1, \dots, 1)$:

$$a(\tau + \theta, t + q\omega) = a(\tau, t) \in C_{\tau, t}^{(0,e)} \left(\mathbb{R} \times \mathbb{R}^m\right), q \in \mathbb{Z}^m, \tag{2.3}$$

where $\mathbb{Z}^m$ is the set of all integer vectors $q = (q_1, \ldots, q_m)$, $q\omega = (q_1\omega_1, \ldots, q_m\omega_m)$; $\omega_0 = \theta, \omega_1, \ldots, \omega_m$ are positive rationally incommensurable periods.

b) The vector coordinates $\nu_1, \ldots, \nu_l$ are positive rationally incommensurable constants, therefore, numbers $\alpha_j = 2\pi\nu_j^{-1}$, $j = \overline{1,l}$ are also incommensurable and fulfilled

$$q_i\alpha_i + q_j\alpha_j \neq 0, q_i^2 + q_j^2 \neq 0, q_i, q_j \in \mathbb{Z}, \left(i, j = \overline{0,l}\right). \tag{2.4}$$

c) The vector-function $g\,(\tau) = (g_1(\tau), \ldots, g_l(\tau))$ with the components

$$g_j\,(\tau) = (\varphi_j(\tau), \psi_j(\tau)), j = \overline{1,l}$$

is θ-periodic and continuous, i.e.

$$g(\tau + \theta) = g(\tau) \in C\,(\mathbb{R}), \tag{2.5}$$

moreover, constants θ and $\alpha_j, j = \overline{1,l}$ also rationally incommensurable:

$$q_0\theta + q_j\alpha_j \neq 0, q_0^2 + q_j^2 \neq 0, j = \overline{1,l};\; q_0, q_j \in \mathbb{Z}. \tag{2.6}$$

Under assumptions (2.3)–(2.6) on the basis of the vector fields

$$\frac{dt}{d\tau} = a(\tau, t), \tag{2.7}$$

$$\frac{d\zeta}{d\tau} = \nu I\zeta + g(\tau), \tag{2.8}$$

the characteristics

$$t = \lambda\left(\tau, \tau^0, t^0\right), \tag{2.9}$$

$$\zeta = Z\left(\tau - \tau^0\right)\left[\zeta^0 - z\left(\tau^0\right)\right] + z\,(\tau) \tag{2.10}$$

are determined, where $(\tau^0, t^0, \zeta^0) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{2l}$, $Z\,(\tau) = diag\,[Z_1(\tau), \ldots, Z_l(\tau)]$, $z\,(\tau) = (z_1(\tau), \ldots, z_l(\tau))$, $Z_j(\tau) = \begin{pmatrix} \cos\nu_j\tau & -\sin\nu_j\tau \\ \sin\nu_j\tau & \cos\nu_j\tau \end{pmatrix}$, $z_j(\tau + \theta) = z_j(\tau)$,

$$z_j(\tau) = \left[Z_j^{-1}(\tau + \theta) - Z_j^{-1}(\tau)\right]^{-1} \int\limits_{\tau}^{\tau+\theta} Z_j^{-1}(s)g_j(s)\,ds, j = \overline{1,l}.$$

Based on relations (2.9) and (2.10), we define the first integrals of vector fields (2.7) and (2.8) in the form

$$t^0 = \lambda\left(\tau^0, \tau, t\right), \tag{2.11}$$

$$\zeta^0 = Z\left(\tau^0 - \tau\right)\left[\zeta - z\,(\tau)\right] + z\left(\tau^0\right) \equiv \mu\left(\tau^0, \tau, \zeta\right), \tag{2.12}$$

where the following properties hold:

$$\begin{gathered} D\lambda\,(\tau^0, \tau, t) = 0, \lambda\,(\tau^0, \tau^0, t) = t, \\ \lambda\,(\tau', \tau'', \lambda\,(\tau'', \tau, t)) = \lambda\,(\tau', \tau, t)\,, \\ \lambda\,(\tau^0 + \theta, \tau + \theta, t + q\omega) = \lambda\,(\tau^0, \tau, t) + q\omega; \end{gathered} \tag{2.13}$$

$$D\mu\left(\tau^0,\tau,\zeta\right)=0,\mu\left(\tau^0,\tau^0,\zeta\right)=\zeta,$$
$$\mu\left(\tau',\tau'',\mu\left(\tau'',\tau,\zeta\right)\right)=\mu\left(\tau',\tau,\zeta\right), \tag{2.14}$$
$$\mu\left(\tau^0+\theta,\tau+\theta,\zeta\right)=\mu\left(\tau^0,\tau,\zeta\right).$$

Also, note that the matrices $Z_j(\tau+\alpha_j)=Z_j(\tau), j=\overline{1,l}$ are periodic with the incommensurable periods $\alpha_j, j=\overline{1,l}$. Consequently, the matrix $Z(\tau)$ is quasiperiodic with the frequency basis $\nu=(\nu_1,\ldots,\nu_l)$, which is represented by the multiperiodic blocks $Z_j(\tau)$ with the period vector $\alpha=(\alpha_1,\ldots,\alpha_l)$:

$$Z\left(\widehat{\tau}+q\alpha\right)=diag\left[Z_1\left(\tau_1+q_1\alpha_1\right),\ldots,Z_l\left(\tau_l+q_l\alpha_l\right)\right]$$
$$=diag\left[Z_1\left(\tau_1\right),\ldots,Z_l\left(\tau_l\right)\right]=Z\left(\widehat{\tau}\right),q\in\mathbb{R}^l,\widehat{\tau}=\left(\tau_1,\ldots,\tau_l\right)\in\mathbb{R}\times\ldots\times\mathbb{R}=\mathbb{R}^l.$$

In conclusion, we note that by virtue of relations (2.11)–(2.14) with an arbitrary smooth vector-function $u^0\left(t,\zeta\right)\in C_{t,\zeta}^{(e,\tilde{e})}\left(\mathbb{R}^m\times\mathbb{R}^{2l}\right)$, we can determine the zero $u\left(\tau^0,\tau,t,\zeta\right)=u^0\left(\lambda\left(\tau^0,\tau,t\right),\mu\left(\tau^0,\tau,\zeta\right)\right)$ of the operator D: $Du\left(\tau^0,\tau,t,\zeta\right)=0$, where e and $\tilde{e}$ are m and $2l$-vectors with the unit coordinates.

It is easy to verify that the initial problem for the linear system

$$Dx=Ax \tag{2.15}$$

with the initial condition

$$x|_{\tau=\tau^0}=u\left(t,\zeta\right)\in C_{t,\zeta}^{(e,\tilde{e})}\left(\mathbb{R}^m\times\mathbb{R}^{2l}\right) \tag{2.16}$$

is determined by the formula

$$x\left(\tau^0,\tau,t,\zeta\right)=X\left(\tau-\tau^0\right)u\left(\lambda\left(\tau^0,\tau,t\right),\mu\left(\tau^0,\tau,\zeta\right)\right), \tag{2.17}$$

where $X(\tau)$ is the matricant of system (2.15): $DX(\tau)=AX(\tau), X(0)=E$ is the identity matrix.

We note that the condition

$$det\left[X\left(\theta\right)-E\right]\neq 0 \tag{2.18}$$

is necessary for system (2.15) to have only zero (θ,ω)-periodic with respect to (τ,t) solution.

It can be proved that if condition (2.18) is satisfied, then problem (2.15)–(2.16) has a nonzero (θ,ω)-periodic solution (2.17) if and only if the fundamental system

$$\begin{gathered}u\left(\lambda\left(\tau^0,\tau^0+\theta,t,\mu\left(\tau^0,\tau^0+\theta,\zeta\right)\right)\right)\\=\left[E-X(\theta)\right]^{-1}\left[u\left(\lambda\left(\tau^0,\tau^0+\theta,t\right),\mu\left(\tau^0,\tau^0+\theta,\zeta\right)\right)-u\left(t,\zeta\right)\right]\end{gathered} \tag{2.19}$$

has a ω-periodic with respect to t nontrivial smooth solution $u\left(t,\zeta\right)$.

Therefore, if, along with condition (2.18), system (2.19) has only zero solution in the class of functions

$$u\left(t+q\omega,\zeta\right)=u\left(t,\zeta\right)\in C_{t,\zeta}^{(e,\tilde{e})}\left(\mathbb{R}^m\times\mathbb{R}^{2l}\right), \tag{2.20}$$

then system (2.15) does not have (θ,ω)-periodic solutions, except for the zero one.

Now we consider the inhomogeneous system

$$Dx=Ax+f\left(\tau,t,\zeta\right) \tag{2.21}$$

with the vector-function

$$f\left(\tau+\theta,t+q\omega,\zeta\right)=f\left(\tau,t,\zeta\right)\in C_{\tau,t,\zeta}^{(0,e,\tilde{e})}\left(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^{2l}\right),q\in\mathbb{Z}^m. \tag{2.22}$$

Suppose that under condition (2.18), system (2.19) has only zero solution from class (2.20). In other words, system (2.15) has only zero (θ,ω)-periodic with respect to (τ,t) solution.

Further, in order to write out an integral representation of the (θ,ω)-periodic solution of system (2.21), we use vector-function (2.22) along with the characteristics $(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))$, $(s,\lambda(s,\tau+\theta,t),\mu(s,\tau+\theta,\zeta))$ and the following vector-function

$$f_\theta\left(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta)\right)=\begin{cases} f\left(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta)\right), \tau\le s\le 0,\\ f\left(s,\lambda(s,\tau+\theta,t),\mu(s,\tau+\theta,\zeta)\right),\\ 0<s\le\tau+\theta.\end{cases} \tag{2.23}$$

Then under the above conditions, the (θ,ω)-periodic solution $x^*(\tau,t,\zeta)$ of system (2.21) has the following integral representation

$$x^*(\tau,t,\zeta)=\left[X^{-1}(\tau+\theta)-X^{-1}(\tau)\right]^{-1}\int\limits_{\tau}^{\tau+\theta}X^{-1}(s)f_\theta(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))\,ds, \tag{2.24}$$

where $f_\theta\left(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta)\right)$ has form (2.23).

Note that vector-function (2.24) is a (θ,ω)-periodic solution of system (2.21) which can be verified by direct calculations. The uniqueness follows from the fact that homogeneous system (2.15) has only zero (θ,ω)-periodic solution.

Now we state *the main goal of this paper*: investigation of the problem of the existence of a unique (θ,ω)-periodic with respect to (τ,t) solution of quasilinear system (2.1) based on the above information on multiperiodic solutions of linear system (2.21).

3 Multiperiodic solution of a quasilinear system with the differentiation operator D

We consider quasilinear system (2.1) under the following assumptions.

1^0. All assumptions with respect to the input data of linear part of system (2.1) remain valid.

2^0. A vector-function $f(\tau,t,\zeta,x)$ satisfies the condition

$$f\left(\tau+\theta,t+q\omega,\zeta,x\right)=f\left(\tau,t,\zeta,x\right)\in C^{(0,e,\tilde e,\hat e)}_{\tau,t,\zeta,x}\left(\mathbb{R}\times\mathbb{R}^m\times\overline{B}^{2l}_\delta\times\overline{B}^n_\Delta\right), \tag{3.1}$$

where $e,\tilde e,\hat e$ are the vectors with the unit components of dimensions $m,2l,n$, respectively.

3^0. Linear system (2.15) has no (θ,ω)-periodic solutions except zero.

Let $\theta,\omega,\delta,\gamma>0$. We introduce the space $S^{\theta,\omega}_{\delta,\gamma}$ of all n-vector-functions $x(\tau,t,\zeta)$ which are (θ,ω)-periodic with respect to $(\tau,t)\in\mathbb{R}\times\mathbb{R}^m$, continuously differentiable with respect to $(\tau,t,\zeta)\in\mathbb{R}\times\mathbb{R}^m\times B^{2l}_\delta$, continuous with the partial derivatives of order one on the closure $\mathbb{R}\times\mathbb{R}^m\times\overline{B}^{2l}_\delta$ and such that $\|x\|<\gamma$, where

$$\|x\|=\|x\|_\circ+\sum_{j=0}^{m}\left\|\frac{\partial x}{\partial t_j}\right\|_\circ+\sum_{k=1}^{l}\left(\left\|\frac{\partial x}{\partial\xi_k}\right\|_\circ+\left\|\frac{\partial x}{\partial\eta_k}\right\|_\circ\right)$$

is the norm of n-vector-functions $x(\tau,t,\zeta)$ of variables $\tau=t_0$, $t=(t_1,\dots,t_m)$, $\zeta=(\zeta_1,\dots,\zeta_l)=((\xi_1,\eta_1),\dots,(\xi_l,\eta_l))$, where $\|x\|_\circ=sup|x(\tau,t,\zeta)|$, the supremum is taken with respect to $(\tau,t,\zeta)\in\mathbb{R}\times\mathbb{R}^m\times\overline{B}^{2l}_\delta$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$.

We consider the integral operators T, P_k', P_k'', Q_j, R with the kernel $K = K(\tau, s)$ of the form

$$(Tx)(\tau,t,\zeta) = \int\limits_{\tau}^{\tau+\theta} K(\tau,s) f_\theta(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta),x(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta)))\,ds, \tag{3.2}$$

$$K(\tau,s) = \left[X^{-1}(\tau+\theta) - X^{-1}(\tau)\right]^{-1} X^{-1}(s), \tag{3.3}$$

$$(P_k' x)(\tau,t,\zeta) = \int\limits_{\tau}^{\tau+\theta} K(\tau,s)\left[\frac{\partial f_\theta}{\partial \zeta} + \frac{\partial f_\theta}{\partial x}\cdot\frac{\partial x}{\partial \zeta}\right]\frac{\partial \zeta}{\partial \zeta_k}\cdot\frac{\partial \mu_k(s,\tau,\zeta_k)}{\partial \xi_k}\,ds, \tag{3.4}$$

$$(P_k'' x)(\tau,t,\zeta) = \int\limits_{\tau}^{\tau+\theta} K(\tau,s)\left[\frac{\partial f_\theta}{\partial \zeta} + \frac{\partial f_\theta}{\partial x}\cdot\frac{\partial x}{\partial \zeta}\right]\frac{\partial \zeta}{\partial \zeta_k}\cdot\frac{\partial \mu_k(s,\tau,\zeta_k)}{\partial \eta_k}\,ds, \tag{3.5}$$

$$(Q_j x)(\tau,t,\zeta) = \int\limits_{\tau}^{\tau+\theta} K(\tau,s)\left[\frac{\partial f_\theta}{\partial t_j} + \frac{\partial f_\theta}{\partial x}\cdot\frac{\partial x}{\partial t}\right]\frac{\partial \lambda(s,\tau,t)}{\partial t_j}\,ds, \tag{3.6}$$

$$\begin{aligned}(Rx)(\tau,t,\zeta) = f(\tau,t,\zeta,x) + \int\limits_{\tau}^{\tau+\theta} K(\tau,s)\left[\frac{\partial f_\theta}{\partial \zeta} + \frac{\partial f_\theta}{\partial x}\cdot\frac{\partial x}{\partial \zeta}\right]\frac{\partial \mu(s,\tau,\zeta)}{\partial \tau}\,ds \\ +A(Tx)(\tau,t,\zeta) + \int\limits_{\tau}^{\tau+\theta} K(\tau,s)\left[\frac{\partial f_\theta}{\partial t} + \frac{\partial f_\theta}{\partial x}\cdot\frac{\partial x}{\partial t}\right]\frac{\partial \lambda(s,\tau,t)}{\partial t}\,ds,\end{aligned} \tag{3.7}$$

in the space $S_{\delta,\gamma}^{\theta,\omega}$. Here in the Jacobi matrices $\frac{\partial x}{\partial t}, \frac{\partial x}{\partial \sigma}, \frac{\partial \sigma}{\partial \sigma_k}, \frac{\partial f_\theta}{\partial x}$ and partial derivatives $\frac{\partial f_\theta}{\partial \xi_k}, \frac{\partial f_\theta}{\partial \eta_k}, \frac{\partial f_\theta}{\partial t_j}$ in square brackets the variables τ, t, ζ are respectively replaced by $s, \lambda(s,\tau,t), \mu(s,\tau,\zeta) = (\mu_1,\ldots,\mu_l)$, where $\mu_k = \mu_k(s,\tau,\sigma_k), \sigma_k = (\xi_k,\eta_k), k = \overline{1,l}$; the vector-function f_θ is constructed similarly to formula (2.23) based on the given vector-function f.

Note that $P_k'x, P_k''x, Q_jx$ and Rx are obtained from Tx by differentiating it with respect to ξ_k, η_k, t_j and τ, respectively.

Integral operators (3.2)–(3.7) depend on the scalar argument τ and the vector arguments $\zeta_k = (\xi_k,\eta_k)$, $\zeta = (\zeta_1,\ldots,\zeta_l) = ((\xi_1,\eta_1),\ldots,(\xi_l,\eta_l))$, $t = (t_1,\ldots,t_m)$, $x = (x_1,\ldots,x_n)$, $p = (p_1,\ldots,p_k)$, $p_k = \frac{\partial x}{\partial \xi_k}$, $q = (q_1,\ldots,q_k)$, $q_k = \frac{\partial x}{\partial \eta_k}$, $k = \overline{1,l}$, $r = (r_1,\ldots,r_j)$, $r_j = \frac{\partial x}{\partial t_j}, j = \overline{1,m}$. Moreover, dependence on p, q, r is linear.

We have following estimates for $\lambda(s,\tau,t)$, $\frac{\partial \lambda(s,\tau,t)}{\partial \tau}$, $\frac{\partial \lambda(s,\tau,t)}{\partial t_j}$

$$\begin{aligned}&|\lambda(s,\tau,t) - t| \le \|a\|_\circ |\tau - s|, \\ &\left|\frac{\partial \lambda(s,\tau,t)}{\partial \tau}\right| \le \|a\|_\circ \exp\left[\left\|\frac{\partial a}{\partial t}\right\|_\circ |\tau - s|\right], \\ &\left|\frac{\partial \lambda(s,\tau,t)}{\partial t_j}\right| \le \exp\left[\left\|\frac{\partial a}{\partial t}\right\|_\circ |\tau - s|\right], j = \overline{1,m}\end{aligned} \tag{3.8}$$

from the equations of characteristics of the operator D. They are based on the theorems on differentiability with respect to the initial data.

Also we have similar estimates for $\mu_k(s,\tau,\zeta_k)$

$$\begin{aligned}
&|\mu_k(s,\tau,\zeta_k)-z_k(s)|\le |Z_k(s-\tau)|\cdot|\zeta_k-z_k(\tau)|,\\
&\left|\frac{\partial\mu_k(s,\tau,\zeta_k)}{\partial\tau}\right|\le |Z_k(s-\tau)|\cdot|-\nu_kI_2\zeta_k-g_k(\tau)|,\\
&\left|\frac{\partial\mu_k(s,\tau,\zeta_k)}{\partial\xi_k}\right|\le |Z_k(s-\tau)|\cdot|e'|,\\
&\left|\frac{\partial\mu_k(s,\tau,\zeta_k)}{\partial\eta_k}\right|\le |Z_k(s-\tau)|\cdot|e''|,
\end{aligned}\tag{3.9}$$

where $|e'|=|e''|=1$.

Assume that $\mu(s,\tau,\zeta)=(\mu_1(s,\tau,\zeta_1),\dots,\mu_l(s,\tau,\zeta_l))$. Suppose that $\mu_k=(\mu_k',\mu_k'')$ and $z_k=(z_k',z_k'')$, then we have the following coordinate representations of the form

$$\begin{pmatrix}\mu_k'(s,\tau,\xi_k,\eta_k)-z_k'(s)\\ \mu_k''(s,\tau,\xi_k,\eta_k)-z_k''(s)\end{pmatrix}=\begin{pmatrix}[\xi_k-z_k'(\tau)]\cos\nu_k(s-\tau)-[\eta_k-z_k''(\tau)]\sin\nu_k(s-\tau)\\ [\xi_k-z_k'(\tau)]\sin\nu_k(s-\tau)+[\eta_k-z_k''(\tau)]\cos\nu_k(s-\tau)\end{pmatrix},$$

$$\begin{pmatrix}\frac{\partial}{\partial\tau}\mu_k'(s,\tau,\xi_k,\eta_k)\\ \frac{\partial}{\partial\tau}\mu_k''(s,\tau,\xi_k,\eta_k)\end{pmatrix}=\begin{pmatrix}[\nu_k\eta_k-\varphi_k(\tau)]\cos\nu_k(s-\tau)+[\nu_k\xi_k+\psi_k(\tau)]\sin\nu_k(s-\tau)\\ [\nu_k\eta_k-\varphi_k(\tau)]\sin\nu_k(s-\tau)-[\nu_k\xi_k+\psi_k(\tau)]\cos\nu_k(s-\tau)\end{pmatrix},$$

$$\begin{pmatrix}\frac{\partial}{\partial\xi_k}\mu_k'(s,\tau,\xi_k,\eta_k)\\ \frac{\partial}{\partial\xi_k}\mu_k''(s,\tau,\xi_k,\eta_k)\end{pmatrix}=\begin{pmatrix}\cos\nu_k(s-\tau)\\ \sin\nu_k(s-\tau)\end{pmatrix},$$

$$\begin{pmatrix}\frac{\partial}{\partial\eta_k}\mu_k'(s,\tau,\xi_k,\eta_k)\\ \frac{\partial}{\partial\eta_k}\mu_k''(s,\tau,\xi_k,\eta_k)\end{pmatrix}=\begin{pmatrix}-\sin\nu_k(s-\tau)\\ \cos\nu_k(s-\tau)\end{pmatrix}.$$

Further, we consider the matrix $\dfrac{\partial x}{\partial\zeta}=\left[\dfrac{\partial x_j}{\partial\xi_k},\dfrac{\partial x_j}{\partial\eta_k}\right]_{\substack{j=\overline{1,n}\\ k=\overline{1,l}}}=\left[\dfrac{\partial x}{\partial\xi_k},\dfrac{\partial x}{\partial\eta_k}\right]_{k=\overline{1,l}}$ whose columns are vectors $p_k=\dfrac{\partial x}{\partial\xi_k}$, $q_k=\dfrac{\partial x}{\partial\eta_k}$, $k=\overline{1,l}$. Moreover,

$$\left|\frac{\partial x}{\partial\zeta}\right|\le\sum_{k=1}^{l}\left(\left|\frac{\partial x}{\partial\xi_k}\right|+\left|\frac{\partial x}{\partial\eta_k}\right|\right)=\sum_{k=1}^{l}(|p_k|+|q_k|).$$

Similarly, we have

$$\left|\frac{\partial x}{\partial t}\right|\le\sum_{j=1}^{m}\left|\frac{\partial x}{\partial t_j}\right|=\sum_{j=1}^{m}|r_j|.$$

The matrix $\dfrac{\partial\zeta}{\partial\zeta_k}=\left[\dfrac{\partial\zeta_j}{\partial\xi_k},\dfrac{\partial\zeta_j}{\partial\eta_k}\right]_{j=\overline{1,l}}$ is a $(l\times 2)$-dimensional matrix, where the k-th row consists of the units, the rest of its elements are zeros, since $\zeta_j=(\xi_j,\eta_j)$, $j=\overline{1,j}$. It is obvious that $\left|\dfrac{\partial\zeta}{\partial\zeta_k}\right|\le\sqrt{2}$.

It is easy to obtain the estimates

$$\left|\frac{\partial\mu_k(s,\tau,\zeta_k)}{\partial\xi_k}\right|\le\sqrt{2},$$

$$\left|\frac{\partial\mu_k(s,\tau,\zeta_k)}{\partial\eta_k}\right|\le\sqrt{2}, \tag{3.10}$$

$$\left|\frac{\partial\mu(s,\tau,\zeta)}{\partial\tau}\right|\le\sqrt{2l},k=\overline{1,l}$$

from the above representations and estimates (3.9).

Further, by virtue of condition (3.1) on twice continuous differentiability, we have the following estimates for the differences and their corollaries

1) $|f_\theta(\tau,t,\zeta,x)-f_\theta(\tau,t,\zeta,y)|\le c_1|x-y|$,

2) $\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial\zeta}-\dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial\zeta}\right|\le c_2|x-y|$,

3) $\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial t_j}-\dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial t_j}\right|\le c_3|x-y|, j=\overline{1,m}$,

4) $\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial x}-\dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial x}\right|\le c_4|x-y|$,

5) $\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial x}\dfrac{\partial x}{\partial\zeta}-\dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial x}\dfrac{\partial y}{\partial\zeta}\right|\le\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial x}-\dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial x}\right|\cdot\left|\dfrac{\partial x}{\partial\zeta}\right|$
$+\left|\dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial x}\right|\cdot\left|\dfrac{\partial x}{\partial\zeta}-\dfrac{\partial y}{\partial\zeta}\right|$,

6) $\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial x}\dfrac{\partial x}{\partial t}-\dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial x}\dfrac{\partial y}{\partial t}\right|\le\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial x}-\dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial x}\right|\cdot\left|\dfrac{\partial x}{\partial t}\right|$
$+\left|\dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial x}\right|\cdot\left|\dfrac{\partial x}{\partial t}-\dfrac{\partial y}{\partial t}\right|$,

7) $|f_\theta(\tau,t,\zeta,x)|\le|f_\theta(\tau,t,\zeta,0)|+|f_\theta(\tau,t,\zeta,x)-f_\theta(\tau,t,\zeta,0)|\le\chi_1+c_1|x|$,

8) $\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial\zeta}\right|\le\left|\dfrac{\partial f_\theta(\tau,t,\zeta,0)}{\partial\zeta}\right|+\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial\zeta}-\dfrac{\partial f_\theta(\tau,t,\zeta,0)}{\partial\zeta}\right|\le\chi_2+c_2|x|$,

9) $\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial t_j}\right|\le\left|\dfrac{\partial f_\theta(\tau,t,\zeta,0)}{\partial t_j}\right|+\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial t_j}-\dfrac{\partial f_\theta(\tau,t,\zeta,0)}{\partial t_j}\right|\le\chi_3+c_3|x|$,
$j=\overline{1,m}$,

10) $\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial x}\right|\le\left|\dfrac{\partial f_\theta(\tau,t,\zeta,0)}{\partial x}\right|+\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial x}-\dfrac{\partial f_\theta(\tau,t,\zeta,0)}{\partial x}\right|\le\chi_4+c_4|x|$,

11) $\left|\dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial x}\dfrac{\partial x}{\partial\zeta}-\dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial x}\dfrac{\partial y}{\partial\zeta}\right|\le c_4|x-y|\sum\limits_{k=1}^{l}(|p_k|+|q_k|)$

$+ (\chi_4 + c_4\gamma) \sum\limits_{k=1}^{l} (|p_k - \tilde{p}_k| + |q_k - \tilde{q}_k|)$, where $\tilde{p}_k = \dfrac{\partial y}{\partial \xi_k}$, $\tilde{q}_k = \dfrac{\partial y}{\partial \eta_k}$, $k = \overline{1, l}$,

12) $\left| \dfrac{\partial f_\theta(\tau,t,\zeta,x)}{\partial x} \dfrac{\partial x}{\partial t} - \dfrac{\partial f_\theta(\tau,t,\zeta,y)}{\partial x} \dfrac{\partial y}{\partial t} \right| \leq c_4|x-y| \sum\limits_{j=1}^{m} |r_j| + (\chi_4 + c_4\gamma) \sum\limits_{j=1}^{m} |r_j - \tilde{r}_j|$,

where $\tilde{r}_j = \dfrac{\partial y}{\partial t_j}$, $j = \overline{1,m}$. Here c_1, c_2, c_3, c_4 are the positive Lipschitz constants of the vector and matrix functions f_θ, $\dfrac{\partial f_\theta}{\partial \zeta}$, $\dfrac{\partial f_\theta}{\partial t_j}$, $j = \overline{1,m}$, $\dfrac{\partial f_\theta}{\partial x}$; $\chi_1, \chi_2, \chi_3, \chi_4 > 0$ are the maximums of the functions $|f_\theta|$, $\left|\dfrac{\partial f_\theta}{\partial \zeta}\right|$, $\left|\dfrac{\partial f_\theta}{\partial t_j}\right|$, $j = \overline{1,m}$, $\left|\dfrac{\partial f_\theta}{\partial x}\right|$ with respect to $(\tau, t, \zeta, 0)$. (For χ_3 the maximum is also with respect to $j = \overline{1,m}$.)

Obviously, the integral operator $T_\circ$ of the form

$$(T_\circ x)(\tau,t,\zeta) = \int\limits_{\tau}^{\tau+\theta} K(\tau,s)x(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))\,ds \tag{3.11}$$

with kernel (3.3) is linear, maps the space $S_{\delta,\gamma}^{\theta,\omega}$ into itself and is bounded by

$$\|T_\circ x\|_\circ = \kappa \|x\|_\circ, \tag{3.12}$$

where

$$\kappa = \sup_{0 \leq \tau \leq \theta} \int\limits_{\tau}^{\tau+\theta} |K(\tau,s)|\,ds.$$

Now we estimate operators (3.2)–(3.7) by using inequalities 1)-12) and (3.8)–(3.12). For $x \in S_{\delta,\gamma}^{\theta,\omega}$

$$\big|(Tx)(\tau,t,\zeta)\big| \leq \kappa\big(\chi_1 + c_1\|x\|_\circ\big),$$

$$\big|(P_k'x)(\tau,t,\zeta)\big| \leq \kappa\Big[\chi_2 + c_2\|x\|_\circ + \big(\chi_4 + c_4\|x\|_\circ\big)\sum_{k=1}^{l}\big(\|p_k\|_\circ - \|q_k\|_\circ\big)\Big]\cdot\sqrt{2}\cdot\sqrt{2},$$

$$\big|(P_k''x)(\tau,t,\zeta)\big| \leq \kappa\Big[\chi_2 + c_2\|x\|_\circ + \big(\chi_4 + c_4\|x\|_\circ\big)\sum_{k=1}^{l}\big(\|p_k\|_\circ - \|q_k\|_\circ\big)\Big]\cdot\sqrt{2}\cdot\sqrt{2},$$

$$\big|(Q_jx)(\tau,t,\zeta)\big| \leq \kappa\Big[\chi_3 + c_3\|x\|_\circ + \big(\chi_4 + c_4\|x\|_\circ\big)\sum_{j=1}^{m}\|r_j\|_\circ\Big]\cdot e^{2\|\frac{\partial a}{\partial t}\|_\circ\theta},$$

$$\big|(Rx)(\tau,t,\zeta)\big| \leq \kappa\Big[\chi_2 + c_2\|x\|_\circ + \big(\chi_4 + c_4\|x\|_\circ\big)\sum_{k=1}^{l}\big(\|p_k\|_\circ - \|q_k\|_\circ\big)\Big]\cdot\sqrt{2l}$$

$$+ \kappa\Big[\chi_3 + c_3\|x\|_\circ + \big(\chi_4 + c_4\|x\|_\circ\big)\sum_{j=1}^{m}\|r_j\|_\circ\Big]\cdot e^{2\|\frac{\partial a}{\partial t}\|_\circ\theta} + |A|\kappa\big(\chi_1 + c_1\|x\|_\circ\big)$$

$+ \big(\chi_1 + c_1\|x\|_\circ\big)$ for $|s| \leq \theta$, $|\tau| \leq \theta$.

Let $max\,\{\chi_1,\chi_2,\chi_3,\chi_4\} = \chi$, $max\,\{c_1,c_2,c_3,c_4\} = c$. Then we obtain the following estimates via γ:

$$\|(Tx)\|_\circ \leq \kappa(\chi + c\gamma)(1+\gamma);$$

$$\|(P_k'x)\|_\circ \leq 4l\kappa(\chi + c\gamma)(1+\gamma), k = \overline{1,l};$$

$\|(P''_k x)\|_\circ \le 4l\kappa(\chi + c\gamma)(1+\gamma), k = \overline{1,l};$

$\|(Q_j x)\|_\circ \le m\varepsilon\kappa(\chi + c\gamma)(1+\gamma), j = \overline{1,m}, \varepsilon = exp\left[2\left\|\dfrac{\partial a}{\partial t}\right\|_\circ \theta\right];$

$\|(Rx)\|_\circ \le \big[1 + \kappa|A| + m\varepsilon\kappa(1+\gamma) + (2l)^{3/2}\kappa(1+\gamma)\big](\chi + c\gamma)$

$\le \big(1 + \kappa|A| + m\varepsilon\kappa + (2l)^{3/2}\kappa\big)(\chi + c\gamma)(1+\gamma).$

Then we have

$$\begin{aligned} \|Tx\| = \|(Tx)\|_\circ + \sum_{k=1}^{l}(\|(P'_k x)\|_\circ + \|(P''_k x)\|_\circ) + \sum_{j=1}^{m}\|(Q_j x)\|_\circ \\ +\|(Rx)\|_\circ \le \big(\kappa + 8l^2\kappa + m^2\varepsilon\kappa + 1 + \kappa|A| + m\varepsilon\kappa \\ +(2l)^{3/2}\kappa\big)(1+\gamma)(\chi + c\gamma) = c_*(\chi + c\gamma) \le \gamma, \end{aligned} \tag{3.13}$$

for sufficiently small χ and c, where

$$c_* = \big(\kappa + 8l^2\kappa + m^2\varepsilon\kappa + 1 + \kappa|A| + m\varepsilon\kappa + (2l)^{3/2}\kappa\big)(1+\gamma).$$

So, for sufficiently small χ and c the operator T maps the space $S^{\theta,\omega}_{\delta,\gamma}$ into itself.

Next, we estimate the differences in the values of operators (3.2)–(3.7) at different points $x, y \in S^{\theta,\omega}_{\delta,\gamma}$ by using 1)-12) and (3.8)–(3.12):

$|(Tx)(\tau,t,\zeta) - (Ty)(\tau,t,\zeta)| \le \kappa c_1 \|x-y\|_\circ,$

$|(P'_k x)(\tau,t,\zeta) - (P'_k y)(\tau,t,\zeta)| \le \kappa\Big[c_2\|x-y\|_\circ + c_4\|x-y\|_\circ \sum\limits_{k=1}^{l}\big(\|p_k\|_\circ + \|q_k\|_\circ\big)$

$+\big(\chi_4 + c_4\|y\|_\circ\big)\cdot \sum\limits_{k=1}^{l}\big(\|p_k - \tilde{p}_k\|_\circ + \|q_k - \tilde{q}_k\|_\circ\big)\Big]\cdot\sqrt{2}\cdot\sqrt{2},$

$|(P''_k x)(\tau,t,\zeta) - (P''_k y)(\tau,t,\zeta)| \le \kappa\Big[c_2\|x-y\|_\circ + c_4\|x-y\|_\circ \sum\limits_{k=1}^{l}\big(\|p_k\|_\circ + \|q_k\|_\circ\big)$

$+\big(\chi_4 + c_4\|y\|_\circ\big)\cdot \sum\limits_{k=1}^{l}\big(\|p_k - \tilde{p}_k\|_\circ + \|q_k - \tilde{q}_k\|_\circ\big)\Big]\cdot\sqrt{2}\cdot\sqrt{2},$

$|(Q_j x)(\tau,t,\zeta) - (Q_j y)(\tau,t,\zeta)| \le \kappa\Big[c_3\|x-y\|_\circ + c_4\|x-y\|_\circ \sum\limits_{j=1}^{m}\|r_j\|_\circ$

$+\big(\chi_4 + c_4\|y\|_\circ\big)\sum\limits_{j=1}^{m}\|r_j - \tilde{r}_j\|_\circ\Big]\cdot e^{2\|\frac{\partial a}{\partial t}\|_\circ\theta},$

$|(Rx)(\tau,t,\zeta) - (Ry)(\tau,t,\zeta)| \le c_1\|x-y\|_\circ + \kappa\,|A|\cdot c_1\|x-y\|_\circ + \kappa\Big[c_2\|x-y\|_\circ$

$+c_4\|x-y\|_\circ \sum\limits_{k=1}^{l}\big(\|p_k\|_\circ + \|q_k\|_\circ\big) + \big(\chi_4 + c_4\|y\|_\circ\big)\sum\limits_{k=1}^{l}\big(\|p_k - \tilde{p}_k\|_\circ + \|q_k - \tilde{q}_k\|_\circ\big)\Big]\sqrt{2l}$

$+\kappa\Big[c_3\|x-y\|_\circ + c_4\|x-y\|_\circ \sum\limits_{j=1}^{m}\|r_j\|_\circ + \big(\chi_4 + c_4\|y\|_\circ\big)\sum\limits_{j=1}^{m}\|r_j - \tilde{r}_j\|_\circ\Big]\|a\|_\circ e^{2\|\frac{\partial a}{\partial t}\|_\circ\theta}.$

Hence, we have

$\|Tx - Ty\|_\circ \le \kappa c\,\|x-y\|_\circ;$

$$\|P'_k x - P'_k y\|_\circ \le 2\kappa\Big[\big(c + c2l\gamma\big)\|x-y\|_\circ + \big(\chi + c\gamma\big)\sum_{k=1}^{l}\big(\|p_k - \tilde{p}_k\|_\circ + \|q_k - \tilde{q}_k\|_\circ\big)\Big],$$
$k = \overline{1,l}$;

$$\|P''_k x - P''_k y\|_\circ \le 2\kappa\Big[\big(c + c2l\gamma\big)\|x-y\|_\circ + \big(\chi + c\gamma\big)\sum_{k=1}^{l}\big(\|p_k - \tilde{p}_k\|_\circ + \|q_k - \tilde{q}_k\|_\circ\big)\Big],$$
$k = \overline{1,l}$;

$$\|Q_j x - Q_j y\|_\circ \le \kappa\epsilon\Big[\big(c + cm\gamma\big)\|x-y\|_\circ + \big(\chi + c\gamma\big)\sum_{j=1}^{m}\|r_j - \tilde{r}_j\|_\circ\Big], j = \overline{1,m};$$

$$\|Rx - Ry\|_\circ \le \Big[c + \kappa c|A| + \kappa c\sqrt{2l} + \kappa c(2l)^{3/2}\gamma\big)\Big]\|x-y\|_\circ$$
$$+ \kappa\sqrt{2l}\big(\chi + c\gamma\big)\sum_{k=1}^{l}\big(\|p_k - \tilde{p}_k\|_\circ + \|q_k - \tilde{q}_k\|_\circ\big) + \kappa\|a\|_\circ\epsilon\big(c + cm\gamma\big)\|x-y\|_\circ$$
$$+ \kappa\|a\|_\circ\epsilon\big(\chi + c\gamma\big)\sum_{j=1}^{m}\|r_j - \tilde{r}_j\|_\circ.$$

Further, summing up these inequalities, we obtain

$$\|Tx - Ty\|_\circ + \sum_{k=1}^{l}\|P'_k x - P'_k y\|_\circ + \sum_{k=1}^{l}\|P''_k x - P''_k y\|_\circ + \sum_{j=1}^{m}\|Q_j x - Q_j y\|_\circ + \|Rx - Ry\|_\circ$$
$$\le c\Big[\kappa + 8l\kappa\big(1 + 2l\gamma\big) + \kappa\varepsilon\big(1 + m\gamma\big) + \big(1 + \kappa|A| + \kappa\sqrt{2l} + \kappa(2l)^{3/2}\gamma\big)$$
$$+ \kappa\|a\|_\circ\varepsilon\big(1 + m\gamma\big)\Big]\|x-y\|_\circ + \big(\chi + c\gamma\big)\big[8l\kappa + \sqrt{2l}\kappa\big]\sum_{k=1}^{l}\big(\|p_k - \tilde{p}_k\|_\circ + \|q_k - \tilde{q}_k\|_\circ\big)$$
$$+ \big(\chi + c\gamma\big)\big[\kappa\varepsilon m + \kappa m\varepsilon\|a\|_\circ\big]\sum_{j=1}^{m}\|r_j - \tilde{r}_j\|_\circ,$$

where $p_k = \dfrac{\partial x}{\partial \xi_k}$, $\tilde{p}_k = \dfrac{\partial y}{\partial \xi_k}$, $q_k = \dfrac{\partial x}{\partial \eta_k}$, $\tilde{q}_k = \dfrac{\partial y}{\partial \eta_k}$, $r_j = \dfrac{\partial x}{\partial t_j}$, $\tilde{r}_j = \dfrac{\partial y}{\partial t_j}$.

Hence, for sufficiently small values of χ and c, we have

$$\|Tx - Ty\| = \|Tx - Ty\|_\circ + \sum_{k=1}^{l}\|P'_k x - P'_k y\|_\circ + \sum_{k=1}^{l}\|P''_k x - P''_k y\|_\circ$$
$$+ \sum_{j=1}^{m}\|Q_j x - Q_j y\|_\circ + \|Rx - Ry\|_\circ \le (\chi + c\gamma)c_0\Big[\|x-y\|_\circ \tag{3.14}$$
$$+ \sum_{k=1}^{l}(\|p_k - \tilde{p}_k\|_\circ + \|q_k - \tilde{q}_k\|_\circ) + \sum_{j=1}^{m}\|r_j - \tilde{r}_j\|_\circ\Big] \le (\chi + c\gamma)c_0\,\|x-y\|\,,$$

where $c_0 = max\Big\{\big[\kappa + 8l\kappa(1 + 2l\gamma) + \kappa\varepsilon(1 + m\gamma) + (1 + \kappa|A| + \kappa\sqrt{2l} + \kappa(2l)^{3/2}\gamma) + \kappa\|a\|_\circ\varepsilon(1 + m\gamma)\big]; \big[8l\kappa + \sqrt{2l}\kappa\big]; \big[\kappa\varepsilon m + \kappa m\varepsilon\|a\|_\circ\big]\Big\}$.

Under the conditions

$$c_*(\chi + c\gamma) \le \gamma, c_0(\chi + c\gamma) < 1 \tag{3.15}$$

by virtue of estimates (3.13) and (3.14), it is clear that operator (3.2) maps the space $S^{\theta,\omega}_{\delta,\gamma}$ into itself and is a contraction operator. The space $S^{\theta,\omega}_{\delta,\gamma}$ is complete.

Therefore, there exists a unique fixed point $x^*(\tau,t,\zeta) \in S^{\theta,\omega}_{\delta,\gamma}$ of the operator T: $(Tx^*)(\tau,t,\zeta) = x^*(\tau,t,\zeta)$ that is

$$x^*(\tau,t,\zeta) \equiv \int\limits_{\tau}^{\tau+\theta} K(\tau,s) f_\theta(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta),x^*(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta)))\, ds.$$

It is the unique (θ,ω)-periodic solution of system (2.1) by condition 3^0. Thus, the following theorem is proved.

Theorem 3.1. *Suppose that*

1) conditions (2.3)–(2.6) *are satisfied with respect to the input data of the differentiation operator D,*

2) the matrix A is such that linear homogeneous system (2.15) *corresponding to the given quasilinear system has no (θ,ω)-periodic solutions, except for the zero one (in particular, if conditions* (2.18)–(2.20) *are satisfied)*
and

3) the vector-function $f(\tau,t,\zeta,x)$ satisfies condition (3.1).

Then, under condition (3.15) *with respect to the parameters c_0, c_*, c, χ, γ quasilinear system* (2.1) *has a unique (θ,ω)-periodic solution $x^*(\tau,t,\zeta) \in S^{\theta,\omega}_{\delta,\gamma}$.*

4 Conclusion

A technique has been developed for investigation of oscillatory solutions of perturbed quasilinear autonomous systems of form (2.1)–(2.2), based on proving the existence of a fixed point of nonlinear operator (3.2), which is an analogue of representation (2.24) in space $S^{\theta,\omega}_{\delta,\gamma}$.

The main essence of the technique for investigation of multiperiodic solutions of the system under consideration is the further development of the methods of papers [15, 16], on the basis of the fundamental methods of works [4, 9, 11, 17] and investigation carried out in [5–8, 10, 12–14]. In conclusion, the developed technique allows to establish sufficient conditions for the existence and uniqueness of multiperiodic solutions of quasilinear systems (2.1) with the differentiation operator (2.2).

Acknowledgments

The authors are thankful to the editors for careful reading the paper and useful comments which allowed to improve the quality of the paper.

References

[1] A.P. Bergamasco, Dattori da Silva, R.B. Gonzalez, *Existence and regularity of periodic solutions to certain first-order partial differential equations.* Journal of Fourier Analysis and Applications 23 (2017), no. 1, 65–90.

[2] A.B. Berzhanov, E.K. Kurmangaliev, *Solution of a countable system of quasilinear partial differential equations multiperiodic in a part of variables.* Ukrainian Mathematical Journal 61 (2009), no. 2, 336–345.

[3] A. Fonda, A. Sfecci, *Multiple periodic solutions of hamiltonian systems confined in a box.* Discrete and Continuous Dynamical Systems-Series A 37 (2017), no. 3, 1425–1436.

[4] V.Kh. Kharasakhal, *Almost-periodic solutions of ordinary differential equations.* Nauka, Alma-Ata, 1970 (in Russian).

[5] A.A. Kulzhumiyeva, Z.A. Sartabanov, *Integration of a linear equation with differential operator, corresponding to the main diagonal in the space of independent variables, and coefficients, constant on the diagonal.* Russian Mathematics 63 (2019), no. 6, 29–41.

[6] A.A. Kulzhumiyeva, Z.A. Sartabanov, *On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables.* Eurasian Mathematical Journal 8 (2017), no. 1, 67–75.

[7] A.A. Kulzhumiyeva, Zh.A. Sartabanov, *General bounded multiperiodic solutions of linear equations with differential operator in the direction of the mail diagonal.* Bulletin of the Karaganda University - Mathematics 92 (2018), no. 4, 44–53.

[8] A.A. Kulzhumiyeva, Zh.A. Sartabanov, *On reducibility of linear D-e-system with constant coefficients on the diagonal to D-e-system with Jordan matrix in the case of equivalence of its higher order one equation.* Bulletin of the Karaganda University - Mathematics 84 (2016), no. 4, 88–93.

[9] A.A. Kulzhumiyeva, Zh.A. Sartabanov, *Periodic solutions of the systems of differential equations with multidimensional time.* Editorial and Publishing Center of the M. Utemisov West Kazakhstan State University, Uralsk, 2013 (in Russian).

[10] A.A. Kulzhumiyeva, Zh.A. Sartabanov, *Reduction of linear homogeneous D-e-systems to the Jordan canonical form.* News of the National Academy of Sciences of the Republic of Kazakhstan. Physico-Mathematical Series 5 (2017), no. 315, 5–12.

[11] A.A. Mukhambetova, Zh.A. Sartabanov, *Stability of solutions of the systems of differential equations with multidimensional time.* Publishing House "Print A", Aktobe, 2007 (in Russian).

[12] B.Zh. Muhambetova, Zh.A. Sartabanov, A.A. Kulzhumiyeva, *Multiperiodic solutions of systems of equations with one quasi-linear differential operator in partial derivatives of the first order.* Bulletin of the Karaganda University - Mathematics 78 (2015), no. 2, 112–117.

[13] Zh.A. Sartabanov, *Pseudoperiodic solutions of a system of integro-differential equations.* Ukrainian Mathemstical Journal 41 (1989), no. 1, 116–120.

[14] Zh.A. Sartabanov, *The multi-period solution of a linear system of equations with the operator of differentiation along the main diagonal of the space of independent variables and delayed arguments.* AIP Conference Proceedings 1880 (2017), 040020-1–040020-5.

[15] Zh.A. Sartabanov, B.Zh. Omarova, *Multiperiodic solutions of autonomous systems with operator of differentiation on the Lyapunov's vector field.* AIP Conference Proceedings 1997 (2018), 020041-1–020041-4.

[16] Zh.A. Sartabanov, B.Zh. Omarova, *On multi-periodic solutions of quasilinear autonomous systems with operator of differentiation on the Lyapunov's vector field.* Bulletin of the Karaganda University - Mathematics 94 (2019), no. 2, 70–83.

[17] D.U. Umbetzhanov, *Almost multiperiodic solutions of partial differential equations.* Nauka, Alma-Ata, 1979 (in Russian).

Bibigul Zharbolovna Omarova, Zhaishylyk Almaganbetovich Sartabanov
Department of Mathematics
K. Zhubanov Aktobe Regional State University
34 A. Moldagulova St
030000 Aktobe, Kazakhstan
E-mails: bibigul_zharbolkyzy@mail.ru, sartabanov42@mail.ru

Received: 14.10.2019