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SHAVKAT ARIFJANOVICH ALIMOV

(to the 75th birthday)

Shavkat Arifjanovich Alimov was born on March 2, 1945 in the city of Nukus, Uzbekistan. In 1968, he graduated from the Department of Mathematics of Physical Faculty of the M.V. Lomonosov Moscow State University (MSU), receiving a diploma with honors. From 1968 to 1970, he was a post-graduate student in the same department under the supervision of Professor V.A. Il'in. He defended his PhD thesis in 1970. In May 1973, at the age of 28, he defended his doctoral thesis devoted to equations of mathematical physics. In 1973, for research on the spectral theory, he was awarded the highest youth prize of the USSR.

From 1974 to 1984, he worked as a professor in the Department of General Mathematics at the Faculty of Computational Mathematics and Cybernetics. In 1984, Sh.A. Alimov joined the Tashkent State University (TSU) as a professor. From 1985 to 1987 he worked as the Rector of the Samarkand State University, from 1987 to 1990 - the Rector of the TSU, from 1990 to 1992 - the Minister of Higher and Secondary Special Education of the Republic of Uzbekistan. From 1992 to 1994, he headed the Department of Mathematical Physics of the TSU.

After some years of diplomatic work, he continued his academic career as a professor of the Department of Mathematical Physics at the National University of Uzbekistan (NUU). From the first days of the opening of the Tashkent branch of the MSU in 2006, he worked as a professor in the Department of Applied Mathematics. From 2012 to 2017, he headed the Laboratory of Mathematical Modeling of the Malaysian Institute of Microelectronic Systems in Kuala Lumpur. From 2017 to 2019, he worked as a professor at the Department of Differential Equations and Mathematical Physics of the NUU. From 2019 to the present, Sh.A. Alimov is a Scientific Consultant at the Center for Intelligent Software Systems, and an adviser to the Rector of the NUU.

The main scientific activity of Sh.A. Alimov is connected with the spectral theory of partial differential equations and the theory of boundary value problems for equations of mathematical physics. He obtained series of remarkable results in these fields. They cover many important problems of the theory of Schrodinger equations with singular potentials, the theory of boundary control of the heat transfer process, the mathematical problems of peridynamics related to the theory of hypersingular integrals.

In 1984, Sh.A. Alimov was elected a corresponding member and in 2000 an academician of the Academy of Sciences of Uzbekistan. He was awarded several prestigious state prizes.

Sh.A. Alimov has over 150 published scientific and a large number of educational works. Among his pupils there are 10 doctors of sciences and more than 20 candidates of sciences (PhD) working at universities of Uzbekistan, Russia, USA, Finland, and Malaysia.

For about thirty years, Sh.A. Alimov has been actively involved in the reform of mathematical school education.

Sh.A. Alimov meets his 75th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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ON SOME DIFFERENTIAL PROPERTIES OF SMALL SMALL SOBOLEV-MORREY SPACES

A.M. Najafov

Communicated by R. Oinarov

Key words: small small Lebesgue-Morrey and Sobolev-Morrey spaces, embedding theorem, Hölder condition.

AMS Mathematics Subject Classification: 46E30, 35A31.

Abstract. In the paper small small Sobolev-Morrey spaces are constructed. Differential and differential-difference properties of functions belonging to the constructed space are studied by the integral representation method.

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1 Introduction

A new type functional spaces that got the name of Lebesgue grand spaces $L_{p)}(G)$ (meas $G < \infty$) was introduced in [8]. These spaces were introduced for the purpose of investigating local integrability of the Jacobian of functions belonging to the Sobolev space $W^{1,p}_{loc}(G, \mathbb{R}^n)$. Further, these spaces were developed and generalized by many mathematicians. For example, the Lebesgue small space $L_{(p}(G)$, Lebesgue-Morrey grand grand spaces $L_{p),\lambda)}(G)$ and Sobolev-Morrey grand spaces $W^l_{p),\varkappa,a}(G)$ were introduced and studied in [1, 3, 4, 6, 7, 9-12, 18, 19].

This paper is devoted to introduction and study of some properties of functions from the space called small small Sobolev-Morrey spaces $W^l_{(p,(\varkappa,a,\alpha}(G)(l \in \mathbb{N}^n$, $p \in (1,\infty)$, $\varkappa \in (0,\infty)^n, \alpha \geq 0$, $a \in [0,1], G \subset \mathbb{R}^n$ is a bounded domain) from the point of view of the embedding theory. In other words, at first the modification of the Lebesgue-Morrey space called small small Lebesgue-Morrey spaces $L_{(p,(\varkappa,a,\alpha}(G)$ is constructed and on the base of the space $L_{(p,(\varkappa,a,\alpha}(G)$ small small spaces $W^l_{(p,(\varkappa,a,\alpha}(G)$ are constructed. Then, by the method of integral representation of functions defined on n-dimensional domains and satisfying the flexible λ-horn condition, the Sobolev-type inequalities are proved in this space, and it is proved that for $f \in W^l_{(p,(\varkappa,a,\alpha}(G)$, generalized mixed derivatives $D^\nu f$ satisfy the Hölder condition in the metric of the space $L_{q-\varepsilon}(G)$ with the exponent σ.

It should be noted that in this paper we not only introduce a new space but also show that the Hölder exponent σ is greater than in previous papers [5], [13]-[16]. Hence, Theorem 1.3 can be more effective when studying smoothness properties of solutions of partial differential equations.

Let G be a bounded domain of the space $\mathbb{R}^n$, for any $t > 0$ and $x \in \mathbb{R}^n$, we denote

$$I_{t^\varkappa}(x) = \left\{ y : |y_j - x_j| < \frac{1}{2} t^{\varkappa_j}, \quad j = 1, 2, ..., n \right\}, \; G_{t^\varkappa}(x) = G \cap I_{t^\varkappa}(x)$$

Definition 1. We denote by $W^l_{(p,(\varkappa,a,\alpha}(G)$ the space of all locally summable functions f on G and having on G the generalized derivatives $D_i^{l_i}f$ $(i=1,2,...,n)$ with the finite norm

$$\|f\|_{W^l_{(p,(\varkappa,a,\alpha}(G)} = \|f\|_{(p,(\varkappa,a,\alpha;G} + \sum_{i=1}^n \left\|D_i^{l_i}f\right\|_{(p,(\varkappa,a,\alpha;G}, \tag{1.1}$$

where

$$\|f\|_{(p,(\varkappa,a,\alpha;G} = \|f\|_{L_{(p,(\varkappa,a,\alpha}(G)}$$

$$= \sup_{\substack{x\in G\\ 0<t\le d}} \inf_{0<\varepsilon<s_m} \left(\frac{1}{t^{|\varkappa|a+\alpha\varepsilon}} \varepsilon^{-\frac{p-\varepsilon}{(p-\varepsilon)'}} \frac{1}{|G_{t^\varkappa}(x)|} \int\limits_{G_{t^\varkappa}(x)} |f(y)|^{p-\varepsilon}\, dy \right)^{\frac{1}{p-\varepsilon}}, \tag{1.2}$$

d is the diameter of G,

$$s_m = \min\left\{p-1, \frac{|\varkappa|\,a}{\alpha}\right\},\ (p-\varepsilon)' = \frac{p-\varepsilon}{p-\varepsilon-1},\ \text{(we suppose that } \frac{0}{0} \text{ is equal to 0).}$$

We note a number of properties of the spaces $L_{(p,(\varkappa,a,\alpha}(G)$ and $W^l_{(p,(\varkappa,a,\alpha}(G)$.

1. For any $\varkappa\in(0,\infty)^n, \alpha\ge 0$, and $a\in[0,1]$ we have the embeddings

$$L_{(p,(\varkappa,a,\alpha}(G) \hookrightarrow L_{(p}(G), \quad W^l_{(p,(\varkappa,a,\alpha}(G) \hookrightarrow W^l_{(p}(G)$$

i.e. there exists $C>0$ such that

$$\|f\|_{(p,G} \le C\,\|f\|_{(p,(\varkappa,a,\alpha;G}\,; \|f\|_{W^l_{(p}(G)} \le C\,\|f\|_{W^l_{(p,(\varkappa,a,\alpha}(G)},$$

$$\|f\|_{W^l_{(p}(G)} = \|f\|_{(p,G} + \sum_{i=1}^n \left\|D_i^{l_i}f\right\|_{(p,G},$$

$$\|f\|_{(p,G} = \|f\|_{L_{(p}(G)} = \inf_{0<\varepsilon<p-1} \left(\varepsilon^{-\frac{p-\varepsilon}{(p-\varepsilon)'}} \frac{1}{|G|} \int\limits_G |f(x)|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}}.$$

Indeed,

$$\|f\|_{(p,(\varkappa,a,\alpha;G} = \sup_{\substack{x\in G\\ 0<t\le d}} \inf_{0<\varepsilon<s_m} \left(\frac{1}{t^{|\varkappa|a+\alpha\varepsilon}} \varepsilon^{-\frac{p-\varepsilon}{(p-\varepsilon)'}} \frac{1}{|G_{t^\varkappa}(x)|} \int\limits_{G_{t^\varkappa}(x)} |f(y)|^{p-\varepsilon}\, dy \right)^{\frac{1}{p-\varepsilon}}$$

$$\ge d^{-\frac{|\varkappa|a+\alpha\varepsilon}{p}} \inf_{0<\varepsilon<p-1} \left(\varepsilon^{-\frac{p-\varepsilon}{(p-\varepsilon)'}} \frac{1}{|G|} \int\limits_G |f|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}} = d^{-\frac{|\varkappa|a+\alpha\varepsilon}{p}}\, \|f\|_{(p,G}$$

2. The spaces $L_{(p,(\varkappa,a,\alpha}(G)$ and $W^l_{(p,(\varkappa,a,\alpha}(G)$ are complete. The proof of the completeness of these spaces can be carried out as in [2, p.398]

3.

$$\|f\|_{(p,(\varkappa,0,0;G} = \|f\|_{(p,G},\ \|f\|_{W^l_{(p,(\varkappa,0,0}(G)} = \|f\|_{W^l_{(p}(G)}.$$

Theorem 1.1. *Let $p \in (1,\infty)$ and let $G \subset \mathbb{R}^n$ be a bounded domain. Let $f \in L_{p),\varkappa),a,\alpha}(G)$ and $g \in L_{(p',(\varkappa,a,\alpha}(G)$. Then for any $x \in G$ we have the inequality*

$$\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} f(y)g(y)dy \le \|f\|_{p),\varkappa)a,\alpha;G}\,\|g\|_{(p'(\varkappa,a,\alpha;G}\,. \tag{1.3}$$

Here $L_{p),\varkappa),a,\alpha}(G)$ is the grand grand Lebesgue-Morrey space introduced in [17]. The norm in this space is defined as:

$$\|f\|_{p),\varkappa),a,\alpha;G} = \|f\|_{L_{p),\varkappa),a,\alpha}(G)} = \sup_{\substack{x\in G,\\ 0<t\le d,\\ 0<\varepsilon<s_m}} \left(\frac{1}{t^{|\varkappa|a-\alpha\varepsilon}}\varepsilon\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} |f(y)|^{p-\varepsilon}dy\right)^{\frac{1}{p-\varepsilon}}.$$

Proof. For any $x \in G$ and $\alpha > 0$, we have

$$\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} f(y)g(y)dy \le \left(\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} |f(y)|^{p-\varepsilon}\,dy\right)^{\frac{1}{p-\varepsilon}} \left(\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} |g(y)|^{\frac{p-\varepsilon}{p-\varepsilon-1}}\,dy\right)^{\frac{p-\varepsilon}{p-\varepsilon-1}}$$

$$\le \varepsilon^{\frac{1}{p-\varepsilon}}\left(\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} |f(y)^{p-\varepsilon}dy\right)^{\frac{1}{p-\varepsilon}} \varepsilon^{-\frac{1}{p-\varepsilon}}\left(\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} |g(y)|^{\frac{p-\varepsilon-1}{p-\varepsilon}}\,dy\right)^{\frac{p-\varepsilon-1}{p-\varepsilon}}$$

$$\le t^{-\frac{|\varkappa|a}{p-\varepsilon}}\varepsilon^{\frac{1}{p-\varepsilon}}\left(\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} |f(y)|^{p-\varepsilon}\,dy\right)^{\frac{1}{p-\varepsilon-1}} t^{-\frac{|\varkappa|a(p-\varepsilon-1)}{p-\varepsilon}}\varepsilon^{-\frac{1}{p-\varepsilon}}\left(\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} |g(y)|^{\frac{p-\varepsilon}{p-\varepsilon-1}}\,dy\right)^{\frac{p-\varepsilon-1}{p-\varepsilon}}$$

$$\times t^{\frac{|\varkappa|a}{p-\varepsilon}+\frac{|\varkappa|a(p-\varepsilon-1)}{p-\varepsilon}} = t^{-\frac{|\varkappa|a}{p-\varepsilon}}\varepsilon^{\frac{1}{p-\varepsilon}}\left(\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} |f(y)|^{p-\varepsilon}\,dy\right)^{\frac{1}{p-\varepsilon}}$$

$$\times t^{-\frac{|\varkappa|a(p-\varepsilon-1)}{p-\varepsilon}}\varepsilon^{-\frac{1}{p-\varepsilon}}\left(\frac{1}{|G_{t^{\varkappa}}(x)|}\int\limits_{G_{t^{\varkappa}}(x)} |g(y)|^{\frac{p-\varepsilon}{p-\varepsilon-1}}\,dy\right)^{\frac{p-\varepsilon-1}{p-\varepsilon}} t^{|\varkappa|a}$$

$$\le t^{-\frac{|\varkappa|a}{p-\varepsilon}}t^{\frac{\alpha\varepsilon}{p-\varepsilon}}\varepsilon^{\frac{1}{p-\varepsilon}}\,\|f\|_{L_{p-\varepsilon}(G_{t^{\varkappa}}(x))}\,t^{-\frac{|\varkappa|a(p-\varepsilon-1)}{p-\varepsilon}}t^{-\frac{\alpha\varepsilon(p-\varepsilon-1)}{p-\varepsilon}}\varepsilon^{-\frac{1}{p-\varepsilon}}$$

$$\times\|g\|_{L_{\frac{p-\varepsilon}{p-\varepsilon-1}}(G_{t^{\varkappa}}(x))}\,t^{|\varkappa|a}t^{-\frac{\alpha\varepsilon}{p-\varepsilon}}t^{\frac{\alpha\varepsilon(p-\varepsilon-1)}{p-\varepsilon}}$$

$$\le \sup_{\substack{x\in G,\\ 0<t\le d,\\ 0<\varepsilon<s_m}}\left(\frac{1}{t^{\frac{|\varkappa|a-\alpha\varepsilon}{p-\varepsilon}}}\varepsilon^{\frac{1}{p-\varepsilon}}\,\|f\|_{L_{p-\varepsilon}(G_{t^{\varkappa}}(x))}\right)$$

$$\times\sup_{\substack{x\in G,\\ 0<t\le d}}\inf_{0<\varepsilon<s_m}\left(\frac{1}{t^{\frac{(|\varkappa|a+\alpha\varepsilon)(p-\varepsilon-1)}{p-\varepsilon}}}\varepsilon^{-\frac{1}{p-\varepsilon}}\,\|g\|_{L_{\frac{p-\varepsilon}{p-\varepsilon-1}}(G_{t^{\varkappa}}(x))}\right),$$

where

$$|\varkappa|\,a+\frac{\alpha\varepsilon(p-\varepsilon-1)-\alpha\varepsilon}{p-\varepsilon}>0.$$

In other words, for any $x\in G$ we have the inequality (1.3). □

Let $\varphi(\cdot,y,z)\in C_0^\infty(R^n)$ be such that

$$S(\varphi)=\operatorname{supp}\varphi\subset I_1=\left\{x:|x_j|<\frac{1}{2},\ j=1,2,...,\right\},$$

$0<T\le 1,\lambda=(\lambda_1,...,\lambda_n),\lambda_j>0,j=1,2,...,n$ and let

$$V=\bigcup_{0<t\le T}\left\{y:\left(\frac{y}{t^\lambda}\right)\in S(\varphi)\right\},$$

where $\frac{y}{t^\lambda}=\left(\frac{y_1}{t^{\lambda_1}},...,\frac{y_n}{t^{\lambda_n}}\right)$.

Let U be an open set contained in the domain G. We will assume that $U+V\subset G$, and denote

$$G_{T^\varkappa}(U)=(U+I_{T^\varkappa}(x))\cap G=Q.$$

Notice that if $0<\varkappa_j\le\lambda_j\ \ (j=1,2,...,n),\ \ I_{T^\lambda}\subset I_{T^\varkappa}$, then, consequently

$$U+V\subset Q.$$

We prove a lemma that we will need in the sequel to prove main theorems.

Lemma 1.1. *Let* $1<p<q\le r\le\infty,\ \ 0<|\varkappa|\le\frac{|\lambda|-\alpha\varepsilon}{a+1},\ 0<t,\eta\le T\le 1,\ 0<\gamma<\gamma_0$ *(γ_0 is a fixed number),* $\nu=(\nu_1,...,\nu_n),\ \nu_j\ge 0$ *be entire* $(j=1,2,...,n),\ \ \psi\in L_{(p,(\varkappa,a,\alpha}(G)$ *and let*

$$\overline{\mu}_\iota=\lambda_\iota l_i-(\nu,\lambda)-(|\lambda|-|\varkappa|\,a-|\varkappa|-\alpha\varepsilon)\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)\tag{1.4}$$

$$(\nu,\lambda)=\sum_{j=1}^n\nu_j\lambda_j,\quad |\lambda|=\sum_{j=1}^n\lambda_j,$$

$$E_\eta^i(x)=\int_0^\eta t^{-1-|\lambda|-(\nu,\lambda)+\lambda_i l_i}\int_{R^n}\psi(x+y)\varphi\left(\frac{y}{t^\lambda},\frac{\rho(t^\lambda,x)}{t^\lambda},\rho'(t^\lambda,x)\right)dydt,\tag{1.5}$$

$$E_{\eta,T}^i(x)=\int_\eta^T t^{-1-|\lambda|-(\nu,\lambda)+\lambda_i l_i}\int_{R^n}\psi(x+y)\varphi\left(\frac{y}{t^\lambda},\frac{\rho(t^\lambda,x)}{t^\lambda},\rho'(t^\lambda,x)\right)dydt,\tag{1.6}$$

$\rho'(u,x)=\frac{\partial}{\partial u}\rho(u,x)$. *Then*

$$\sup_{\overline{x}\in U}\left\|E_\eta^i\right\|_{q-\varepsilon,U_{\gamma^\varkappa}(\overline{x})}\le C_1\|\psi\|_{(p,(\varkappa,a,\alpha;G}\,\varepsilon^{\frac{1}{(p-\varepsilon)'}}\eta^{\overline{\mu}_\iota}\quad(\overline{\mu}_\iota>0),\tag{1.7}$$

$$\sup_{\overline{x}\in U}\left\|E_{\eta,T}^i\right\|_{q-\varepsilon,U_{\gamma^\varkappa}(x)}\le C_2\|\psi\|_{(p,(\varkappa,a,\alpha;G}\,\varepsilon^{\frac{1}{(p-\varepsilon)'}}\quad\begin{cases}T^{\overline{\mu}_\iota}, & \overline{\mu}_\iota>0\\ \ln\frac{T}{\eta}, & \overline{\mu}_\iota=0\\ \eta^{\overline{\mu}_\iota}, & \overline{\mu}_\iota<0.\end{cases}\tag{1.8}$$

Here $U_{\gamma^\varkappa}(\overline{x})=\left\{x:|x_j-\overline{x}_j|<\frac{1}{2}\gamma^{\varkappa_j},\ \ ,j=1,2,...,n\right\}$, C_1 *and* C_2 *are positive constants independent of* ψ,γ,η,T.

Proof. Applying the generalized Minkowski inequality, for any $\overline{x} \in U$ and $0 < \varepsilon < s_m$ we get

$$\left\| E_\eta^i \right\|_{q-\varepsilon, U_{\gamma^\varkappa}(\overline{x})} \leq \int_0^\eta t^{-1-|\lambda|-(\nu,\lambda)+\lambda_i l_i} \left\| A(\cdot, t) \right\|_{q-\varepsilon, U_{\gamma^\varkappa}(\overline{x})} dt, \tag{1.9}$$

where

$$A(x,t) = \int_{R^n} \psi(x+y) \varphi\left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x) \right) dy.$$

By the Hölder inequality ($q \leq r$) we have

$$\left\| A(\cdot, t) \right\|_{q-\varepsilon, U_{\gamma^\varkappa}(\overline{x})} \leq \left\| A(\cdot, t) \right\|_{r-\varepsilon, U_{\gamma^\varkappa}(\overline{x})} \gamma^{|\varkappa|\left(\frac{1}{q-\varepsilon} - \frac{1}{r-\varepsilon}\right)}. \tag{1.10}$$

Let χ be the characteristic function of the set $S(\varphi)$ and $1 < p < r \leq \infty$, $s \leq r$ $\left(\frac{1}{s} = 1 - \frac{1}{p-\varepsilon} + \frac{1}{r-\varepsilon} \right)$,

$$|\psi\varphi| = \left(|\psi|^{p-\varepsilon} |\varphi|^s \right)^{\frac{1}{r-\varepsilon}} \left(|\psi|^{p-\varepsilon} \chi \right)^{\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon}} \left(|\varphi|^s \right)^{\frac{1}{s} - \frac{1}{r-\varepsilon}}.$$

Applying the Hölder inequality ($\frac{1}{r-\varepsilon} + \left(\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon} \right) + \frac{1}{s} - \frac{1}{r-\varepsilon} = 1$) and taking into account that $|\varphi(x,y,z)| \leq C_1 |\varphi_1(x)|$, we get

$$\left\| A(\cdot, t) \right\|_{r-\varepsilon, U_{\gamma^\varkappa}(\overline{x})} \leq C_1 \sup_{x \in U_{\gamma^\varkappa}(\overline{x})} \left(\int_{R^n} |\psi(x+y)|^{p-\varepsilon} \chi\left(\frac{y}{t^\lambda} \right) dy \right)^{\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon}} \times$$

$$\times \sup_{y \in V} \left(\int_{U_{\gamma^\varkappa}(\overline{x})} |\psi(x+y)|^{p-\varepsilon} dx \right)^{\frac{1}{r-\varepsilon}} \left(\int_{R^n} \left| \varphi_1\left(\frac{y}{t^\lambda} \right) \right|^s dy \right)^{\frac{1}{s}}. \tag{1.11}$$

For any $0 < t \leq 1, \varkappa \leq \lambda,\ x \in U$, then $Q_{t^\lambda}(x) \subset Q_{t^\varkappa}(x)$ and we have

$$\int_{R^n} |\psi(x+y)|^{p-\varepsilon} \chi\left(\frac{y}{t^\lambda} \right) dy \leq \int_{Q_{t^\lambda}(x)} |\psi(y)|^{p-\varepsilon} dy \leq \int_{Q_{t^\varkappa}(x)} |\psi(y)|^{p-\varepsilon} dy =$$

$$= \|\psi\|_{p-\varepsilon, Q_{t^\varkappa}(x)}^{p-\varepsilon} \leq \|\psi\|_{(p, Q_{t^\varkappa}(x)}^{p-\varepsilon} |Q_{t^\varkappa}(x)| \, \varepsilon^{\frac{p-\varepsilon}{(p-\varepsilon)'}} \leq \|\psi\|_{(p,(\varkappa,a,\alpha;Q}^{p-\varepsilon} \varepsilon^{\frac{p-\varepsilon}{(p-\varepsilon)'}} t^{|\varkappa|a+\alpha\varepsilon+|\varkappa|}, \tag{1.12}$$

for $y \in V$

$$\int_{U_{\gamma^\varkappa}(\overline{x})} |\psi(x+y)|^{p-\varepsilon} dx \leq \int_{Q_{\gamma^\varkappa}(\overline{x}+y)} |\psi(x)|^{p-\varepsilon} dx = \|\psi\|_{p-\varepsilon, Q_{\gamma^\varkappa}(\overline{x}+y)}^{p-\varepsilon} \leq$$

$$\leq \|\psi\|_{(p, Q_{\gamma^\varkappa}(\overline{x}+y)}^{p-\varepsilon} \varepsilon^{\frac{p-\varepsilon}{(p-\varepsilon)'}} |Q_{\gamma^\varkappa}(x)| \leq \|\psi\|_{(p,(\varkappa,a,\alpha;Q}^{p-\varepsilon} \varepsilon^{\frac{p-\varepsilon}{(p-\varepsilon)'}} \gamma^{|\varkappa|a+|\varkappa|+\alpha\varepsilon}, \tag{1.13}$$

$$\int_{R^n} \left| \varphi_1\left(\frac{y}{t^\lambda} \right) \right|^s dy = t^{|\lambda|} \|\varphi_1\|_s^s. \tag{1.14}$$

From inequalities (1.10) - (1.14) for $r = q$ we get

$$\left\| A(\cdot, t) \right\|_{q-\varepsilon, U_{\gamma^\varkappa}(\overline{x})} \leq C_1 \|\varphi_1\|_s \|\psi\|_{(p,\varkappa,a,\alpha;Q} \varepsilon^{\frac{1}{(p-\varepsilon)'}} \gamma^{\frac{|\varkappa|a+|\varkappa|+\alpha\varepsilon}{q-\varepsilon}} \eta^{\overline{\mu_\iota}} \quad (\overline{\mu}_\iota > 0). \tag{1.15}$$

Substituting (1.15) in (1.9), we arrive at (1.7). Inequality (1.8) is proved in the same way. □

We prove two theorems on properties of functions belonging to the space $W^l_{(p,(\varkappa,a,\alpha}(G)$.

Theorem 1.2. *Let an open bounded set $G \subset \mathbb{R}^n$ satisfy the flexible λ-horn condition (see [2]); $1 < p < q \leq \infty; |\varkappa| \leq \frac{|\lambda|-\alpha\varepsilon}{1+a}$; $\nu = (\nu_1, ..., \nu_n)$, $\nu_j \geq 0$ be nonnegative integers $(j = 1, 2, ..., n)$; $\overline{\mu}_i > 0$ $(i = 1, 2, ..., n)$ and let $f \in W^l_{(p,(\varkappa,a,\alpha}(G)$.*

Then $D^\nu : W^l_{(p,(\varkappa,a,\alpha}(G) \hookrightarrow L_{q-\varepsilon}(G)$ $(0 < \varepsilon < s_m)$ and the following inequality is valid

$$\|D^\nu f\|_{q-\varepsilon,G} \leq C(\varepsilon)\left(T^{\overline{\mu}_0}\|f\|_{(p,(\varkappa,a,\alpha;G} + \sum_{i=1}^n T^{\overline{\mu}_i}\left\|D_i^{l_i} f\right\|_{(p,(\varkappa,a,\alpha;G}\right), \tag{1.16}$$

where $\overline{\mu}_0 = \overline{\mu}_i - \lambda_i l_i$

In particular, if

$$\overline{\mu}_{i,0} = \lambda_i l_i - (\nu, \lambda) - (|\lambda| - |\varkappa|\, a - |\varkappa| - \alpha\varepsilon)\frac{1}{p-\varepsilon} > 0 \ (i = 1, 2, ..., n),$$

then $D^\nu f$ is equivalent to a continuous function on G and

$$ess \sup_{x\in G}|D^\nu f(x)| \leq C(\varepsilon)\left(T^{\overline{\mu}_{0,0}}\|f\|_{(p,(\varkappa,a,\alpha;G} + \sum_{i=1}^n T^{\overline{\mu}_{\iota,0}}\left\|D_i^{l_i} f\right\|_{(p,(\varkappa,a,\alpha;G}\right), \tag{1.17}$$

where $\overline{\mu}_{0,0} = \overline{\mu}_{i,0} - \lambda_i l_i$.

In (1.16) and (1.17) $0 < T \leq 1$ and $C(\varepsilon) = C\varepsilon^{\frac{1}{(p-\varepsilon)'}}$, where $C > 0$ is a constant independent of f and T.

Proof. Under the assumptions of the theorem there exist the generalized derivatives $D^\nu f$. Indeed, if $\overline{\mu}_i > 0$, $p < q$, $|\varkappa| < \frac{\lambda-\alpha\varepsilon}{|\varkappa|+a}$, then $\lambda_i l_i - (\nu, \lambda) > 0$ $(i = 1, ..., n)$. Since $f \in W^l_{(p,(\varkappa,a,\alpha}(G) \hookrightarrow W^l_{(p}(G) \hookrightarrow W^l_{p-\varepsilon}(G)$ $(p - \varepsilon > 1)$, for almost every point $x \in G$, we have the identity obtained by O.V. Besov [2]

$$D^\nu f(x) = f^{(\nu)}_{T^\lambda}(x) + \sum_{i=1}^n E^i_T, \tag{1.18}$$

where

$$f^{(\nu)}_{T^\lambda}(x) = T^{-|\lambda|-(\nu,\lambda)}\int_{\mathbb{R}^n} f(x+y)\Omega^{(\nu)}\left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda}\right)dy \tag{1.19}$$

and

$$E^i_T = \int_0^T \int_{\mathbb{R}^n} t^{-1-|\lambda|-(\nu,\lambda)+\lambda_i l_i} L_i^{(\nu)}\left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda}, \rho'(t^\lambda, x)\right) D^\nu f(x+y)dydt.$$

Here $0 < T \leq \min(1, T_0)$, the functions $\Omega^\nu(\cdot, y)$, $L_\iota^{(\nu)}(\cdot, y, z)$ are the functions of the class $C_0^\infty(\mathbb{R}^n)$, and their supports are contained in I_1, the supports of the kernels in representations (1.18), (1.19) are in the flexible horn $x + V(\lambda) \subset G$. Based on the Minkowski inequality, from (1.18), (1.19) we have

$$\|D^\nu f\|_{q-\varepsilon,G} \leq \left\|f^{(\nu)}_{T^\lambda}\right\|_{q-\varepsilon,G} + \sum_{i=1}^n \left\|E^i_T\right\|_{q-\varepsilon,G}. \tag{1.20}$$

By means of inequality (1.15) for $U = G, t = T, f = \psi, \varphi = \Omega^{(\nu)}$, we get

$$\left\|f^{(\nu)}_{T^\lambda}\right\|_{q-\varepsilon,G} \leq C_1(\varepsilon)T^{\overline{\mu}_0}\|f\|_{(p,(\varkappa,a,\alpha;G}, \tag{1.21}$$

and by means of inequality (1.7), for $U = G$, $D_i^{l_i} f = \psi, \varphi = L_i^{(\nu)}, \eta = T$, we get

$$\left\| E_T^i \right\|_{q-\varepsilon,G} \le C_2(\varepsilon) T^{\overline{\mu}_i} \left\| D_i^{l_i} f \right\|_{(p,(\varkappa,a,\alpha;G}. \tag{1.22}$$

Taking into account inequalities (1.20)-(1.22), we get inequality (1.16).

Now let $\overline{\mu}_{i,0} > 0 (i = 1, 2, ..., n)$. We show that $D^\nu f$ is equivalent to a continuous function on G. From equalities (1.18), (1.19) and inequality (1.22) for $q = \infty$, $\overline{\mu}_i(q = \infty) = \overline{\mu}_{i,0} > 0 \ \ (i = 1, 2, ..., n)$ we have

$$\left\| D^\nu f - f_{T^\lambda}^{(\nu)} \right\|_{\infty,G} \le \sum_{i=1}^{n} T^{\overline{\mu}_{i,0}} \left\| D_i^{l_i} f \right\|_{(p,(\varkappa,a,\alpha;G}.$$

Hence it follows that as $T \to 0$ the left hand side of the inequality tends to zero. As $f_{T^\lambda}^{\nu}$ are continuous on G and converge to $D^\nu f$ in $L_\infty(G)$, the limit function $D^\nu f$ is equivalent to a continuous function on G. □

Theorem 1.3. *Let the assumtions of Theorem 1.2 be satisfied. Then for* $\overline{\mu}_i > 0 (i = 1, 2, ..., n)$ *the derivative* $D^\nu f$ *satisfies on* G *the Hölder condition in the metric* $L_{q-\varepsilon}$ *with the exponent* σ*, namely*

$$\left\| \Delta(\zeta, G) D^\nu f \right\|_{q-\varepsilon,G} \le C(\varepsilon) \left\| f \right\|_{W^l_{(p,(\varkappa,a,\alpha}(G)} |\zeta|^\sigma, \tag{1.23}$$

where $C(\varepsilon)$ *is defined in Theorem 1.2 and* σ *is a number satisfying the inequalities*

$$\begin{gathered} 0 \le \sigma \le 1, \ \ if\ \frac{\overline{\mu}^0}{\lambda_0} > 1, \\ 0 \le \sigma < 1, \ \ if\ \frac{\overline{\mu}^0}{\lambda_0} = 1 \\ 0 \le \sigma < \ \frac{\overline{\mu}^0}{\lambda_0}\ \ if\ \frac{\overline{\mu}^0}{\lambda_0} < 1, \end{gathered} \tag{1.24}$$

$\overline{\mu}^0 = \min \overline{\mu}_i \ \ (i = 1, ..., n); \lambda_0 = \max \lambda_j (j = 1, ..., n).$

If $\overline{\mu}_{i,0} > 0 \ \ (i = 1, ..., n)$*, then*

$$ess \sup_{x \in G} |\Delta(\zeta, G) D^\nu f(x)| \le C(\varepsilon) \left\| f \right\|_{W^l_{(p,(\varkappa,a,\alpha}(G)} |\zeta|^{\sigma_0}, \tag{1.25}$$

where σ_0 *satisfies the same conditions as* σ*, but with* $\overline{\mu}_i$ *replaced by* $\overline{\mu}_{i,0}$.

Proof. Let $\zeta \in \mathbb{R}^n$. By Lemma 8.6 from [2] there exists a domain

$$G_\omega \subset G \ \ (\omega = \xi r_\lambda(x), \xi > 0, r_\lambda(x) = \rho_\lambda(x, \partial G), x \in G).$$

Suppose $|\zeta|_\lambda < \omega$. Then for any $x \in G_\omega$, the segment connecting the points $x, x + \zeta$ is contained in G. Then for all points of this segment, equalities (1.18), (1.19) with the same kernels are valid. After some transformations we get

$$|\Delta(\zeta, G) D^\nu f(x)| \le T^{-|\lambda| - (\nu,\lambda)} \int_{R^n} f(x+y) \times$$

$$\times \left| \Omega^{(\nu)} \left(\frac{y-\zeta}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda} \right) - \Omega^{(\nu)} \left(\frac{y}{T^\lambda}, \frac{\rho(T^\lambda, x)}{T^\lambda} \right) \right| dy +$$

$$+\sum_{i=1}^{n}\left\{\int_{0}^{|\zeta|^{\frac{1}{\lambda_0}}} t^{-1-|\lambda|-(\nu,\lambda)+\lambda_i l_i}\int_{R^n}\left(\left|D_i^{l_i}f(x+\zeta+y)\right|+\left|D_i^{l_i}f(x+y)\right|\right)\times\right.$$

$$\times\left|L_i^{(\nu)}\left(\frac{y}{t^\lambda},\frac{\rho(t^\lambda,x)}{t^\lambda},\rho'(t^\lambda,x)\right)\right|dydt+\int_{|\zeta|^{\frac{1}{\lambda_0}}}^{T} t^{-1-|\lambda|-(\nu,\lambda)+\lambda_i l_i}\times$$

$$\times\int_{R^n}\left|D_i^{l_i}f(x+y)\right|\left|L_i^{(\nu)}\left(\frac{y-\zeta}{t^\lambda},\frac{\rho(t^\lambda,x)}{t^\lambda},\rho'(t^\lambda,x)\right)-\right.$$

$$\left.-L_i^{(\nu)}\left(\frac{y}{t^\lambda},\frac{\rho(t^\lambda,x)}{t^\lambda},\rho'(t^\lambda,x)\right)\right|dydt=E(x,\zeta)+$$

$$+\sum_{i=1}^{n}\left(E^i_{|\zeta|^{\frac{1}{\lambda_0}}}(x,\zeta)+E^i_{|\zeta|^{\frac{1}{\lambda_0}},T}(x,\zeta)\right),$$

where $0<T\le\min(1,T_0)$, $|\zeta|^{\frac{1}{\lambda_0}}<T$, consequently, $|\zeta|<\min(\omega,T^{\lambda_0})$.

If $x\in G$ $x\in G\backslash G_\omega$, then by the definition $\Delta(\zeta,g)D^\nu f(x)=0$.

Then

$$\|\Delta(\zeta,G)D^\nu f\|_{q-\varepsilon,G}=\|\Delta(\zeta,G)D^\nu f\|_{q-\varepsilon,G_\omega}\le$$

$$\le\|E(\cdot,\zeta)\|_{q-\varepsilon,G}+\sum_{\iota=1}^{n}\left(\left\|E^i_{|\zeta|^{\frac{1}{\lambda_0}}}(\cdot,\zeta)\right\|_{q-\varepsilon,G}+\left\|E^i_{|\zeta|^{\frac{1}{\lambda_0}},T}(\cdot,\zeta)\right\|_{q-\varepsilon,G}\right). \tag{1.26}$$

Notice that

$$\left|\Omega^{(\nu)}\left(\frac{y-\zeta}{T^\lambda},\frac{\rho(T^\lambda,x)}{T^\lambda}\right)-\Omega^{(\nu)}\left(\frac{y}{T^\lambda},\frac{\rho(T^\lambda,x)}{T^\lambda}\right)\right|\le$$

$$\le\sum_{j=1}^{n}T^{-\lambda_j}\int_0^{|\zeta|}\left|D_j\Omega^{(\nu)}\left(\frac{y-\eta_{e_\zeta}}{T^\lambda},\frac{\rho(T^\lambda,x)}{T^\lambda}\right)\right|d\eta,$$

where $e_\zeta=\frac{\zeta}{|\zeta|}$, then

$$E(x,\zeta)\le\sum_{j=1}^{n}T^{-\lambda_j-|\lambda|-(\nu,\lambda)}\int_0^{|\zeta|}d\eta\int_{R^n}|f(x+\eta e_\zeta+y|\left|D_j\Omega^{(\nu)}\left(\frac{y}{T^\lambda},\frac{\rho(T^\lambda,x)}{T^\lambda}\right)\right|dy,$$

$$E^i_{|\zeta|^{\frac{1}{\lambda_0}},T}(x,\zeta)\le\sum_{j=1}^{n}\int_0^{|\zeta|}t^{-1-|\lambda|-(\nu,\lambda)-\lambda_j+\lambda_i l_i}dt\int_{R^n}\left|D_i^{l_i}f(x+\eta e_\zeta+y)\right|\times$$

$$\times\left|D_jL_i^{(\nu)}\left(\frac{y}{t^\lambda},\frac{\rho(t^\lambda,x)}{t^\lambda},\rho'(t^\lambda,x)\right)\right|dy.$$

By means of inequality (1.15) for $U=G$, $t=T$, $f=\psi$, $\varphi=\Omega^{(\nu)}$, we have

$$\|E(\cdot,\zeta)\|_{q-\varepsilon,G}\le C_1^{(\xi)}|\zeta|\,\|f\|_{(p,(\varkappa,a,\alpha;G}. \tag{1.27}$$

From (1.7) for $U = G, \eta = |\zeta|^{\frac{1}{\lambda_0}}, \quad D_i^{l_i} f = \psi, \quad L_i^{(\nu)} = \varphi$ we get

$$\left\| E^i_{|\zeta|^{\frac{1}{\lambda_0}}}(\cdot,\zeta) \right\|_{q-\varepsilon,G} \leq C_2(\varepsilon)\, |\zeta|^{\frac{\overline{\mu_i}}{\lambda_0}} \left\| D_i^{l_i} f \right\|_{(p,(\varkappa,a,\alpha;G}, \tag{1.28}$$

and from (1.8) for $U = G, \eta = |\zeta|^{\frac{1}{\lambda_0}}, \quad D_i^{l_i} f = \psi, \quad L_i^{(\nu)} = \varphi$ we get

$$\left\| E^i_{|\zeta|^{\frac{1}{\lambda_0}},T}(\cdot,\zeta) \right\|_{q-\varepsilon,G} \leq C_3(\varepsilon)\, |\zeta|^{\sigma} \left\| D_\iota^{l_\iota} f \right\|_{(p,(\varkappa,a,\alpha;G}. \tag{1.29}$$

From inequalities (1.26)-(1.29), we have

$$\|\Delta(\zeta,G) D^\nu f\|_{q-\varepsilon,G} \leq C(\varepsilon)\, \|f\|_{W^l_{(p,(\varkappa,a,\alpha}(G)}\, |\zeta|^\sigma .$$

Also if $|\zeta| \geq \min(\omega, T^{\lambda_0})$, then

$$\|\Delta(\zeta,G) D^\nu f\|_{q-\varepsilon,G} \leq 2\, \|D^\nu f\|_{q-\varepsilon,G} \leq C(\omega,T)\, \|D^\nu f\|_{q-\varepsilon,G}\, |\zeta|^\sigma .$$

Estimating $\|D^\nu f\|_{q-\varepsilon,G}$ by using inequality (1.16), we again get the required inequality. $\square$

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References

[1] G. Anatriello, M.R. Formica, R. Giova, *Fully measurable small Lebesgue spaces.* Jour. of Mathematical Analysis and Applications. 447 (2017), no. 1, 550- 563.

[2] O.V. Besov, V.P. Il'in, S.M. Nikol'skii, *Integral representation of functions and imbedding theorems.* M. Nauka, 1996, 480 pp. (in Russian).

[3] C. Capone, A. Fiorenza, *On small Lebesgue spaces.* Jour. of Function Spaces and Applications 3 (2005), no. 1, 73-89.

[4] D. Cruz-Uribe, A. Fiorenza, O.M. Guzmán, *Imbeddings of grand Morrey spaces, small Lebesgue spaces and Lebesgue spaces with variable exponent.* Matematicheskiye Zametki. 102 (2017), no. 5-6, 736-748 (in Russian).

[5] A. Eroglu, J.V. Azizov, V.S. Guliyev, *Fractional maximal operator and its commutators in generalized Morrey spaces on Heisenberg group.* Proc. of Inst. Math. and Mechanics NAS Azerb. 44 (2018), no. 2, 304-317.

[6] A. Fiorenza, C.E. Karadzhov, *Grand and small Lebesgue spaces and their anologues.* J. Anal. Appl. 23 (2004), no. 4, 657-681.

[7] A. Fiorenza, M.R. Formica, A. Gogatishvili, *On grand and small Lebesgue spaces and some applications to PDE's.* Differential Equations and Applications 10 (2018), no. 1, 21-46.

[8] T. Iwaniec, C. Sbordone *On the inerrability of the Jacobian under minimal hypotheses.* Arch. Ration. Mech. Anal. 119 (1992), 129-143.

[9] V.M. Kokilashvili, A. Meskhi, *Trace inequalities for fractional integrals in grand Lebesgue spaces.* Studia Math. 210 (2012), no. 2, 159-176.

[10] Y. Liang, D. Yang, W. Yuan, S. Sawano, T. Ullrich, *A new framework for generalized Besov-type and Triebel-Lizorkin-type spaces.* Dissertationes Mathematicae 489 (2013), 1-114.

[11] A. Meskhi, Y. Sawano *Density, duality and preduality in grand variable exponent Lebesgue and Morrey spaces.* arxiv:17/0.02383 .v1 [math. FA] 06 0ct. 2017.

[12] Y. Mizuta, T. Ohno, *Trudingers exponential integrability for Reisz potentials of function in generalized grand Morrey spaces.* J. Math. Anal. Appl. 420 (2014), no. 1, 268-278.

[13] A.M. Najafov, *On some properties of functions from Sobolev-Morrey type spaces.* Sibirskii Matem. Zhurnal 46 (2005), no. 3, 634-648 (in Russian).

[14] A.M. Najafov, A.T. Orujova, *On the solution of a class of partial differential equations.* Electron. J. Qual. Theory Differ. Equ. (2017), no. 44, 1-9.

[15] A.M. Najafov, N.R. Rustamova, *On properties of functions from Sobolev-Morrey type spaces with dominant mixed derivatives.* Trans. of National Academy of Sciences of Azerbaijan, Issue Math. 37 (2017), no. 4, 132-141.

[16] A.M. Najafov, N.R. Rustamova, *Some differential properties of anisotropic grand Sobolev-Morrey spaces.* Trans. of A. Razmadze Mathematical Institute 172 (2018), 82-89.

[17] H. Rafeiro, *A note on boundedness of operators in grand grand Morrey spaces.* Advances in Harmonic Analysis and Operator Theory (2013), 349-356. DOI: io.1007/978-3-0348-0516

[18] C. Sbordone, *Grand Sobolev spaces and their applications to variational problems.* Le Mathematiche 1 (1996), no. 11, 335-347.

[19] S.M. Umarkhadziev, *The boundedness of the Riesz potential operator from generalized grand Lebesgue spaces to generalized grand Morrey spaces.* Operator theory: Advances and Applications 242 (2014), 363-373.

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