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SHAVKAT ARIFJANOVICH ALIMOV

(to the 75th birthday)

Shavkat Arifjanovich Alimov was born on March 2, 1945 in the city of Nukus, Uzbekistan. In 1968, he graduated from the Department of Mathematics of Physical Faculty of the M.V. Lomonosov Moscow State University (MSU), receiving a diploma with honors. From 1968 to 1970, he was a post-graduate student in the same department under the supervision of Professor V.A. Il'in. He defended his PhD thesis in 1970. In May 1973, at the age of 28, he defended his doctoral thesis devoted to equations of mathematical physics. In 1973, for research on the spectral theory, he was awarded the highest youth prize of the USSR.

From 1974 to 1984, he worked as a professor in the Department of General Mathematics at the Faculty of Computational Mathematics and Cybernetics. In 1984, Sh.A. Alimov joined the Tashkent State University (TSU) as a professor. From 1985 to 1987 he worked as the Rector of the Samarkand State University, from 1987 to 1990 - the Rector of the TSU, from 1990 to 1992 - the Minister of Higher and Secondary Special Education of the Republic of Uzbekistan. From 1992 to 1994, he headed the Department of Mathematical Physics of the TSU.

After some years of diplomatic work, he continued his academic career as a professor of the Department of Mathematical Physics at the National University of Uzbekistan (NUU). From the first days of the opening of the Tashkent branch of the MSU in 2006, he worked as a professor in the Department of Applied Mathematics. From 2012 to 2017, he headed the Laboratory of Mathematical Modeling of the Malaysian Institute of Microelectronic Systems in Kuala Lumpur. From 2017 to 2019, he worked as a professor at the Department of Differential Equations and Mathematical Physics of the NUU. From 2019 to the present, Sh.A. Alimov is a Scientific Consultant at the Center for Intelligent Software Systems, and an adviser to the Rector of the NUU.

The main scientific activity of Sh.A. Alimov is connected with the spectral theory of partial differential equations and the theory of boundary value problems for equations of mathematical physics. He obtained series of remarkable results in these fields. They cover many important problems of the theory of Schrodinger equations with singular potentials, the theory of boundary control of the heat transfer process, the mathematical problems of peridynamics related to the theory of hypersingular integrals.

In 1984, Sh.A. Alimov was elected a corresponding member and in 2000 an academician of the Academy of Sciences of Uzbekistan. He was awarded several prestigious state prizes.

Sh.A. Alimov has over 150 published scientific and a large number of educational works. Among his pupils there are 10 doctors of sciences and more than 20 candidates of sciences (PhD) working at universities of Uzbekistan, Russia, USA, Finland, and Malaysia.

For about thirty years, Sh.A. Alimov has been actively involved in the reform of mathematical school education.

Sh.A. Alimov meets his 75th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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ONE-PHASE SPHERICAL STEFAN PROBLEM WITH TEMPERATURE DEPENDENT COEFFICIENTS

S.N. Kharin, T.A. Nauryz

Communicated by V.I. Burenkov

Key words: Stefan problem, nonlinear thermal coefficients, explicit solution, nonlinear integral equation, melting.

AMS Mathematics Subject Classification: 80A22, 35K05, 45D05.

Abstract. The one-phase spherical Stefan problem with coefficients depending on the temperature is considered. The method of solving is based on the similarity principle, which enables us to reduce this problem to a nonlinear ordinary differential equation, and then to an equivalent nonlinear integral equation of the Volterra type. It is shown that the obtained integral operator is a contraction operator and a unique solution exists.

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1 Introduction

The method of similarity for solving the Stefan problem (automodel solution) with thermal coefficients depending on the temperature has been widely developed in recent years. The one-dimensional Stefan problem with given temperature and heat flux condition at fixed face for a semi-infinite material is considered in papers [2]-[3].

Recently, Huntul and Lesnic also discussed an inverse problem of determining the time-dependent thermal conductivity and the transient temperature satisfying the heat equation with initial data [7]. The inverse Stefan problems for finding the time-dependent thermal conductivity using shifted Chebyshev polynomials [5] and the latent heat depending on the position using Kummer functions [1] are considered successfully on the base of the similarity principle. The detailed information concerning this approach can be found in the references of papers [2]-[3].

Mathematical modeling of the arc erosion in electrical contacts should take into account the temperature dependence of all thermal and electrical coefficients, which is very essential for the correct description of melting and boiling dynamics [6].The method of similarity is applied in this paper to modeling of the temperature field of a liquid spherical metal zone between two free moving boundaries related to the melting and boiled isotherms.

2 Mathematical model

The temperature distribution in a liquid metal zone at the interaction of electrical contacts with the arc can be described by the spherical model introduced by R. Holm [4]

$$c(T)\gamma(T)\frac{\partial T}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\lambda(T)\frac{\partial T}{\partial r}\right],\quad \alpha(t)<r<\beta(t),\quad t>0, \tag{2.1}$$

$$T(\alpha(t),t)=T_b, \tag{2.2}$$

$$T(\beta(t),t)=T_m, \tag{2.3}$$

and Stefan's conditions

$$-\lambda_b\frac{\partial T}{\partial r}(\alpha(t),t)=L_b\gamma_b\alpha'(t), \tag{2.4}$$

$$-\lambda_m\frac{\partial T}{\partial r}(\beta(t),t)=L_m\gamma_m\beta'(t). \tag{2.5}$$

Here $T(r,t)$ is the temperature distribution in a liquid zone, T_b is the temperature of boiling, T_m is the temperature of melting, $c(T),\gamma(T)$ and $\lambda(T)$ are given coefficients of the heat capacity, density, heat conductivity correspondingly, L_b, L_m are the specific heats of evaporation and melting, $r=\alpha(t), r=\beta(t)$ are the radii of boiling and melting isotherms, $\lambda_b=\lambda(T_b), \lambda_m=\lambda(T_m), \gamma_b=\gamma(T_b)$ and $\gamma_m=\gamma(T_m)$. After the substitution

$$\theta(r,t)=\frac{T(r,t)-T_m}{T_b-T_m}, \tag{2.6}$$

we get the following new problem

$$\tilde{c}(\theta)\tilde{\gamma}(\theta)\frac{\partial\theta}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\tilde{\lambda}(\theta)\frac{\partial\theta}{\partial r}\right],\quad \alpha(t)<r<\beta(t),\quad t>0, \tag{2.7}$$

$$\theta(\alpha(t),t)=1, \tag{2.8}$$

$$\theta(\beta(t),t)=0, \tag{2.9}$$

$$-\lambda_b\frac{\partial\theta}{\partial r}(\alpha(t),t)=L_b\gamma_b\alpha'(t)/(T_b-T_m), \tag{2.10}$$

$$-\lambda_m\frac{\partial\theta}{\partial r}(\beta(t),t)=L_m\gamma_m\beta'(t)/(T_b-T_m). \tag{2.11}$$

where

$$\tilde{c}(\theta)=c((T_b-T_m)\theta+T_m),\ \ \tilde{\gamma}(\theta)=\gamma((T_b-T_m)\theta+T_m),\ \ \tilde{\lambda}(\theta)=\lambda((T_b-T_m)\theta+T_m).$$

Using the similarity principle [2], the solution of problem (2.1)-(2.11) can be represented in the following form

$$\theta(r,t)=u(\eta),\quad \alpha(t)=\alpha_0\sqrt{t},\quad \beta(t)=\beta_0\sqrt{t},\quad \eta=\frac{r}{2\alpha_0\sqrt{t}} \tag{2.12}$$

for some $\beta_0>\alpha_0>0$. Then

$$\frac{\partial\theta}{\partial t}=-\frac{1}{2t}\eta\frac{du}{d\eta},\qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\tilde{\lambda}(\theta)\frac{\partial\theta}{\partial r}\right]=\frac{1}{4\alpha_0^2t}\frac{1}{\eta^2}\frac{d}{d\eta}\left[\tilde{\lambda}(u)\eta^2\frac{du}{d\eta}\right] \tag{2.13}$$

and problem (2.7)-(2.11) takes the form

$$\frac{d}{d\eta}\left[L(u)\eta^2\frac{du}{d\eta}\right]+2\alpha_0^2\eta^3N(u)\frac{du}{d\eta}=0,\quad \frac{1}{2}<\eta<\frac{\beta_0}{2\alpha_0}, \tag{2.14}$$

$$u(1/2)=1, \tag{2.15}$$

$$u(\beta_0/2\alpha_0)=0, \tag{2.16}$$

$$\frac{du}{d\eta}(1/2)=-(L_b\gamma_b\alpha_0^2)/(\lambda_b(T_b-T_m)), \tag{2.17}$$

$$\frac{du}{d\eta}(\beta_0/2\alpha_0) = -(L_m\gamma_m\beta_0^2)/(\lambda_m(T_b - T_m)), \tag{2.18}$$

where

$$L(u) = \lambda((T_b - T_m)u + T_m), \quad N(u) = c((T_b - T_m)u + T_m) \cdot \gamma((T_b - T_m)u + T_m).$$

Let us consider the obtained differential equation

$$\left[L(u)\eta^2 u'\right]' + 2\alpha_0^2\eta^3 N(u)u' = 0. \tag{2.19}$$

By using the substitution $L(u(\eta))\eta^2 u'(\eta) = \nu(\eta)$ we get the following equation

$$\nu'(\eta) + P(\eta, u(\eta))\nu(\eta) = 0, \tag{2.20}$$

where

$$P(\eta, u(\eta)) = \frac{2\alpha_0^2\eta N(u(\eta))}{L(u(\eta))}.$$

Solving equation (2.20) with respect to $\nu(\eta)$ we get

$$\nu(\eta) = \nu(1/2)\exp\left[-\int_{1/2}^{\eta} P(s, u(s))ds\right],$$

where, by the definition of the function ν and conditions (2.15) and (2.17),

$$\nu(1/2) = (L(1)u'(1/2))/4 = (\lambda_b u'(1/2))/4 = -(L_b\gamma_b\alpha_0^2)/(4(T_b - T_m)). \tag{2.21}$$

By the substitution $L(u(\eta))\eta^2 u'(\eta) = \nu(\eta)$ using condition (2.15) we get the following non-linear integral equation of the Volterra type with respect to $u(\eta)$

$$u(\eta) - 1 = \nu(1/2)\int_{1/2}^{\eta} \frac{1}{v^2 L(u(v))}\exp\left[-\int_{1/2}^{v} P(s, u(s))ds\right]dv,$$

which we can rewrite as follows

$$u(\eta) = 1 + \Phi[\eta, L(u), N(u)], \tag{2.22}$$

where

$$\Phi[\eta, L(u), N(u)] = \nu(1/2)\int_{1/2}^{\eta} E[t, u(t)]/(t^2 L(u(t))dt,$$

$$E[t, u] = \exp\Big(-\int_{1/2}^{t} P(s, u(s))ds\Big) = \exp\Big(-2\alpha_0^2\int_{1/2}^{t} sN(u(s))/L(u(s))ds\Big).$$

Integral equation (2.22) is equivalent to differential equation (2.14) plus conditions (2.15) and (2.17), and the initial problem of finding a solution to differential equation (2.14), satisfying conditions (2.15)-(2.18), is equivalent to the problem of finding a solution to integral equation (2.22), satisfying conditions (2.16) and (2.18).

If u is a solution to nonlinear integral equation (2.22), satisfying conditions (2.16) and (2.18), then by (2.12) the desired temperature distribution in a liquid zone $T(r,t)$ has the form

$$T(r,t) = T_m + (T_b - T_m)u(r/(2\alpha_0\sqrt{t})).$$

3 Main results

Using the fixed point theorem we find conditions ensuring that integral equation (2.22) has a unique solution if $\beta_0 > \alpha_0$. Let us denote $\Phi[\eta, u] \equiv \Phi[\eta, L(u), N(u)]$. We suppose that there exist positive constants N_m, N_M, L_s and L_M, such that for all $\xi > 0$

$$L_s \leq L(\xi) \leq L_M, \quad N_m \leq N(\xi) \leq N_M. \tag{3.1}$$

We assume that the specific heat and dimensionless thermal conductivity are Lipschitz functions and there exist positive constants $\tilde{L}$ and $\tilde{N}$ such that

$$\begin{aligned} ||L(g) - L(f)|| &\leq \tilde{L}||g - f||, \quad \forall g, f \in C(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+), \\ ||N(g) - N(f)|| &\leq \tilde{N}||g - f||, \quad \forall g, f \in C(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+), \end{aligned} \tag{3.2}$$

where $||f|| = \sup_{\eta\in\mathbb{R}^+} |f(\eta)|$ and $\mathbb{R}^+ = [0, \infty)$.

Lemma 3.1. *For $\eta > \frac{1}{2}$ we have*

$$\exp\left(-\alpha_0^2 \frac{N_M}{L_s}\left(\eta^2 - \frac{1}{4}\right)\right) \leq E[\eta, u] \leq \exp\left(-\alpha_0^2 \frac{N_m}{L_M}\left(\eta^2 - \frac{1}{4}\right)\right).$$

Proof. To prove, for example, the right-hand-side inequality it suffices to note that

$$E[\eta, u] \leq \exp\left(-2\alpha_0^2(N_m/L_M)\int\limits_{1/2}^{\eta} sds\right) = \exp\left(-\alpha_0^2(N_m/L_M)\left(\eta^2 - \frac{1}{4}\right)\right).$$

□

Lemma 3.2. *For $\beta_0 > \alpha_0$ we have*

$$\begin{aligned} &\frac{|\nu(1/2)|\alpha_0\sqrt{N_M}}{L_M\sqrt{L_s}} \exp((\alpha_0^2 N_M)/(4L_s))\Big(-\sqrt{\pi}\,\mathrm{erf}(\alpha_0\sqrt{N_M/L_s}\eta) \\ &+\sqrt{\pi}\,\mathrm{erf}(\alpha_0/2\sqrt{N_M/L_s}) - \frac{1}{\alpha_0\eta}\sqrt{\frac{L_s}{N_M}}\exp(-\alpha_0^2\eta^2 N_M/L_s) \\ &+\frac{2}{\alpha_0}\sqrt{L_s/N_M}\exp(-(\alpha_0^2 N_M)/(4L_s)) \leq \Phi[\eta, u] \\ &\leq \frac{|\nu(1/2)|\alpha_0\sqrt{N_m}}{L_s\sqrt{L_M}} \exp((\alpha_0^2 N_m)/(4L_M))\Big(-\sqrt{\pi}\,\mathrm{erf}(\alpha_0\sqrt{N_m/L_M}\eta) \\ &+\sqrt{\pi}\,\mathrm{erf}(\alpha_0/2\sqrt{N_m/L_M}) - \frac{1}{\alpha_0\eta}\sqrt{\frac{L_M}{N_m}}\exp(-\alpha_0^2\eta^2 N_m/L_M) \\ &+\frac{2}{\alpha_0}\sqrt{L_M/N_m}\exp(-(\alpha_0^2 N_m)/(4L_M)), \end{aligned}$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int\limits_0^x e^{-t^2}dt$ and $\nu(1/2)$ is defined by (2.21).

Proof. By Lemma 3.1 we have

$$\Phi[\eta,u] \leq (|\nu(1/2)|/L_s) \int_{1/2}^{\eta} \exp(-\alpha_0^2 N_m(v^2-1/4)/L_M)/(v^2)dv$$

$$= (|\nu(1/2)|/L_s)\exp((\alpha_0^2 N_m)/(4L_s)) \int_{1/2}^{\eta} \exp(-\alpha_0^2 N_m v^2/L_M)/(v^2)dv.$$

By making the substitution $t = \alpha_0 v\sqrt{N_m/L_M}$ we obtain

$$\Phi[\eta,u] \leq \frac{|\nu(1/2)|\alpha_0\sqrt{N_m}}{L_s\sqrt{L_M}} \exp((\alpha_0^2 N_m)/(4L_M))$$

$$\cdot\Big[-\sqrt{\pi}\,\mathrm{erf}(\alpha_0\eta\sqrt{N_m/L_M}) + \sqrt{\pi}\,\mathrm{erf}(\alpha_0/2\sqrt{N_m/L_M})$$

$$-\frac{1}{\alpha_0\eta}\sqrt{\frac{L_M}{N_m}}\exp(-\alpha_0^2\eta^2 N_m/L_M) + \frac{2}{\alpha_0}\sqrt{\frac{L_M}{N_m}}\exp(-(\alpha_0^2 N_m)/(4L_M))\Big].$$

Analogously we can obtain the left-hand-side inequality. □

Lemma 3.3. *Let $\beta_0 > \alpha_0$. If (3.1)-(3.2) hold, then for all $u, u^* \in C[1/2, \beta_0/2\alpha_0]$ we have*

$$|E[\eta,u] - E[\eta,u^*]| \leq (\alpha_0^2/L_s)(\eta^2 - 1/4)(\tilde{N} + N_M\tilde{L}/L_s)||u^* - u||, \quad \forall \eta \in (1/2, \beta_0/2\alpha_0).$$

Proof. By using the inequality $|\exp(-x) - \exp(-y)| \leq |x-y|, \ \forall x, y \geq 0$ we obtain

$$|E[\eta,u] - E[\eta,u^*]| = \Bigg| \exp\Bigg(-2\alpha_0^2 \int_{1/2}^{\eta} sN(u(s))/L(u(s))ds \Bigg)$$

$$-\exp\Bigg(-2\alpha_0^2 \int_{1/2}^{\eta} sN(u^*(s))/L(u^*(s))ds \Bigg) \Bigg| \leq 2\alpha_0^2 \Bigg| \int_{1/2}^{\eta} sN(u(s))/L(u(s))ds$$

$$-\int_{1/2}^{\eta} sN(u^*(s))/L(u^*(s))ds \Bigg| \leq 2\alpha_0^2 \int_{1/2}^{\eta} \Bigg| N(u)/L(u) - N(u^*)/L(u^*) \Bigg| sds$$

$$\leq 2\alpha_0^2 \int_{1/2}^{\eta} \Bigg| N(u)/L(u) - N(u^*)/L(u) + N(u^*)/L(u) - N(u^*)/L(u^*) \Bigg| sds$$

$$\leq 2\alpha_0^2 \int_{1/2}^{\eta} \Bigg(|N(u) - N(u^*)|/|L(u)| + |N(u^*)| \cdot |L(u^*) - L(u)|/(|L(u)| \cdot |L(u^*)|) \Bigg) sds$$

$$\leq 2\alpha_0^2 \Bigg(\tilde{N}/L_s + N_M(\tilde{L}/L_s^2) \Bigg) ||u^* - u|| \int_{1/2}^{\eta} sds \leq (\alpha_0^2/L_s)(\eta^2 - 1/4)(\tilde{N} + N_M\tilde{L}/L_s)||u^* - u||.$$

□

Lemma 3.4. *Let $\beta_0 > \alpha_0$. Suppose that (3.1)-(3.2) hold. Then for all $u, u^* \in C[1/2, \beta_0/2\alpha_0]$ we have*

$$|\Phi[\eta,u]-\Phi[\eta,u^*]| \leq (|\nu(1/2)|/L_s^2)||u^*-u||\Big(\alpha_0^2(\tilde{N}+N_M\tilde{L}/L_s)(\eta+1/4\eta-1)+\tilde{L}(2-1/\eta)\Big),$$

$$\forall\eta\in(1/2,\beta_0/2\alpha_0),$$

where $\nu(1/2)$ defined by (2.21).

Proof.

$$|\Phi[\eta,u]-\Phi[\eta,u^*]| \leq |\nu(1/2)|\int\limits_{1/2}^{\eta}\Big|\exp\Big(-2\alpha_0^2\int\limits_{1/2}^{\eta}uN(v(u))/L(v(u))du\Big)$$

$$-\exp\Big(-2\alpha_0^2\int\limits_{1/2}^{\eta}uN(v^*(u))/L(v^*(u))du\Big)\Big|/v^2L(u(v))dv$$

$$+|\nu(1/2)|\int\limits_{1/2}^{\eta}\Big|\frac{1}{L(u(v))}-\frac{1}{L(u^*(v))}\Big|\frac{1}{v^2}\exp\Big(-2\alpha_0^2\int\limits_{1/2}^{\eta}uN(u^*)/L(u^*)du\Big)dv$$

$$\equiv T_1(\eta)+T_2(\eta).$$

From Lemma 3.3 we get

$$T_1(\eta)\leq|\nu(1/2)|\int\limits_{1/2}^{\eta}|E[v,u]-E[v,u^*]|/(v^2L_s)dv$$

$$\leq|\nu(1/2)|\int\limits_{1/2}^{\eta}(\alpha_0^2/L_s)(\eta^2-1/4)(\tilde{N}+N_M\tilde{L}/L_s)||u^*-u||/(v^2L_s)dv$$

$$\leq|\nu(1/2)|(\alpha_0^2/L_s^2)(\tilde{N}+(N_M/L_s)\tilde{L})||u^*-u||\int\limits_{1/2}^{\eta}\frac{v^2-1/4}{v^2}dv$$

$$=|\nu(1/2)|(\alpha_0^2/L_s^2)(\tilde{N}+(N_M/L_s)\tilde{L})||u^*-u||(\eta+1/4\eta-1)$$

and

$$T_2(\eta)\leq|\nu(1/2)|\int\limits_{1/2}^{\eta}\Big|\frac{1}{L(u(v))}-\frac{1}{L(u^*(v))}\Big|\frac{dv}{v^2}$$

$$\leq|\nu(1/2)|\int\limits_{\nu}^{\eta}\frac{|L(u^*(v))-L(u(v))|}{|L(u(v)L(u^*(v)))|}\frac{dv}{v^2}\leq|\nu(1/2)|(\tilde{L}/L_s^2)||u^*-u||\int\limits_{1/2}^{\eta}\frac{1}{v^2}dv$$

$$=|\nu(1/2)|(\tilde{L}/L_s^2)||u^*-u||(2-1/\eta).$$

Then we have

$$T_1(\eta)+T_2(\eta)=(|\nu(1/2)|/L_s^2)||u^*-u||\Big(\alpha_0^2(\tilde{N}+(N_M/L_s)\tilde{L})(\eta+1/4\eta-1)+\tilde{L}(2-1/\eta)\Big).$$

□

Theorem 3.1. *Let $\beta_0 > \alpha_0$. Suppose that* (3.1)-(3.2) *hold. If the following inequality is satisfied*

$$b(\alpha_0, \beta_0) = \frac{|\nu(1/2)|}{L_s^2}\left(\alpha_0^2\left(\tilde{N} + \frac{N_M \tilde{L}}{L_s}\right)\left(\frac{\beta_0}{2\alpha_0} + \frac{\alpha_0}{2\beta_0} - 1\right) + \tilde{L}\left(2 - \frac{2\alpha_0}{\beta_0}\right)\right) < 1, \tag{3.3}$$

where $\nu(1/2)$ is defined by (2.21)*, then there exists a unique solution $u \in C[1/2, \beta_0/2\alpha_0]$ of integral equation* (2.22).

Proof. Let $W : C[1/2, \beta_0/2\alpha_0] \to C[1/2, \beta_0/2\alpha_0]$ be the operator defined by

$$W(u)(\eta) = 1 + \Phi[\eta, L(u), N(u)], \quad u \in C[1/2, \beta_0/2\alpha_0].$$

A solution of (2.22) is a fixed point of the operator W, that is

$$W(u)(\eta) = u(\eta), \quad 1/2 < \eta < \beta_0/2\alpha_0.$$

Let $u, u^* \in C[1/2, \beta_0/2\alpha_0]$, then we obtain

$$||W(u) - W(u^*)|| = \max_{\eta\in[1/2,\beta_0/2\alpha_0]} |W(u(\eta)) - W(u^*(\eta))| \leq \max_{\eta\in[1/2,\beta_0/2\alpha_0]} |\Phi[\eta, u^*] - \Phi[\eta, u]|.$$

Finally, by using Lemmas 3.2 - 3.4 we have

$$||W(u) - W(u^*)|| \leq b(\alpha_0, \beta_0)||u^* - u||.$$

Hence, if condition (3.3) is satisfied, W is a contraction operator and by the fixed point theorem there exists a unique solution of integral equation (2.22).
Remark. Clearly, condition (3.3) is satisfied for $\beta_0 > \alpha_0$ sufficiently close to α_0 and is not satisfied for sufficiently large β_0. □

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