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SHAVKAT ARIFJANOVICH ALIMOV

(to the 75th birthday)

Shavkat Arifjanovich Alimov was born on March 2, 1945 in the city of Nukus, Uzbekistan. In 1968, he graduated from the Department of Mathematics of Physical Faculty of the M.V. Lomonosov Moscow State University (MSU), receiving a diploma with honors. From 1968 to 1970, he was a post-graduate student in the same department under the supervision of Professor V.A. Il'in. He defended his PhD thesis in 1970. In May 1973, at the age of 28, he defended his doctoral thesis devoted to equations of mathematical physics. In 1973, for research on the spectral theory, he was awarded the highest youth prize of the USSR.

From 1974 to 1984, he worked as a professor in the Department of General Mathematics at the Faculty of Computational Mathematics and Cybernetics. In 1984, Sh.A. Alimov joined the Tashkent State University (TSU) as a professor. From 1985 to 1987 he worked as the Rector of the Samarkand State University, from 1987 to 1990 - the Rector of the TSU, from 1990 to 1992 - the Minister of Higher and Secondary Special Education of the Republic of Uzbekistan. From 1992 to 1994, he headed the Department of Mathematical Physics of the TSU.

After some years of diplomatic work, he continued his academic career as a professor of the Department of Mathematical Physics at the National University of Uzbekistan (NUU). From the first days of the opening of the Tashkent branch of the MSU in 2006, he worked as a professor in the Department of Applied Mathematics. From 2012 to 2017, he headed the Laboratory of Mathematical Modeling of the Malaysian Institute of Microelectronic Systems in Kuala Lumpur. From 2017 to 2019, he worked as a professor at the Department of Differential Equations and Mathematical Physics of the NUU. From 2019 to the present, Sh.A. Alimov is a Scientific Consultant at the Center for Intelligent Software Systems, and an adviser to the Rector of the NUU.

The main scientific activity of Sh.A. Alimov is connected with the spectral theory of partial differential equations and the theory of boundary value problems for equations of mathematical physics. He obtained series of remarkable results in these fields. They cover many important problems of the theory of Schrodinger equations with singular potentials, the theory of boundary control of the heat transfer process, the mathematical problems of peridynamics related to the theory of hypersingular integrals.

In 1984, Sh.A. Alimov was elected a corresponding member and in 2000 an academician of the Academy of Sciences of Uzbekistan. He was awarded several prestigious state prizes.

Sh.A. Alimov has over 150 published scientific and a large number of educational works. Among his pupils there are 10 doctors of sciences and more than 20 candidates of sciences (PhD) working at universities of Uzbekistan, Russia, USA, Finland, and Malaysia.

For about thirty years, Sh.A. Alimov has been actively involved in the reform of mathematical school education.

Sh.A. Alimov meets his 75th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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BOUNDEDNESS OF RIEMANN-LIOUVILLE OPERATOR FROM WEIGHTED SOBOLEV SPACE TO WEIGHTED LEBESGUE SPACE

A. Kalybay, R. Oinarov

Communicated by V.D. Stepanov

Key words: Sobolev space, Riemann-Liouville operator, weight function, boundedness, weighted inequality.

AMS Mathematics Subject Classification: 26D10, 47B38.

Abstract. In this paper, we obtain a criterion for the boundedness of the Riemann-Liouville fractional integration operator from a weighted Sobolev space to a weighted Lebesgue space.

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1 Introduction

Let $I = (0, \infty)$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Let υ, ρ be positive functions and ω be a nonnegative function on I such that υ^p, ρ^p, ω^q, $\rho^{-p'}$ and $\omega^{-q'}$ are locally summable on I.

Denote by $W_p^1(\rho, \upsilon) \equiv W_p^1(\rho, \upsilon, I)$ the space of all functions locally absolutely continuous on I having the finite norm

$$\|f\|_{W_p^1} = \|\rho f'\|_p + \|\upsilon f\|_p,$$

where $\|\cdot\|_p$ is the standard norm of the Lebesgue space $L_p(I)$.

Let $\overset{\circ}{AC}(I)$ be the set of all locally absolutely continuous functions with compact supports on I.

Denote by $\overset{\circ}{W}{}_p^1(\rho, \upsilon) \equiv \overset{\circ}{W}{}_p^1(\rho, \upsilon, I)$ the closure of the set $\overset{\circ}{AC}(I) \cap W_p^1(\rho, \upsilon)$ with respect to the norm of the space $W_p^1(\rho, \upsilon)$.

Let $L_{p,\upsilon} \equiv L_p(\upsilon, I)$ be the space of all measurable functions I with the finite norm $\|f\|_{p,\upsilon} \equiv \|\upsilon f\|_p$.

We consider the Riemann-Liouville fractional integration operator I_α, $\alpha > 0$:

$$I_\alpha f(x) = \int_0^x (x-s)^{\alpha-1} f(s) ds, \quad x \in I. \tag{1.1}$$

The main aim of this paper is to establish a criterion of the boundedness of the Riemann-Liouville operator I_α from $\overset{\circ}{W}{}_p^1(\rho, \upsilon)$ to $L_q(\omega, I)$, i.e., the validity of the inequality:

$$\|\omega I_\alpha f\|_q \le C(\|\rho f'\|_p + \|\upsilon f\|_p), \quad f \in \overset{\circ}{W}{}_p^1(\rho, \upsilon). \tag{1.2}$$

In papers [2], [5] and [7], inequalities of type (1.2) are studied for certain classes of Volterra type integral operators. In the case where $\rho \equiv 0$, the validity of inequality (1.2) means the boundedness of the Riemann-Liouville operator I_α from $L_{p,\upsilon}$ to $L_{q,\omega}$. There are many recent works devoted to this problem. For example, in the case $0 < q < \infty$, $1 < p < \infty$, $\alpha > \frac{1}{p}$ and $\upsilon(\cdot) \equiv 1$, explicit criteria for the boundedness of operators (1.1) from L_p to $L_{q,\omega}$ are independently found in the works by A.A. Meskhi [3] and D.V. Prokhorov [9]. When the function $\upsilon(\cdot)$ does not increase, a generalization of these results is given in the work by S.M. Farsani [1]. In paper [10], D.V. Prokhorov and V.D. Stepanov find criteria for the boundedness and compactness of (1.1) from $L_{p,\upsilon}$ to $L_{q,\omega}$ for $1 < p \leq q < \infty$ and two cases: (a) $1 - \frac{q'}{p'} < \alpha \leq 1$ and the function υ does not increase; (b) $1 - \frac{p}{q} < \alpha \leq 1$ and the function ω does not increase. Generalizations of these results for the convolution type operators are presented in works [8] and [11].

This paper is organized as follows: In Section 2 we collect all required notations, definitions and statements. In Sections 3 we state and prove our main result concerning the validity of inequality (1.2).2

2 Preliminaries

In the sequel, the relation $A \ll B$ means $A \leq CB$ with a constant C depending only on the parameters p and q. Moreover, if $A \ll B \ll A$ we write $A \approx B$.

As in [4] (see also [5, 7]), we introduce the following function

$$\delta(x,y) = \sup\left\{ d > 0 : \int\limits_{x-d}^{x} \rho^{-p'}(t)dt \leq \int\limits_{x}^{x+y} \rho^{-p'}(t)dt, (x-d, x] \subset I \right\},$$

with the domain $D(\delta) = \{(x,y) : x \in I, y > 0, [x, x+y) \subset I\}$. If we fix $x \in I$, then at least for a sufficiently small $y > 0$ we have

$$\int\limits_{x-\delta(x,y)}^{x} \rho^{-p'}(t)dt = \int\limits_{x}^{x+y} \rho^{-p'}(t)dt. \tag{2.1}$$

Let $x \in I$ and D_x be the set of $y > 0$ such that $x + y \in I$ and (2.1) holds. For all $x \in I$ we define

$$d^+(x) = \sup\{d : \|\rho^{-1}\|_{p',(x-\delta(x,d),x+d)} \|\upsilon\|_{p,(x-\delta(x,d),x+d)} \leq 1, \ d \in D_x\}$$

and $d^-(x) = \delta(x, d^+(x))$. Moreover, we assume that $\mu^-(x) = x - d^-(x)$ and $\mu^+(x) = x + d^+(x)$.

Let $a = \inf\{x \in I : \mu^-(x) > 0\}$ and $b = \sup\{x \in I : \mu^+(x) < \infty\}$. Let $h_0 = \|\rho^{-1}\|_{p',(0,c)} \|\upsilon\|_{p,(0,c)}$ and $h_\infty = \|\rho^{-1}\|_{p',(c,\infty)} \|\upsilon\|_{p,(c,\infty)}$ for some $c \in I$. In [4] it is shown that

$$a = 0 \Leftrightarrow h_0 = \infty, \quad b = \infty \Leftrightarrow h_\infty = \infty,$$

$$\|\rho^{-1}\|_{p',\Delta(x)} \|\upsilon\|_{p,\Delta(x)} = 1, \ \ \forall x \in (a,b),$$

and hence

$$\|\rho^{-1}\|_{p',\Delta(x)} \|\upsilon\|_{p,\Delta(x)} = 1, \ \ \forall x \in I, \tag{2.2}$$

in the case

$$h_0 = \|\rho^{-1}\|_{p',(0,c)} \|\upsilon\|_{p,(0,c)} = \infty, \quad h_\infty = \|\rho^{-1}\|_{p',(c,\infty)} \|\upsilon\|_{p,(c,\infty)} = \infty, \tag{2.3}$$

where $\Delta(x) = [\mu^-(x), \mu^+(x)]$.

By Lemma 1.1 in [4], the functions $\mu^-(x) = x - d^-(x)$ and $\mu^+(x) = x + d^+(x)$ are continuous and strictly increasing on (a, ∞) and $(0, b)$, respectively. In addition, if $h_0 < \infty$, then

$$\mu^-(x) = 0 = \lim_{z \to a^+} \mu^-(z), \ \ \forall x \in (0, a),$$

if $h_\infty < \infty$, then

$$\mu^+(x) = \infty = \lim_{z\to b^-} \mu^+(z), \ \ \forall x \in (b,\infty),$$

and if (2.3) holds, then

$$\lim_{x\to 0^+} \mu^\pm(x) = 0, \ \lim_{x\to\infty} \mu^\pm(x) = \infty. \tag{2.4}$$

From the last condition it follows that $0 < \mu^\pm(x) < \infty$ for any $x \in I$. Moreover, condition (2.4) follows from (2.2).

For simplicity, we assume that (2.3) holds. Then, in view of the above, the functions $\mu^-(x)$ and $\mu^+(x)$ are continuous and strictly increasing on I and (2.4) holds. The validity of (2.3) is equivalent to the condition $\mathring{W}_p^1(\rho,\upsilon) = W_p^1(\rho,\upsilon)$ (see [4]). How to overcome the difficulties that arise when condition (2.3) does not hold is also given in [4].

We need Lemma 2.1 of [7].

Lemma 2.1. ([7, Lemma 2.1]) *Let condition (2.3) hold. Then the functions $\mu^-(x)$ and $\mu^+(x)$ are locally absolutely continuous on I.*

Denote by φ^+ and φ^- the inverses of the functions μ^- and μ^+, respectively. Then the functions φ^+ and φ^- are continuous and strictly increasing on I. Moreover, $\varphi^+(x) > \varphi^-(x)$ for any $x \in I$ and $\lim\limits_{x\to 0^+} \varphi^+(x) = 0$, $\lim\limits_{x\to\infty} \varphi^-(x) = \infty$.

Let us consider the extremal problem of finding the quantity

$$J(\rho,\upsilon,g) = \sup_{0\neq f\in \mathring{W}_p^1} \frac{\int\limits_0^\infty f(x)g(x)dx}{\|f\|_{W_p^1}}, \tag{2.5}$$

where g is a nonnegative function locally summable on I. The solution of problem (2.5) is given in Theorem 2.1 obtained in [5].

Theorem A. ([5, Theorem 2.1]) *Let $1 < p < \infty$. Let g be a nonnegative function locally integrable on I. Then*

$$J(\rho,\upsilon,g) \approx \left(\int\limits_0^\infty \left(\int\limits_{\varphi^-(x)}^{\varphi^+(x)} g(t)dt \right)^{p'} \rho^{-p'}(x)dx \right)^{\frac{1}{p'}},$$

where the equivalence constants depend only on p.

For an arbitrary positive operator T we consider the inequality

$$\|\omega Tf\|_q \le C(\|\rho f'\|_p + \|\upsilon f\|_p), \ \ f \in \mathring{W}_p^1(\rho,\upsilon). \tag{2.6}$$

On the basis of Theorem A we have Theorem B.

Theorem B. ([5, Theorem 2.2]) *Let $1 < p,q < \infty$. Inequality (2.6) for all functions $f \in \mathring{W}_p^1(\rho,\upsilon)$ is equivalent to the inequality*

$$\left(\int\limits_a^b \left(\int\limits_{\varphi^-(x)}^{\varphi^+(x)} (T^*g)(t)dt \right)^{p'} \rho^{-p'}(x)dx \right)^{\frac{1}{p'}} \le C_1 \left(\int\limits_a^b \omega^{-q'}(t) g^{q'}(t)dt \right)^{\frac{1}{q'}} \tag{2.7}$$

for all non-negative functions $g \in L_{q'}(\omega^{-1}, I)$, where T^* is the dual operator to the operator T with respect to the bilinear form $\int\limits_a^b f(t)g(t)dt$. Moreover, $C \approx C_1$, where $C > 0$ and $C_1 > 0$ are the best constants in (2.6) and (2.7), respectively.

Indeed, we have

$$\sup_{g\geq 0}\frac{\|\frac{1}{\rho}\int_{\varphi^-}^{\varphi^+}T^*g\|_{p'}}{\|\frac{g}{\omega}\|_{q'}}=\sup_{g\geq 0}\frac{1}{\|\frac{g}{\omega}\|_{q'}}\sup_{f\in\mathring{W}_p^1}\frac{\int_I f\cdot T^*g}{\|f\|_{W_p^1}}=\sup_{g\geq 0}\frac{1}{\|\frac{g}{\omega}\|_{q'}}\sup_{0\leq f\in\mathring{W}_p^1}\frac{\int_I f\cdot T^*g}{\|f\|_{W_p^1}}$$

$$=\sup_{0\leq f\in\mathring{W}_p^1}\frac{1}{\|f\|_{W_p^1}}\sup_{g\geq 0}\frac{\int_I g\cdot Tf}{\|\frac{g}{\omega}\|_{q'}}=\sup_{0\leq f\in\mathring{W}_p^1}\frac{\|\omega Tf\|_q}{\|f\|_{W_p^1}}=\sup_{f\in\mathring{W}_p^1}\frac{\|\omega Tf\|_q}{\|f\|_{W_p^1}}.$$

On the basis of Theorem B Lemma 3.3 in [2] was obtained that presents the inequality dual to (2.6).

Lemma 2.2. ([2, Lemma 3.3]) *Let $1<p,q<\infty$. Inequality (2.6) for all functions $f\in\mathring{W}_p^1(\rho,\upsilon)$ is equivalent to the inequality*

$$\left(\int_0^\infty\left(\omega(x)T\left(\int_{\mu^-(\cdot)}^{\mu^+(\cdot)}f(t)dt\right)(x)\right)^q dx\right)^{\frac{1}{q}}\leq C_1\left(\int_0^\infty\rho^p(t)f^p(t)dt\right)^{\frac{1}{p}} \tag{2.8}$$

for all nonnegative functions $f\in L_p(\rho,I)$. Moreover, $C\approx C_1$, where $C>0$ and $C_1>0$ are the best constants in (2.6) and (2.8), respectively.

Lemma 2.2 is proved as follows

$$\sup_{g\geq 0}\frac{\|\frac{1}{\rho}\int_{\varphi^-}^{\varphi^+}T^*g\|_{p'}}{\|\frac{g}{\omega}\|_{q'}}=\sup_{g\geq 0}\frac{1}{\|\frac{g}{\omega}\|_{q'}}\sup_{f\geq 0}\frac{\int_I\left[f\cdot\int_{\varphi^-}^{\varphi^+}T^*g\right]}{\|f\rho\|_p}$$

$$=\sup_{f\geq 0}\frac{1}{\|f\rho\|_p}\sup_{g\geq 0}\frac{\int_I\left[g\cdot T\left(\int_{\mu^-}^{\mu^+}f\right)\right]}{\|\frac{g}{\omega}\|_{q'}}=\sup_{f\geq 0}\frac{\|\omega T\left(\int_{\mu^-}^{\mu^+}f\right)\|_q}{\|f\rho\|_p}.$$

Let $\alpha(x)$ and $\beta(x)$ be locally absolutely continuous and strictly increasing functions on I such that $\alpha(x)<\beta(x)$ for any $x\in I$ and $\lim\limits_{x\to 0+}\alpha(x)=\lim\limits_{x\to 0+}\beta(x)=0$, $\lim\limits_{x\to\infty}\alpha(x)=\lim\limits_{x\to\infty}\beta(x)=\infty$. Consider the Hardy-type operator

$$\mathcal{H}f(x)=\int_0^{\beta(x)}f(s)ds,\quad x\in I, \tag{2.9}$$

and the integral operator

$$\mathcal{K}f(x)=\int_{\alpha(x)}^{\beta(x)}K(x,s)f(s)ds,\quad x\in I. \tag{2.10}$$

Let

$$E=\sup_{z\in I}\left(\int_z^\infty\omega^q(x)dx\right)^{\frac{1}{q}}\left(\int_0^{\beta(z)}\rho^{-p'}(t)dt\right)^{\frac{1}{p'}}.$$

From the results of work [12] we have Theorem C.

Theorem C. ([12, Theorem 4.1]) *Let $1<p\leq q<\infty$. Then operator (2.9) is bounded from $L_p(\rho,I)$ to $L_q(\omega,I)$ if and only if the condition $E<\infty$ holds. Moreover, for the norm $\|\mathcal{H}\|_{p\to q}$ of the operator $\mathcal{H}$ from $L_p(\rho,I)$ to $L_q(\omega,I)$ the relation $\|\mathcal{H}\|_{p\to q}\approx E$ holds.*

Let $\Omega = \{(x,s) : 0 < x < \infty,\ \alpha(x) \le s \le \beta(x)\}$. Let a function $K(\cdot,\cdot) \ge 0$ be defined and measurable on Ω. Moreover, assume that $K(\cdot,\cdot)$ does not decrease in the first argument. Let us define the class $\mathcal{O}_1(\Omega)$ of kernels of operator (2.10). The function $K(\cdot,\cdot)$ belongs to the class $\mathcal{O}_1(\Omega)$ if and only if for $K(\cdot,\cdot)$ there exist functions $K_{1,0}(x,s)$ and $u(s)$ defined and measurable on Ω and the relation

$$K(x,s) \approx K_{1,0}(x,t)u(s) + K(t,s) \tag{2.11}$$

holds for $0 < t \le x < \infty$ and $\alpha(x) \le s \le \beta(t)$, where the constants of equivalency in (2.11) do not depend on x, t and s. Let us note that the class $\mathcal{O}_1(\Omega)$ is the class $\mathcal{O}_1^+(\alpha, \beta(\cdot), \Omega^+)$ of paper [6].

Denote

$$D_1^+ = \sup_{z\in I} \sup_{y\in\Delta^+(z)} \left(\int_y^z \omega^q(x) \left(\int_{\alpha(z)}^{\beta(y)} K^{p'}(x,s)\rho^{-p'}(s)ds \right)^{\frac{q}{p'}} dx \right)^{\frac{1}{q}},$$

$$D_2^+ = \sup_{z\in I} \sup_{y\in\Delta^+(z)} \left(\int_{\alpha(z)}^{\beta(y)} \rho^{-p'}(s) \left(\int_y^z K^q(x,s)\omega^q(x)dx \right)^{\frac{p'}{q}} ds \right)^{\frac{1}{p'}},$$

where $\Delta^+(z) = [\beta^{-1}(\alpha(z)), z]$. From the results of work [6] we have one more theorem.

Theorem D. ([6, Theorem 3]) *Let $1 < p \le q < \infty$ and the kernel of operator (2.10) belong to $\mathcal{O}_1(\Omega)$. Then operator (2.10) is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if the condition $D_i < \infty$ holds at least for one of $i = 1, 2$. Moreover, for the norm $\|\mathcal{K}\|_{p\to q}$ of the operator $\mathcal{K}$ from $L_p(\rho, I)$ to $L_q(\omega, I)$ the relation $\|\mathcal{K}\|_{p\to q} \approx D_1 \approx D_2$ holds.*

3 Criterion for the validity of inequality (1.2) for operator (1.1)

Here and in the sequel we assume that condition (2.3) holds.

Let

$$A_1 = \sup_{z\in I} \left(\int_z^\infty \omega^q(x)x^{q(\alpha-1)}dx \right)^{\frac{1}{q}} \left(\int_0^{\mu^-(z)} \rho^{-p'}(t) \left[\varphi^+(t) - \varphi^-(t)\right]^{p'} dt \right)^{\frac{1}{p'}},$$

$$A_{2,1} = \sup_{z\in I} \sup_{y\in\Delta^+(z)} \left(\int_{\mu^-(z)}^{\mu^+(y)} \rho^{-p'}(x) \left(\int_y^z (t-\varphi^-(x))^{q\alpha}\omega^q(t)dt \right)^{\frac{p'}{q}} dx \right)^{\frac{1}{p'}},$$

$$A_{2,2} = \sup_{z\in I} \sup_{y\in\Delta^+(z)} \left(\int_y^z \omega^q(t) \left(\int_{\mu^-(z)}^{\mu^+(y)} \left(t-\varphi^-(x)\right)^{p'\alpha} \rho^{-p'}(x)dx \right)^{\frac{q}{p'}} dt \right)^{\frac{1}{q}},$$

where $\Delta^+(z) = [\varphi^-(\mu^-(z)), z]$.

Suppose that

$$U(t) = \frac{d}{dt} \int_0^{\mu^-(t)} |\varphi^+(x) - \varphi^-(x)|^{p'} \rho^{-p'}(x)dx.$$

Theorem 3.1. *Let $1 < p \le q < \infty$ and $\alpha > \frac{1}{p}$. Let the function U be non-increasing for $t > 0$. Then Riemann-Liouville operator (1.1) is bounded from $\mathring{W}_p^1(\rho, \upsilon)$ to $L_q(\omega, I)$ if and only if $\mathcal{A}_j = \max\{A_1, A_{2,j}\} < \infty$ at least for one of $j = 1, 2$. Moreover, for the norm $\|I_\alpha\|_{W\to q}$ of operator (1.1) from $\mathring{W}_p^1(\rho, \upsilon)$ to $L_q(\omega, I)$, the relation $\|I_\alpha\|_{W\to q} \approx \mathcal{A}_j$, $j = 1, 2$, is valid.*

Proof of Theorem 3.1. By Lemma 2.2 inequality (1.2) holds if and only if inequality (2.8) holds for $T = I_\alpha$, i.e., the operator

$$\widetilde{\mathcal{I}}_\alpha f(s) \equiv I_\alpha \left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)} f(x)dx \right)(s)$$

is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$. Moreover, $\|I_\alpha\|_{W\to q} \approx \|\widetilde{\mathcal{I}}_\alpha\|_{p\to q}$, where $\|\widetilde{\mathcal{I}}_\alpha\|_{p\to q}$ is the norm of the operator $\widetilde{\mathcal{I}}_\alpha$ from $L_p(\rho, I)$ to $L_q(\omega, I)$. Let $0 \le f \in L_p(\rho, I)$. Since

$$I_\alpha \left(\int\limits_{\mu^-(\cdot)}^{\mu^+(\cdot)} f(x)dx \right)(s) = \int\limits_0^s (s-t)^{\alpha-1} \int\limits_{\mu^-(t)}^{\mu^+(t)} f(x)dxdt, \tag{3.1}$$

by changing the order of integration, we get

$$\int\limits_0^s (s-t)^{\alpha-1} \int\limits_{\mu^-(t)}^{\mu^+(t)} f(x)dxdt = \int\limits_0^{\mu^-(s)} f(x) \int\limits_{\varphi^-(x)}^{\varphi^+(x)} (s-t)^{\alpha-1}dtdx$$

$$+ \int\limits_{\mu^-(s)}^{\mu^+(s)} f(x) \int\limits_{\varphi^-(x)}^{s} (s-t)^{\alpha-1}dtdx = \int\limits_0^{\mu^-(s)} f(x) \int\limits_{\varphi^-(x)}^{\varphi^+(x)} (s-t)^{\alpha-1}dtdx$$

$$+\frac{1}{\alpha} \int\limits_{\mu^-(s)}^{\mu^+(s)} (s-\varphi^-(x))^\alpha f(x)dx = \widetilde{\mathcal{I}}_{1,\alpha} f(s) + \frac{1}{\alpha}\widetilde{\mathcal{I}}_{2,\alpha} f(s). \tag{3.2}$$

From (2.8), (3.1) and (3.2) it follows that the operator $\widetilde{\mathcal{I}}_\alpha$ is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if the operators

$$\widetilde{\mathcal{I}}_{1,\alpha} f(s) \equiv \int\limits_0^{\mu^-(s)} f(x) \int\limits_{\varphi^-(x)}^{\varphi^+(x)} (s-t)^{\alpha-1}dtdx \tag{3.3}$$

and

$$\widetilde{\mathcal{I}}_{2,\alpha} f(s) \equiv \int\limits_{\mu^-(s)}^{\mu^+(s)} (s-\varphi^-(x))^\alpha f(x)dx \tag{3.4}$$

are bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$. Moreover, for the norms of the operators $\|\widetilde{\mathcal{I}}_\alpha\|_{p\to q}$, $\|\widetilde{\mathcal{I}}_{1,\alpha}\|_{p\to q}$ and $\|\widetilde{\mathcal{I}}_{2,\alpha}\|_{p\to q}$ from $L_p(\rho, I)$ to $L_q(\omega, I)$ the following relation holds:

$$\|\widetilde{\mathcal{I}}_\alpha\|_{p\to q} \approx \|\widetilde{\mathcal{I}}_{1,\alpha}\|_{p\to q} + \|\widetilde{\mathcal{I}}_{2,\alpha}\|_{p\to q}. \tag{3.5}$$

Theorem 3.1 will be proved if we prove the following two assertions.

Theorem 3.2. *Let $1 < p \leq q < \infty$. Then operator (3.4) is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if the condition $A_{2,j} < \infty$ holds at least for one of $j = 1, 2$. Moreover, for the norm $\|\widetilde{\mathcal{I}}_{2,\alpha}\|_{p\to q}$ of the operator $\widetilde{\mathcal{I}}_{2,\alpha}$ from $L_p(\rho, I)$ to $L_q(\omega, I)$, the relation $\|\widetilde{\mathcal{I}}_{2,\alpha}\|_{p\to q} \approx A_{2,1} \approx A_{2,2}$ is valid.*

Theorem 3.3. *Let $1 < p \leq q < \infty$ and $\alpha > \frac{1}{p}$. Let the function U do not increase for $t > 0$. Then operator (3.3) is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if the condition $A_1 < \infty$ holds. Moreover, for the norm $\|\widetilde{\mathcal{I}}_{1,\alpha}\|_{p\to q}$ of the operator $\widetilde{\mathcal{I}}_{1,\alpha}$ from $L_p(\rho, I)$ to $L_q(\omega, I)$, the relation $\|\widetilde{\mathcal{I}}_{1,\alpha}\|_{p\to q} \approx A_1$ is valid.*

Proof of Theorem 3.2. We first consider the kernel $K(s, x) = (s - \varphi^-(x))^\alpha$ of the operator $\widetilde{\mathcal{I}}_{2,\alpha}$.

Let $0 < t \leq s < \infty$ and $\mu^-(s) \leq x \leq \mu^+(t)$. Then from $x \leq \mu^+(t)$ it follows that $\varphi^-(x) \leq t \leq s < \infty$. Hence, we have that

$$K(s, x) = (s - \varphi^-(x))^\alpha \approx (s - t)^\alpha + (t - \varphi^-(x))^\alpha = K_{1,0}(s, t)u(x) + K(t, x),$$

where $K_{1,0}(s, t) = (s - t)^\alpha$ and $u(x) \equiv 1$, i.e., relation (2.11) holds. Therefore, the kernel $K(s, x) = (s - \varphi^-(x))^\alpha$ of the operator $\widetilde{\mathcal{I}}_{2,\alpha}$ belongs to the class $\mathcal{O}_1(\Omega)$. Then on the basis of Theorem D the operator $\widetilde{\mathcal{I}}_{2,\alpha}$ is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if $A_{2,j} < \infty$ at least for one of $j = 1, 2$. Moreover,

$$\|\widetilde{\mathcal{I}}_{2,\alpha}\|_{p\to q} \approx A_{2,1} \approx A_{2,2}. \tag{3.6}$$

□

Proof of Theorem 3.3. Let the operator $\widetilde{\mathcal{I}}_{1,\alpha}$ be bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$. Since for $u(x) = \varphi^+(x) - \varphi^-(x)$ we have that

$$\left(\int_0^\infty \omega^q(s) s^{q(\alpha-1)} \left|\int_0^{\mu^-(s)} u(x) f(x) dx\right|^q ds\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^\infty \omega^q(s) \left|\int_0^{\mu^-(s)} f(x) \int_{\varphi^-(x)}^{\varphi^+(x)} (s - t)^{\alpha-1} dt dx\right|^q ds\right)^{\frac{1}{q}}$$

$$\leq C \left(\int_0^\infty |\rho(x) f(x)|^p dx\right)^{\frac{1}{p}},$$

then the Hardy-type operator $\mathcal{H}_{\mu^-,\alpha} f(s) = s^{\alpha-1} \int_0^{\mu^-(s)} u(x) f(x) dx$ is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$. Moreover, in view of Theorem B, we have

$$\|\widetilde{\mathcal{I}}_{1,\alpha}\|_{p\to q} \geq \|\mathcal{H}_{\mu^-,\alpha}\|_{p\to q} \gg A_1. \tag{3.7}$$

Let $A_1 < \infty$. Then we have

$$\left(\int_0^\infty \omega^q(s) \left|\int_0^{\mu^-(s)} f(x) \int_{\varphi^-(x)}^{\varphi^+(x)} (s - t)^{\alpha-1} dt dx\right|^q ds\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^\infty \omega^q(s) \left| \int_0^{\mu^-(s)} (s-\varphi^+(x))^{\alpha-1} u(x) f(x) dx \right|^q ds \right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^\infty \omega^q(s) \left| \int_{\mu^-(\frac{s}{2})}^{\mu^-(s)} (s-\varphi^+(x))^{\alpha-1} u(x) f(x) dx \right|^q ds \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^\infty \omega^q(s) \left| \int_0^{\mu^-(\frac{s}{2})} (s-\varphi^+(x))^{\alpha-1} u(x) f(x) dx \right|^q ds \right)^{\frac{1}{q}}$$

$$= F_1 + F_2. \tag{3.8}$$

Let us estimate F_2.

$$F_2 = \left(\int_0^\infty \omega^q(s) \left| \int_0^{\mu^-(\frac{s}{2})} (s-\varphi^+(x))^{\alpha-1} u(x) f(x) dx \right|^q ds \right)^{\frac{1}{q}}$$

$$\ll \left(\int_0^\infty \omega^q(s) s^{q(\alpha-1)} \left| \int_0^{\mu^-(\frac{s}{2})} u(x) f(x) dx \right|^q ds \right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^\infty \omega^q(s) s^{q(\alpha-1)} \left| \int_0^{\mu^-(s)} u(x) f(x) dx \right|^q ds \right)^{\frac{1}{q}}.$$

Thus, by Theorem C we have

$$F_2 \ll A_1 \left(\int_0^\infty |\rho(x) f(x)|^p dx \right)^{\frac{1}{p}}. \tag{3.9}$$

Now we estimate F_1. In the expression F_1 we change the variables $x = \mu^-(t)$ in the inner integral. Then we get

$$F_1 = \left(\int_0^\infty \omega^q(s) \left| \int_{\frac{s}{2}}^{s} (s-t)^{\alpha-1} u(\mu^-(t)) f(\mu^-(t)) \frac{d\mu^-(t)}{dt} dt \right|^q ds \right)^{\frac{1}{q}}$$

$$= \left(\int_0^\infty \omega^q(s) \left| \int_{\frac{s}{2}}^{s} (s-t)^{\alpha-1} u(\mu^-(t)) \rho^{-1}(\mu^-(t)) \right.\right.$$

$$\left.\left. \cdot \left(\frac{d\mu^-(t)}{dt} \right)^{\frac{1}{p'}} \rho(\mu^-(t)) \left(\frac{d\mu^-(t)}{dt} \right)^{\frac{1}{p}} f(\mu^-(t)) dt \right|^q ds \right)^{\frac{1}{q}}$$

$$= \left(\int_0^\infty \omega^q(s) \left| \int_{\frac{s}{2}}^s (s-t)^{\alpha-1} U^{\frac{1}{p'}}(t) \widetilde{\rho}(t) \widetilde{f}(t) dt \right|^q ds \right)^{\frac{1}{q}}, \tag{3.10}$$

where $\widetilde{\rho}(t) = \rho(\mu^-(t)) \left(\frac{d\mu^-(t)}{dt}\right)^{\frac{1}{p}}$ and $\widetilde{f}(t) = f(\mu^-(t))$.

Since the function U is non-increasing, from (3.10) we have

$$F_1^q = \sum_k \int_{2^k}^{2^{k+1}} \omega^q(s) \left| \int_{\frac{s}{2}}^s (s-t)^{\alpha-1} U^{\frac{1}{p'}}(t) \widetilde{\rho}(t) \widetilde{f}(t) dt \right|^q ds$$

$$\leq \sum_k U^{\frac{q}{p'}}(2^{k-1}) \int_{2^k}^{2^{k+1}} \omega^q(s) \left(\int_{\frac{s}{2}}^s (s-t)^{p'(\alpha-1)} \right)^{\frac{q}{p'}} ds \left(\int_{2^{k-1}}^{2^{k+1}} |\widetilde{\rho}(t) \widetilde{f}(t)|^p dt \right)^{\frac{q}{p}}$$

$$\ll \sum_k \left(U(2^{k-1}) 2^{k-1} \right)^{\frac{q}{p'}} \int_{2^k}^{2^{k+1}} \omega^q(s) s^{q(\alpha-1)} ds \left(\int_{2^{k-1}}^{2^{k+1}} |\widetilde{\rho}(t) \widetilde{f}(t)|^p dt \right)^{\frac{q}{p}}$$

$$\leq \sum_k \int_{2^k}^\infty \omega^q(s) s^{q(\alpha-1)} ds \left(\int_0^{\mu^-(2^k)} u^{p'}(x) \rho^{-p'}(x) dx \right)^{\frac{q}{p'}} \left(\int_{2^{k-1}}^{2^{k+1}} |\widetilde{\rho}(t) \widetilde{f}(t)|^p dt \right)^{\frac{q}{p}}$$

$$\leq A_1^q \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} |\widetilde{\rho}(t) \widetilde{f}(t)|^p dt \right)^{\frac{q}{p}} \leq A_1^q \left(\int_0^\infty |\rho(x) f(x)|^p dx \right)^{\frac{q}{p}}.$$

From the last inequality and inequalities (3.8), (3.9) it follows that the operator $\widetilde{\mathcal{I}}_{1,\alpha}$ is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ and the estimate $\|\widetilde{\mathcal{I}}_{1,\alpha}\|_{p\to q} \ll A_1$ holds. This, together with (3.7), gives that the operator $\widetilde{\mathcal{I}}_{1,\alpha}$ is bounded from $L_p(\rho, I)$ to $L_q(\omega, I)$ if and only if $A_1 < \infty$ and $\|\widetilde{\mathcal{I}}_{1,\alpha}\|_{p\to q} \approx A_1$. The proof of Theorem 3.3 is completed, which also completes the proof of Theorem 3.1. □

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References

[1] S.M. Farsani, *On boundedness and compactness of Riemann-Liouville fractional operators.* Siberian Math. J. 54 (2013), no. 2, 368–378.

[2] A.A. Kalybay, R. Oinarov, *Kernel operators and their boundedness from weighted Sobolev space to weighted Lebesgue space.* Turk. J. Math. 43 (2019), 301–315.

[3] A.A. Meskhi, *Solution of some weight problems for the Riemenn-Liouville and Weyl operators.* Georgian Math. J. 5 (1998), no. 6, 565–574.

[4] R. Oinarov, *On weighted norm inequalities with three weights.* J. London Math. Soc. 48 (1993), no. 2, 103–116.

[5] R. Oinarov, *Boundedness of integral operators from weighted Sobolev space to weighted Lebesgue space.* Complex Variables and Elliptic Equations 56 (2011), no. 10-11, 1021–1038.

[6] R. Oinarov, *Boundedness and compactness in weighted Lebesgue spaces of integral operators with variable integration limits.* Siberian Math. J. 52 (2011), no. 6, 1042–1055.

[7] R. Oinarov, *Boundedness of integral operators in weighted Sobolev spaces.* Izvestiya: Mathematics 78 (2014), no. 4, 836–853.

[8] R. Oinarov, *Boundedness and compactness of a class of convolution integral operators of fractional integration type.* Proc. Steklov Inst. Math. 293 (2016), 255–271.

[9] D.V. Prokhorov, *On the boundedness and compactness of a class of integral operators.* J. London Math. Soc. 64:2 (2000), 617–628.

[10] D.V. Prokhorov, V.D. Stepanov, *Weighted estimates for the Riemann–Liouville operators and applications.* Proc. Steklov Inst. Math. 243 (2003), 278–301.

[11] N.A. Rautian, *On the boundedness of a class of fractional type integral operators.* Sbornik. Math. 200 (2009), no. 12, 1807–1832.

[12] V.D. Stepanov, E.P. Ushakova, *Kernel operators with variable intervals of integration in Lebesgue spaces and applications.* Math. Ineq. Appl. 13 (2010), no. 3, 449–510.

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