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SHAVKAT ARIFJANOVICH ALIMOV

(to the 75th birthday)

Shavkat Arifjanovich Alimov was born on March 2, 1945 in the city of Nukus, Uzbekistan. In 1968, he graduated from the Department of Mathematics of Physical Faculty of the M.V. Lomonosov Moscow State University (MSU), receiving a diploma with honors. From 1968 to 1970, he was a post-graduate student in the same department under the supervision of Professor V.A. Il'in. He defended his PhD thesis in 1970. In May 1973, at the age of 28, he defended his doctoral thesis devoted to equations of mathematical physics. In 1973, for research on the spectral theory, he was awarded the highest youth prize of the USSR.

From 1974 to 1984, he worked as a professor in the Department of General Mathematics at the Faculty of Computational Mathematics and Cybernetics. In 1984, Sh.A. Alimov joined the Tashkent State University (TSU) as a professor. From 1985 to 1987 he worked as the Rector of the Samarkand State University, from 1987 to 1990 - the Rector of the TSU, from 1990 to 1992 - the Minister of Higher and Secondary Special Education of the Republic of Uzbekistan. From 1992 to 1994, he headed the Department of Mathematical Physics of the TSU.

After some years of diplomatic work, he continued his academic career as a professor of the Department of Mathematical Physics at the National University of Uzbekistan (NUU). From the first days of the opening of the Tashkent branch of the MSU in 2006, he worked as a professor in the Department of Applied Mathematics. From 2012 to 2017, he headed the Laboratory of Mathematical Modeling of the Malaysian Institute of Microelectronic Systems in Kuala Lumpur. From 2017 to 2019, he worked as a professor at the Department of Differential Equations and Mathematical Physics of the NUU. From 2019 to the present, Sh.A. Alimov is a Scientific Consultant at the Center for Intelligent Software Systems, and an adviser to the Rector of the NUU.

The main scientific activity of Sh.A. Alimov is connected with the spectral theory of partial differential equations and the theory of boundary value problems for equations of mathematical physics. He obtained series of remarkable results in these fields. They cover many important problems of the theory of Schrodinger equations with singular potentials, the theory of boundary control of the heat transfer process, the mathematical problems of peridynamics related to the theory of hypersingular integrals.

In 1984, Sh.A. Alimov was elected a corresponding member and in 2000 an academician of the Academy of Sciences of Uzbekistan. He was awarded several prestigious state prizes.

Sh.A. Alimov has over 150 published scientific and a large number of educational works. Among his pupils there are 10 doctors of sciences and more than 20 candidates of sciences (PhD) working at universities of Uzbekistan, Russia, USA, Finland, and Malaysia.

For about thirty years, Sh.A. Alimov has been actively involved in the reform of mathematical school education.

Sh.A. Alimov meets his 75th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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SOME NEW STATEMENTS FOR NONLINEAR PARABOLIC PROBLEMS

N.L. Gol'dman

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Key words: parabolic equations, boundary value problems of the first kind, unique solvability, Hölder spaces, *a priori* estimates, inverse problems, quasisolution.

AMS Mathematics Subject Classification: 35K59, 35K61, 35R30.

Abstract. The work is connected with investigation of nonlinear problems for parabolic equations with an unknown coefficient at the derivative with respect to time. The considered statements are new subjects in the theory of parabolic equations which essentially differ from usual boundary value problems. One of the statements is a system containing a boundary value problem of the first kind and an equation for a time dependence of the sought coefficient. For such a nonlinear system we determine the faithful character of differential relations in a class of smooth functions and establish conditions of unique solvability. The obtained results are then used for investigation of another statement in which, moreover, it is required to determine a boundary function in one of the boundary conditions by using an additional information about the sought coefficient at the final time.

The nonlinear parabolic problems considered in the present work are important not only as new theoretical subjects but also as the mathematical models of physical-chemical processes with changeable inner characteristics.

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1 Introduction

We study nonlinear problems for parabolic equations with an unknown coefficient at the derivative with respect to time. There is a rapidly growing interest in such new subjects in the theory of parabolic equations. This interest is connected, in particular, with modern needs of the mathematical modeling of physical-chemical processes, where inner characteristics of materials are subjected to changes (see, e.g., [2]). In the present paper, the main attention is given to such new parabolic problems in the Hölder spaces for a case of the boundary conditions of the first kind.

In Section 2, we analyze one of the statements formulating it as a system that involves a boundary value problem for a quasilinear parabolic equation with an unknown coefficient at the time derivative and, moreover, an additional relationship for a time dependence of this coefficient. Justification of the present mathematical statement is an important task since such a statement essentially differs from usual boundary value problems of the first kind for parabolic equations, where all the input data must be given (see the well known monographs [4, 8]). Therefore, a considerable theoretical interest is to obtain conditions for existence and uniqueness of its smooth solution. Investigation of such conditions is carried out by using the Rothe method and *a priori* estimates in the difference-continuous Hölder spaces for the corresponding differential-difference nonlinear system that approximates the original system by the Rothe method. The approach that is proposed in the present work allows one to avoid additional assumptions of the smoothness of the input data, which have usually been imposed by the Rothe method (see, e.g., [8]). Thus, the faithful character of differential relations between the input data and the solution in the chosen function spaces is determined. Moreover, in Section 2 the error estimates for the approximate solutions of the Rothe method are given.

In Section 3, we analyze a nonlinear parabolic problem which is formulated as an inverse problem to the statement of Section 2: it is necessary to find a boundary function in one of the boundary conditions by using the given final observation of the sought coefficient. This inverse problem belongs to a class of ill-posed boundary inverse problems but it has an essential distinction from usual statements of such ones for parabolic equations with final observation. In this statement, besides the boundary function, the unknown coefficient must be determined in the nonlinear system of Section 2.

In order to investigate the present boundary inverse problem, it is reduced in Section 3 to an operator equation in the corresponding function spaces. The choice of such spaces relies on the faithful differential relations in the Hölder classes established in Section 2. This operator equation is equivalent to the minimization problem for the residual functional on the corresponding set of boundary functions. The estimates in the Hölder spaces obtained in Section 2 allow us to prove the continuity of this functional. This property is then used for regularization of the present ill-posed minimization problem. To this end we modify the known quasisolution method [5, 6] on a system of the extending compact sets. Results for the stability of the regularized solutions in the corresponding Hölder spaces complete Section 3.

Section 4 is a short conclusion summarizing the content of this work. The following remarks must be added.

In our analysis we use standard definitions for the function spaces from [8]. In particular, the Hölder class $H^{2+\lambda,1+\lambda/2}(\overline{Q})$ $(0<\lambda<1)$ is determined as the space of functions $u(x,t)$ continuous on the closed set $\overline{Q}=\{0\le x\le l, 0\le t\le T\}$ together with their derivatives u_{xx}, u_t which satisfy the Hölder condition as functions of x, t with the corresponding exponents λ and $\lambda/2$.

For a convenient presentation, the following notation is also used.

$H^{1,\lambda/2,1}(\overline{D})$ is the space of all functions which are continuous for $(x,t,u)\in\overline{D}=\overline{Q}\times[-M_0,M_0]$ together with their derivatives with respect to x, u and, moreover, satisfy the Hölder condition as functions of t with the exponent $\lambda/2$.

Moreover, in connection with application of the Rothe method we use analogues of the Hölder classes in the case of the grid functions $\hat{u}=(u_0,\dots,u_n,\dots,u_N)$ defined on the grid $\overline{\omega}_\tau=\{t_n\}=\{n\tau, n=\overline{0,N}, \tau=TN^{-1}\}$ and in the case of the grid-continuous functions $\hat{u}(x)=(u_0(x),\dots,u_n(x),\dots,u_N(x))$ defined on the set $\overline{Q}_\tau=\{0\le x\le l, t_n\in\overline{\omega}_\tau\}$. Just as in [3] these analogues are determined in the following way.

$H_\tau^{1+\lambda/2}(\overline{\omega}_\tau)$ is a difference analogue of the space $H^{1+\lambda/2}[0,T]$ (see [8]) of all functions $\hat{u}$ having the finite norm

$$|\hat{u}|^{1+\lambda/2}_{\overline{\omega}_\tau}=\max_{0\le n\le N}|u_n|+\max_{1\le n\le N}|u_{n\bar{t}}|+\langle\hat{u}_{\bar{t}}\rangle^{\lambda/2}_{\overline{\omega}_\tau},$$

$$u_{n\bar{t}}=(u_n-u_{n-1})\tau^{-1},\ n=\overline{1,N},\ \langle\hat{u}_{\bar{t}}\rangle^{\lambda/2}_{\overline{\omega}_\tau}=\max_{1\le n<n'\le N}\{|u_{n\bar{t}}-u_{n'\bar{t}}||t_n-t_{n'}|^{-\lambda/2}\}.$$

$H_\tau^{\lambda,\lambda/2}(\overline{Q}_\tau)$ is a difference-continuous analogue of the space $H^{\lambda,\lambda/2}(\overline{Q})$ (see [8]) of all functions $\hat{u}(x)$ continuous in x for $(x,t_n)\in\overline{Q}_\tau$ and having the finite norm

$$\begin{aligned}
|\hat{u}(x)|^{\lambda,\lambda/2}_{\overline{Q}_\tau} &= \max_{(x,t_n)\in\overline{Q}_\tau}|u_n(x)|+\langle\hat{u}(x)\rangle^{\lambda}_{x,\overline{Q}_\tau}+\langle\hat{u}(x)\rangle^{\lambda/2}_{t,\overline{Q}_\tau},\\
\langle\hat{u}(x)\rangle^{\lambda}_{x,\overline{Q}_\tau} &= \sup_{(x,t_n),(x',t_n)\in\overline{Q}_\tau}\{|u_n(x)-u_n(x')||x-x'|^{-\lambda}\},\\
\langle\hat{u}(x)\rangle^{\lambda/2}_{t,\overline{Q}_\tau} &= \sup_{(x,t_n),(x,t'_n)\in\overline{Q}_\tau}\{|u_n(x)-u_{n'}(x)||t_n-t_{n'}|^{-\lambda/2}\}.
\end{aligned}$$

$H_\tau^{1+\lambda,\frac{1+\lambda}{2}}(\overline{Q}_\tau)$ is a difference-continuous analogue of the space $H^{1+\lambda,\frac{1+\lambda}{2}}(\overline{Q})$ (see [8]) of all functions $\hat{u}(x)$ continuous in x together with their derivatives with respect to x for $(x,t_n)\in\overline{Q}_\tau$ and having the finite norm

$$|\hat{u}(x)|_{\overline{Q}_\tau}^{1+\lambda,\frac{1+\lambda}{2}} = \max_{(x,t_n)\in\overline{Q}_\tau}|u_n(x)| + |\hat{u}_x(x)|_{\overline{Q}_\tau}^{\lambda,\lambda/2} + \langle\hat{u}(x)\rangle_{t,\overline{Q}_\tau}^{\frac{1+\lambda}{2}},$$

where $\hat{u}_x(x) = (u_{0x}(x),\dots,u_{nx}(x),\dots,u_{Nx}(x))$.

$H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau)$ is a difference-continuous analogue of the space $H^{2+\lambda,1+\lambda/2}(\overline{Q})$ of all functions $\hat{u}(x)$ continuous in x together with their derivatives $\hat{u}_{xx}(x)$ and $\hat{u}_{\bar{t}}(x)$ for $(x,t_n)\in\overline{Q}_\tau$ and having the finite norm

$$|\hat{u}(x)|_{\overline{Q}_\tau}^{2+\lambda,1+\lambda/2} = \max_{(x,t_n)\in\overline{Q}_\tau}|u_n(x)| + \max_{(x,t_n)\in\overline{Q}_\tau}|u_{nx}(x)| + |\hat{u}_{xx}(x)|_{\overline{Q}_\tau}^{\lambda,\lambda/2} + |\hat{u}_{\bar{t}}(x)|_{\overline{Q}_\tau}^{\lambda,\lambda/2},$$

where

$$\begin{aligned}\hat{u}_{xx}(x) &= (u_{0xx}(x),\dots,u_{nxx}(x),\dots,u_{Nxx}(x)),\\ \hat{u}_{\bar{t}}(x) &= (u_{1\bar{t}}(x),\dots,u_{n\bar{t}}(x),\dots,u_{N\bar{t}}(x)),\\ u_{n\bar{t}}(x) &= (u_n(x)-u_{n-1}(x))\tau^{-1}, n=\overline{1,N}.\end{aligned}$$

2 Unique solvability of a nonlinear parabolic problem with an unknown coefficient at the derivative with respect to time

2.1. The statement for a quasilinear parabolic equation. We formulate this problem as a system for determining the functions $\{u(x,t),\rho(x,t)\}$ in the domain $\overline{Q}=\{0\le x\le l, 0\le t\le T\}$ that satisfy the boundary value problem of the first kind

$$c(x,t,u)\rho(x,t)u_t - Lu = f(x,t), \quad (x,t)\in Q, \tag{1}$$

$$u(x,t)|_{x=0} = w(t), \quad u(x,t)|_{x=l} = v(t), \quad 0<t\le T, \tag{2}$$

$$u(x,t)|_{t=0} = \varphi(x), \quad 0\le x\le l, \tag{3}$$

and the additional relation

$$\rho_t(x,t) = \gamma(x,t,u), \quad (x,t)\in Q, \quad \rho(x,t)|_{t=0} = \rho^0(x), \quad 0\le x\le l, \tag{4}$$

where the uniformly elliptic operator L has the form

$$Lu \equiv (a(x,t,u)u_x)_x - b(x,t,u)u_x - d(x,t,u)u.$$

All the input data in equation (1), boundary conditions (2), initial condition (3), and in relationship (4) are the known functions of their arguments; $a\ge a_{\min}>0$, $c\ge c_{\min}>0$, $\rho^0\ge\rho^0_{\min}>0$, $a_{\min}$, $c_{\min}$, $\rho^0_{\min}=\mathrm{const}>0$.

In what follows, we assume that the function $\gamma(x,t,u)$ is of constant sign for $(x,t,u)\in\overline{D}=\overline{Q}\times[-M_0,M_0]$ (where $M_0\ge\max_{(x,t)\in\overline{Q}}|u|$, M_0 is the constant from the maximum principle for boundary value problem (1)–(3)). In order to ensure the parabolic form of equation (1) the sought coefficient $\rho(x,t)$ must satisfy some requirements depending on the sign of $\gamma(x,t,u)$ for $(x,t,u)\in\overline{D}$. These requirements have the form

$$0<\rho^0_{\min}<\rho(x,t)\le\max_{0\le x\le l}\rho^0(x)+T\max_{(x,t,u)\in\overline{D}}\gamma(x,t,u) \text{ for } \gamma(x,t,u)>0, \tag{5}$$

$$0<\rho^0_{\min}-T\max_{(x,t,u)\in\overline{D}}|\gamma(x,t,u)|\le\rho(x,t)\le\max_{0\le x\le l}\rho^0(x) \text{ for } \gamma(x,t,u)\le 0. \tag{6}$$

If $\gamma(x,t,u) \leq 0$ in the domain $\overline{D}$, then condition(6) leads to the restriction to the time interval $[0,T]$, where the solution $\{u(x,t),\rho(x,t)\}$ of system (1)–(4) is sought: $0 < T < \rho^0_{\min}(\max_{(x,t,u)\in\overline{D}} |\gamma(x,t,u)|)^{-1}$.

The represented statement involving the quasilinear parabolic equation is especially important in the mathematical modeling of high temperature processes since it allows one to take into account the dependence of thermophysical characteristics upon the temperature.

2.2. Conditions of unique solvability in the Hölder spaces. The result for finding a smooth solution $\{u(x,t),\rho(x,t)\}$ of system (1)–(4) is given in the following theorem.

Theorem 2.1. *Let the following conditions be satisfied.*

1. *For* $(x,t) \in \overline{Q}$ *and any* u, $|u| < \infty$, *the input data of boundary value problem* (1)–(3) *are uniformly bounded functions of their arguments, where the coefficient* $a(x,t,u)$ *— together with the derivatives* $a_x(x,t,u)$ *and* $a_u(x,t,u)$, *moreover,* $0 < a_{\min} \leq a(x,t,u) \leq a_{\max}$, $0 < c_{\min} \leq c(x,t,u) \leq c_{\max}$.

2. *For* $(x,t,u) \in \overline{D} = \overline{Q} \times [-M_0, M_0]$ *the functions* $a(x,t,u)$, $a_x(x,t,u)$, $a_u(x,t,u)$, $b(x,t,u)$, *and* $d(x,t,u)$ *have the uniformly bounded derivatives with respect to* u *and Hölder continuous in* x *and* t *with the corresponding exponents* λ *and* $\lambda/2$; *moreover, the functions* $c(x,t,u)$ *and* $f(x,t)$ *belong to* $H^{1,\lambda/2,1}(\overline{D})$ *and* $H^{\lambda,\lambda/2}(\overline{Q})$, *respectively.*

3. *The functions* $w(t)$ *and* $v(t)$ *belong to* $H^{1+\lambda/2}[0,T]$, *the functions* $\varphi(x)$ *and* $\rho^0(x)$ *are in* $H^{2+\lambda}[0,l]$ *and* $C^1[0,l]$, *respectively,* $0 < \rho^0_{\min} \leq \rho^0(x) \leq \rho^0_{\max}$, $\rho^0_{\min}$ *and* $\rho^0_{\max}$ *are positive constants, and the following matching conditions hold:*

$$c(x,0,\varphi)\rho^0(x)w_t - L\varphi|_{x=0,t=0} = f(x,0)|_{x=0},$$

$$c(x,0,\varphi)\rho^0(x)v_t - L\varphi|_{x=l,t=0} = f(x,0)|_{x=l}.$$

4. *The function* $\gamma(x,t,u)$ *in condition* (4) *is of constant sign for* $(x,t,u) \in \overline{D}$ *and belongs to* $H^{1,\lambda/2,1}(\overline{D})$.

Then there exists a unique solution $\{u(x,t),\rho(x,t)\}$ *of nonlinear system* (1)–(4) *which has properties*

$$u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q}), \quad |u(x,t)|^{2+\lambda,1+\lambda/2}_{\overline{Q}} \leq M, \quad M = \text{const} > 0,$$

$$\rho(x,t) \in C(\overline{Q}), \quad \rho_x(x,t) \in C(\overline{Q}), \quad \rho_t(x,t) \in H^{\lambda,\lambda/2}(\overline{Q})$$

and satisfies restrictions (5), (6) *depending on the sign of the function* $\gamma(x,t,u)$.

In order to prove Theorem 2.1 and to establish the existence of a smooth solution of system (1)–(4) we approximate this system using the discretization procedure of the Rothe method on the uniform grid $\overline{\omega}_\tau = \{t_n\} \in [0,T]$ with time-step $\tau = TN^{-1}$:

$$c_n\rho_n u_{n\bar{t}} - (a_n u_{nx})_x + b_n u_{nx} + d_n u_n = f_n, \quad (x,t_n) \in Q_\tau = \{0 < x < l\} \times \omega_\tau, \tag{7}$$

$$u_n|_{x=0} = w_n, \quad u_n|_{x=l} = v_n, \quad 0 < t_n \leq T, \tag{8}$$

$$u_0(x) = \varphi(x), \quad 0 \leq x \leq l, \tag{9}$$

$$\rho_{n\bar{t}} = \gamma_{n-1}, \ (x,t_n) \in Q_\tau, \quad \rho_n(x)|_{n=0} = \rho^0(x), \quad 0 \leq x \leq l. \tag{10}$$

The approximating system can be constructed as follows. Find $\{u_n(x),\rho_n(x)\}$ — approximate values of the functions $u(x,t)$ and $\rho(x,t)$ for $t = t_n$ — satisfying conditions (7)–(10) in which a_n, b_n, c_n, and

d_n are the values of the corresponding coefficients at the point (x, t_n, u_n); $f_n = f(x, t_n)$, $w_n = w(t_n)$, $v_n = v(t_n)$, and $\gamma_{n-1} = \gamma(x, t_{n-1}, u_{n-1})$. In system (7)–(10) the following notations are also used: $u_{n\bar{t}} = (u_n(x) - u_{n-1}(x))\tau^{-1}$, $u_{nx} = du_n(x)/dx$, $\rho_{n\bar{t}} = (\rho_n(x) - \rho_{n-1}(x))\tau^{-1}$.

The proof of solvability of system (1)–(4) by the Rothe method involves several stages.

Stage 1. Investigation of differential-difference boundary value problem (7)–(9) in the difference-continuous Hölder space $H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau)$ under the assumption that $\rho_n(x)$ is the known function. The aim of this stage is to prove unique solvability of problem (7)–(9) and to drive the corresponding *a priori* estimates for the solution $u_n(x)$ (independent of x, τ, n).

Stage 2. The proof of the existence and uniqueness of a solution $\{u_n(x), \rho_n(x)\}$ to differential-difference system (7)–(10) in the corresponding function spaces by using the results of stage 1.

Stage 3. The passage to the limit as time-step τ goes to 0 (i.e., $n \to \infty$) in conditions (7)–(10) by using the compactness of the set $\{u_n(x), \rho_n(x)\}$ thanks to the estimates obtained at stage 2. The aim of this last stage is to show that original system (1)–(4) has at least one solution in the corresponding Hölder spaces.

2.3. A priori estimates in the difference-continuous Hölder spaces. Passing to these stages we give the proof in details only if the justification of the Rothe method must take into account specific properties of system (1)–(4). Otherwise, we only sketch the proof referring to the known results.

The conditions of unique solvability of problem (7)–(9) in $H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau)$ are formulated by the next lemma under the assumption that the coefficient $\rho_n(x)$ in differential-difference equation (7) is a given function continuous in x together with the derivative $\rho_{nx}(x)$ on the domain $\overline{Q}_\tau$ and satisfying the Hölder condition as a function of t_n with the exponent $\lambda/2$.

Lemma 2.1. *Assume that the conditions* 1–3 *of Theorem* 2.1 *hold and let $\rho_n(x)$ be a coefficient with the above mentioned properties. Then differential-difference boundary value problem* (7)–(9) *has a unique solution $u_n(x)$ in the domain $\overline{Q}_\tau$ (for any sufficiently small time-step τ of the grid $\overline{\omega}_\tau$) and the following estimates are valid*

$$\max_{(x,t_n)\in\overline{Q}_\tau} |u_n(x)| \le M_0, \qquad \max_{(x,t_n)\in\overline{Q}_\tau} |u_{nx}(x)| \le M_1,$$

$$|\hat{u}(x)|^{\lambda,\lambda/2}_{\overline{Q}_\tau} \le M_2, \quad |\hat{u}_x(x)|^{\lambda,\lambda/2}_{\overline{Q}_\tau} \le M_3, \quad |\hat{u}(x)|^{2+\lambda,1+\lambda/2}_{\overline{Q}_\tau} \le M_4, \tag{11}$$

where $M_i > 0$ $(i = \overline{0,4})$ are positive constants independent of x, τ, and n.

The conclusion of Lemma 2.1 is based on results of Theorem 4.3.3 in [3] about the unique solvability of the differential-difference boundary value problems of the first kind in the Hölder class $H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau)$. For problem (7)–(9) the constant M_0 from the maximum principle has the form

$$M_0 = \left\{c_{\min}^{-1}\rho_{\min}^{-1} f_{\max} T + \max(w_{\max}, v_{\max}, \varphi_{\max})\right\} \exp(K_1 T),$$

$$K_1 \ge (1+\varepsilon) d_{\max} c_{\min}^{-1}\rho_{\min}^{-1}, \quad \varepsilon > 0 \text{ is arbitrary}, \quad \tau \le \tau_0 = \varepsilon K_1^{-1}. \tag{12}$$

Passing to stage 2 we consider system (7)–(10) in order to find $\{u_n(x), \rho_n(x)\}$. The values of $\rho_n(x)$ are beforehand unknown and are simultaneously determined with $u_n(x)$. This requires additional reasonings for proving the solvability of system (7)–(10).

Lemma 2.2. *Assume that the input data of system* (1)–(4) *satisfy the hypotheses of Theorem* 2.1*. Then in the domain $\overline{Q}_\tau$ for any time-step $\tau \le \tau_0$ ($\tau_0 > 0$ is the constant defined by estimate* (12)*) there exists a unique solution $\{u_n(x), \rho_n(x)\}$ of differential-difference system* (7)–(10) *having the properties*

$$u_n(x) \in H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau), \quad |\hat{u}(x)|^{2+\lambda,1+\lambda/2}_{Q_\tau} \le M_4,$$

$$0<\rho^0_{\min}<\rho_n(x)\le\rho_{\max},\ \ \rho_{\max}=\rho^0_{\max}+T\max_{(x,t,u)\in\overline{D}}\gamma(x,t,u)\ \mathit{for}\ \gamma(x,t,u)>0,$$
$$0<\rho^0_{\min}-T\max_{(x,t,u)\in\overline{D}}|\gamma(x,t,u)|\le\rho_n(x)\le\rho^0_{\max}\ \mathit{for}\ \gamma(x,t,u)\le 0, \tag{13}$$

$$\max_{(x,t_n)\in\overline{Q}_\tau}|\rho_{nx}(x)|+\max_{(x,t_n)\in\overline{Q}_\tau}|\rho_{n\bar{t}}(x)|+\max_{(x,t_n)\in\overline{Q}_\tau}|\rho_{nx\bar{t}}(x)|+\langle\hat{\rho}_{\bar{t}}(x)\rangle^{\lambda/2}_{t,\overline{Q}_\tau}\le M_5, \tag{14}$$

where M_5 is a positive constant independent of x, τ, and n; its value depends, in particular, on $\max_{0\le x\le l}|\rho^0_x(x)|$, $\max_{(x,t,u)\in\overline{D}}(|\gamma_x(x,t,u)|,|\gamma_u(x,t,u)|)$, and on the values of the constants M_1 and M_2.

Proof. To prove Lemma 2.2 we start with the initial conditions for $t_0=0$ and assume that for each of the time layers t_j $(j=\overline{1,n-1})$ the solutions $\{u_j(x),\rho_j(x)\}$ are found and the corresponding estimates are established. The conditions of Theorem 2.1 concerning the functions $\gamma(x,t,u)$ and $\rho^0(x)$ allow one to conclude that for $0\le x\le l$, $t=t_n$ the following inequalities are valid

$$|\rho_{n\bar{t}}(x)|\le\max_{(x,t,u)\in\overline{D}}|\gamma(x,t,u)|,$$

$$|\rho_{nx\bar{t}}(x)|\le\max_{(x,t,u)\in\overline{D}}|\gamma_x(x,t,u)|+\max_{(x,t,u)\in\overline{D}}|\gamma_u(x,t,u)|\max_{(x,t_{n-1})\in\overline{Q}_\tau}|u_{n-1x}(x)|,$$

$$\langle\hat{\rho}_{\bar{t}}(x)\rangle^{\lambda/2}_{t,\overline{Q}_\tau}\le\langle\hat{\gamma}(x,t_{n-1},u_{n-1})\rangle^{\lambda/2}_{t,\overline{D}_\tau}+\max_{(x,t,u)\in\overline{D}_\tau}|\gamma_u(x,t_{n-1},u_{n-1})|\,\langle\hat{u}_{n-1}(x)\rangle^{\lambda/2}_{t,\overline{Q}_\tau}.$$

From here it follows that

$$|\rho_{nx\bar{t}}(x)|\le\max_{(x,t,u)\in\overline{D}}|\gamma_x(x,t,u)|+\max_{(x,t,u)\in\overline{D}}|\gamma_u(x,t,u)|\,M_1,$$

$$\langle\hat{\rho}_{\bar{t}}(x)\rangle^{\lambda/2}_{t,\overline{Q}_\tau}\le\langle\gamma\rangle^{\lambda/2}_{t,\overline{D}}+\max_{(x,t,u)\in\overline{D}}|\gamma_u(x,t,u)|\,M_2.$$

Moreover, from (10) it is not difficult to obtain that

$$\rho_n(x)=\rho_{n-1}(x)+\tau\gamma(x,t_{n-1},u_{n-1})=\rho^0(x)+\sum_{j=0}^{n-1}\tau\gamma(x,t_j,u_j). \tag{15}$$

Hence, depending on the sign of the function $\gamma(x,t,u)$, the following inequalities are valid

$$0<\rho^0_{\min}<\rho_n(x)\le\rho^0_{\max}+t_{n-1}\gamma_{\max}\ \text{for}\ \gamma(x,t,u)>0,\ \ (x,t,u)\in\overline{D},$$

$$0<\rho^0_{\min}-t_{n-1}\gamma_{\max}\le\rho_n(x)\le\rho^0_{\max}\ \text{for}\ \gamma(x,t,u)\le 0,\ \ (x,t,u)\in\overline{D},$$

where $\gamma_{\max}=\max_{(x,t,u)\in\overline{D}}|\gamma(x,t,u)|$. The required estimates (13) for $\rho_n(x)$ are an easy corollary of these inequalities. Next we note that by (15) that the following representation holds

$$\rho_{nx}(x)=\rho^0_x(x)+\sum_{j=0}^{n-1}\tau\{\gamma_x(x,t_j,u_j)+\gamma_u(x,t_j,u_j)\,u_{jx}(x)\},$$

which leads to the bound

$$|\rho_{nx}(x)|\le\max_{0\le x\le l}|\rho^0_x(x)|+t_{n-1}\big\{\max_{(x,t,u)\in\overline{D}}|\gamma_x(x,t,u)|+\max_{(x,t,u)\in\overline{D}}|\gamma_u(x,t,u)|\,M_1\big\}.$$

Thus, for $t = t_n$ the bounds of $|\rho_{nx}(x)|$, $|\rho_{n\bar{t}}(x)|$, $|\rho_{nx\bar{t}}(x)|$, and $\langle \hat{\rho}_{\bar{t}}(x)\rangle^{\lambda/2}_{t,\overline{Q}_\tau}$ are obtained. This allows one to prove estimate (14) since we assume that the corresponding estimates for t_j $(j = \overline{1, n-1})$ are already known.

As a result of (13), (14) the grid-continuous function $\rho_n(x)$, which is determined from (10) by using the given values of $\rho_{n-1}(x)$ and $u_{n-1}(x)$, satisfies the conditions of Lemma 2.1. This means that the differential-difference boundary value problem of the first kind (7)–(9) with such a coefficient $\rho_n(x)$ has a unique solution $u_n(x)$ in $H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau)$ for which bound (11) holds. Thus Lemma 2.2 is proved.

Passing to stage 3 we note that uniform estimates (11), (13), and (14) (independent of x, τ, and n) mean the compactness of the set $\{u_n(x), \rho_n(x)\}$ in the corresponding spaces. By taking the limit as τ goes to 0 (i.e., as $n \to \infty$) in equations (7)–(10), we can show in a standard way that original problem (1)–(4) has at least one solution $\{u(x,t), \rho(x,t)\}$ such that $u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q})$, $\rho(x,t) \in C(\overline{Q})$, $\rho_t(x,t) \in H^{\lambda,\lambda/2}(\overline{Q})$. Moreover, estimates (13) allow one to establish that $\rho(x,t)$ satisfies inequalities (5) and (6) depending on the sign of $\gamma(x,t,u)$ in condition (4). Next we note thanks to the supposed smoothness of the functions $\gamma(x,t,u)$ and $\rho^0(x)$ that $\rho(x,t)$ has the derivative $\rho_x(x,t)$ continuous everywhere on the closed set $\overline{Q}$. Indeed, from (4) it follows that for $0 \le x \le l$, $0 \le t \le T$

$$\rho(x,t) = \int_0^t \gamma(x,\tau,u(x,\tau))\, d\tau + \rho^0(x), \tag{16}$$

$$\rho_x(x,t) = \int_0^t \{\gamma_x(x,\tau,u(x,\tau)) + \gamma_u(x,\tau,u(x,\tau))u_x(x,\tau)\}\, d\tau + \rho^0_x(x).$$

From here it is easily seen that $\rho_x(x,t) \in C(\overline{Q})$ since $u_x(x,t) \in C(\overline{Q})$, $\gamma(x,t,u) \in H^{1,\lambda/2,1}(\overline{D})$, and $\rho^0(x) \in C^1[0,l]$.

Thus the proof of the solvability in the Hölder spaces of nonlinear boundary value problem (1)–(4) by the Rothe method is completed.

2.4. Proof of the uniqueness of the solution $\{u(x,t), \rho(x,t)\}$. In order to complete the proof of Theorem 2.1, it remains to show that the solution of problem (1)–(4) is unique in the class of smooth functions

$$\sup_{(x,t)\in\overline{Q}} |u, u_x, u_{xx}, u_t| < \infty, \qquad \sup_{(x,t)\in\overline{Q}} |\rho, \rho_x, \rho_t| < \infty.$$

Assume that for $t \in [0, t^0]$, $0 \le t^0 < T$, the uniqueness is already proved. Let us show the uniqueness result for $t \in [t^0, t^0 + \Delta t]$, where $\Delta t > 0$ is a sufficiently small but bounded time interval, that allows us to exhaust all the segment $[0,T]$ by a finite number of steps. We will use a contradiction argument. Assume to the contrary that for $t \in [t^0, t^0 + \Delta t]$ there exist two solutions of system (1)–(4) $\{u(x,t), \rho(x,t)\}$ and $\{\overline{u}(x,t), \overline{\rho}(x,t)\}$. By (16) expressions for $\rho(x,t)$ and $\overline{\rho}(x,t)$ have the form

$$\rho(x,t) = \int_{t^0}^t \gamma(x,\tau,u(x,\tau))\, d\tau + \rho(x,t^0), \quad \overline{\rho}(x,t) = \int_{t^0}^t \gamma(x,\tau,\overline{u}(x,\tau))\, d\tau + \overline{\rho}(x,t^0).$$

By taking into account that $\rho(x,t^0) = \overline{\rho}(x,t^0)$, the differences

$$\eta(x,t) = u(x,t) - \overline{u}(x,t), \quad \zeta(x,t) = \rho(x,t) - \overline{\rho}(x,t)$$

satisfy the following estimate in the domain $\overline{Q}_{t^0} = \{0 \le x \le l, t^0 \le t \le t^0 + \Delta t\}$

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)| \le \Delta t \max_{(x,t,u)\in\overline{D}} |\gamma_u(x,t,u)| \max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)|. \tag{17}$$

Moreover, due to (1)–(3) $\eta(x,t)$ and $\zeta(x,t)$ satisfy the relations

$$c(x,t,u)\rho(x,t)\eta_t - (a(x,t,u)\eta_x)_x + \mathcal{A}_0\eta_x + \mathcal{A}_1\eta = c(x,t,\overline{u})\overline{u}_t\zeta(x,t), \quad (x,t) \in Q_{t^0},$$

$$\eta|_{x=0} = 0, \quad \eta|_{x=l} = 0, \quad t^0 < t \leq t^0 + \Delta t,$$

$$\eta(x,t^0) = 0, \quad 0 \leq x \leq l,$$

where the coefficients $\mathcal{A}_0$ and $\mathcal{A}_1$ depend in the corresponding way on the derivatives a_u, a_{xu}, a_{uu}, b_u, c_u, and d_u at the point $(x,t,\sigma u + (1-\sigma)\overline{u})$ $(0 < \sigma < 1)$. Moreover, $\mathcal{A}_0$ and $\mathcal{A}_1$ depend on $u(x,t)$, $\rho(x,t)$, and the derivatives $u_x(x,t)$, $u_{xx}(x,t)$, and $u_t(x,t)$.

All the input data of this linear boundary value problem of the first kind are uniformly bounded in the domain $\overline{Q}_{t^0}$ as functions of (x,t). This allows one to apply the maximum principle that leads to the following estimate

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)| \leq K_2\Delta t \max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)|, \quad K_2 = \text{const} > 0.$$

From here by taking into account (17) we obtain

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)| \leq (\Delta t)^2 K_2 \max_{(x,t,u)\in\overline{D}} |\gamma_u(x,t,u)| \max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)|.$$

Choosing then $\Delta t > 0$ such that

$$(\Delta t)^2 K_2 \max_{(x,t,u)\in\overline{D}} |\gamma_u(x,t,u)| \leq 1 - \mu, \quad 0 < \mu < 1,$$

we deduce the following inequality

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)| \leq (1-\mu) \max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)|,$$

hence $\max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)| = 0$. Due to (17) from here it is easily seen that $\max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)| = 0$. Thus, the uniqueness result is completely proved for $t \in [t^0, t^0 + \Delta t]$.

By repeating the analogous arguments for $t \in [t^1, t^2]$ $(t^1 = t^0 + \Delta t,\ t^2 = t^1 + \Delta t)$, $t \in [t^2, t^3]$, etc., up to the final time T, we drive the uniqueness result for problem (1)–(4) on all the segment $[0,T]$.

Thus, there exists a unique solution $\{u(x,t), \rho(x,t)\}$ of nonlinear system (1)–(4) in the class of smooth functions. Theorem 2.1 is completely proved.

2.5. Error estimates of the Rothe method. Our next aim is to show that the Rothe method is applicable for construction of approximate solutions of the considered nonlinear system. It is required to estimate the differences

$$\omega_n(x) = u_n(x) - u(x,t_n), \quad \xi_n(x) = \rho_n(x) - \rho(x,t_n),$$

where $\{u(x,t_n), \rho(x,t_n)\}$ solves original problem (1)–(4) for $t = t_n$, $\{u_n(x), \rho_n(x)\}$ solves approximating system (7)–(10).

Theorem 2.2. *Assume that the input data satisfy the conditions of Theorem* 2.1. *Then for any sufficiently small time-step τ of the grid $\overline{\omega}_\tau$ the following error estimates for the Rothe method hold*

$$\max_{(x,t_n)\in\overline{Q}_\tau} |\omega_n(x)| \leq K_3(\Psi + \psi), \quad \max_{(x,t_n)\in\overline{Q}_\tau} |\xi_n(x)| \leq K_4(\Psi + \psi), \tag{18}$$

where $\Psi = \max_{(x,t_n)\in\overline{Q}_\tau} \Psi_n(x)$, $\psi = \max_{(x,t_n)\in\overline{Q}_\tau} \psi_n(x)$, $\Psi_n(x)$ is the discretization error for differential-difference boundary value problem (7)–(9) *and $\psi_n(x)$ is the discretization error for equation* (10), *K_3 and K_4 are positive constants independent of x, t, τ, and n.*

The proof repeats — with the appropriate modification — the above proof of the uniqueness result in Theorem 2.1. We only note that estimates (18) are shown step by step for the bounded time intervals $[0, t_{n_0}]$, $[t_{n_0}, t_{n_1}]$, $[t_{n_1}, t_{n_2}]$, etc., up to the final time $t_N = T$. Existence of such estimates allows one to apply the Rothe method for approximate solving nonlinear problem (1)–(4) with the unknown coefficient at the time derivative. The solution $\{u(x,t), \rho(x,t)\}$ can be obtained as the limit of the solution $\{u_n(x), \rho_n(x)\}$ of approximating system (7)–(10) as the time-step τ of the grid $\overline{\omega}_\tau$ goes to 0.

3 Investigation of the nonlinear problem with the unknown boundary function in system (1)–(4)

3.1. The boundary inverse problem with final observation. Assume that in system (1)–(4) the function $v(t)$ in the boundary condition at $x = l$ is unknown but the additional information is given in the form of a final observation for the coefficient $\rho(x,t)$

$$\rho(x,T) = g(x), \quad g(x) > 0, \quad 0 \le x \le l. \tag{19}$$

Then the following problem arises that is inverse to the statement of system (1)–(4): it is required to find the functions $u(x,t)$, $\rho(x,t)$ in the domain $\overline{Q}$ and the boundary function $v(t)$ for $0 \le t \le T$ that satisfy relations (1)–(4) and final condition (19), where all the other input data are given. The considered inverse problem belongs to a class of ill-posed boundary inverse problems for parabolic equations with final observation. However, usual statements of such ones are related to parabolic equations with the given coefficients. The essential distinction of the considered inverse problem is a requirement to find a boundary function for parabolic equation (1) with the unknown coefficient $\rho(x,t)$.

For this inverse problem it is important to choose the appropriate function spaces for the input data and the solution: if they are not chosen properly, the exact solution may not exist. Our choice relies on the faithful differential relations in the Hölder spaces established in Theorem 2.1 between the input data and the solution $\{u(x,t), \rho(x,t)\}$ of system (1)–(4). By Theorem 2.1 this nonlinear system has the unique solution $\{u(x,t), \rho(x,t)\}$ in the corresponding spaces for any boundary function $v(t)$ that belongs to the class $H^{1+\lambda/2}[0,T]$ and satisfies the matching condition at $t = 0$:

$$c(x,0,\varphi)\rho^0(x)v_t - L\varphi|_{x=l,t=0} = f(x,0)|_{x=l}. \tag{20}$$

Taking this into account we define the solution of the considered inverse problem as a collection of the functions $\{u(x,t), \rho(x,t), v(t)\}$ having the properties

$$u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q}),\ v(t) \in H^{1+\lambda/2}[0,T],$$

$$\rho(x,t) \in C(\overline{Q}),\ \rho_x(x,t) \in C(\overline{Q}),\ \rho_t(x,t) \in H^{\lambda,\lambda/2}(\overline{Q})$$

and satisfying relationships (1)–(4), (19), and (20) in the usual sense.

We represent this inverse problem by the operator equation

$$Av = g, \quad v \in V \subset L_2[0,T], \quad g \in G \subset L_2[0,l], \tag{21}$$

where $A : V \to G$ is a nonlinear operator that maps each element $v \in V$ to $\rho(x,t)|_{t=T}$, here $\{u(x,t), \rho(x,t)\}$ is the solution of system (1)–(4) corresponding to this element. An exact solution of operator equation (21) is an element $v^0 \in V$ such that the corresponding coefficient $\rho(x,t;v^0)$ coincides at $t = T$ with the given element $g \in G$.

Possibility to define the operator A for each $v \in V$ and to realize the insertion $AV \subset G$ is ensured by the corresponding choice of the sets V and G based on Theorem 2.1:

$$V = \left\{ v(t) \in W_2^2[0,T],\ c(x,0,\varphi)\rho^0(x)v_t - L\varphi|_{x=l,t=0} = f(x,0)|_{x=l} \right\},$$

$$V \subset H^{1+\lambda/2}[0,T],\ G = \left\{ w(x) \in C^1[0,l], w(x) > 0, x \in [0,l] \right\}. \tag{22}$$

Operator equation (21) is equivalent to the minimization problem for the residual functional on the chosen set of boundary functions

$$\inf_{v \in V} J_g(v), \quad J_g(v) = \|Av - g\|_{L_2[0,l]}.$$

Taking this into account, below we consider the regularization variational method for obtaining approximate solutions of our boundary inverse problem in the chosen spaces.

3.2. Justification of the variational quasisolution method. One of the efficient methods for solving ill-posed inverse problems is the variational quasisolution method [5, 6]. Carrying out the corresponding modification, we justify its applicability for construction of stable approximations in the class of smooth functions.

Namely, to regularize the ill-posed minimization problem for the functional $J_g(v)$ we use the quasisolution method for the system of extending sets

$$V_R = \{v \in V,\ \|v\|_{W_2^2[0,T]} \le R\}, \quad R = \text{const} > 0,$$

compact in the metric of the Hölder space $H^{1+\lambda/2}[0,T]$ $(0 < \lambda < 1)$ by the corresponding embedding theorem [10].

We call the set

$$V_R^* = \{v_R \in V_R,\ J_g(v_R) = \inf_{v \in V_R} J_g(v)\} \tag{23}$$

a quasisolution of operator equation (21) on the compact set V_R.

Theorem 3.1. *For any fixed $R > 0$ the minimization problem for the residual functional $J_g(v)$ on V_R is well-posed, namely, the set V_R^* is not empty and for any minimizing sequence $\{v^n\} \subset V_R$ the following relation holds*

$$\inf_{v_R \in V_R^*} |v^n - v_R|_{[0,T]}^{1+\lambda/2} \to 0 \text{ for } n \to \infty.$$

The well-known Weierstrass theorem implies the proof of this theorem as a result of the compactness of the set V_R in $H^{1+\lambda/2}[0,T]$ $(0 < \lambda < 1)$ and the following property of $J_g(v)$.

Theorem 3.2. *Under the assumptions of Theorem 2.1 for the input data the residual functional $J_g(v)$ is continuous in $H^{1+\lambda/2}[0,T]$ $(0 < \lambda < 1)$ on the set V_R and is weakly continuous in $W_2^2[0,T]$ on the sets V_R and V.*

Proof. Let $\{v^n(t)\} \subset V_R$ be a sequence convergent in $H^{1+\lambda/2}[0,T]$ to a point $v(t) \in V_R$:

$$|v^n - v|_{[0,T]}^{1+\lambda/2} \to 0 \text{ for } n \to \infty. \tag{24}$$

Denote

$$\Delta v(t) = v^n(t) - v(t),\ \Delta u(x,t) = u^n(x,t) - u(x,t),\ \Delta\rho(x,t) = \rho^n(x,t) - \rho(x,t),$$

where $\{u^n(x,t),\rho^n(x,t)\}$ and $\{u(x,t),\rho(x,t)\}$ are the solutions of system (1)–(4) corresponding to the boundary functions $v^n(t)$ and $v(t)$. It is obvious that

$$|J_g(v^n)-J_g(v)| = \big|\|\rho^n(x,T)-g(x)\|_{L_2[0,l]} - \|\rho(x,T)-g(x)\|_{L_2[0,l]}\big| \le \|\Delta\rho(x,T)\|_{L_2[0,l]}. \tag{25}$$

We show that in the domain $\overline{Q}$ the following estimates hold

$$\max_{(x,t)\in\overline{Q}} |\Delta u(x,t)| \le K_5 \max_{0\le t\le T} |\Delta v(t)|, \quad K_5 = \text{const} > 0,$$

$$\max_{(x,t)\in\overline{Q}} |\Delta \rho(x,t)| \le K_6 \max_{0\le t\le T} |\Delta v(t)|, \quad K_6 = \text{const} > 0. \tag{26}$$

Assume that for $t\in[0,t^0]$, $0\le t^0 < T$, these estimates are already established:

$$\max_{0\le x\le l, 0\le t\le t^0} |\Delta u(x,t)| \le K_5 \max_{0\le t\le t^0} |\Delta v(t)|,$$

$$\max_{0\le x\le l, 0\le t\le t^0} |\Delta \rho(x,t)| \le K_6 \max_{0\le t\le t^0} |\Delta v(t)|. \tag{27}$$

We prove that the analogous estimates hold for $t\in[t^0,t^0+\Delta t]$, where $\Delta t>0$ is a sufficiently small but fixed value. In the domain $\overline{Q}_{t^0}=\{0\le x\le l, t^0\le t\le t^0+\Delta t\}$ the differences $\Delta u(x,t)$ and $\Delta\rho(x,t)$ satisfy the relations

$$c(x,t,u)\rho(x,t)\Delta u_t - (a(x,t,u)\Delta u_x)_x + \mathcal{B}_0\Delta u_x + \mathcal{B}_1\Delta u$$
$$= c(x,t,u^n)u^n_t\Delta\rho(x,t), (x,t)\in Q_{t^0},$$
$$\Delta u|_{x=0}=0, \quad \Delta u|_{x=l}=\Delta v(t), \quad t^0<t\le t^0+\Delta t, \tag{28}$$

$$\Delta\rho(x,t) = \int_{t^0}^{t} \gamma_u(x,\tau,u)\Delta u(x,\tau)\,d\tau + \Delta\rho(x,t^0), \quad 0\le x\le l. \tag{29}$$

All the coefficients of the present parabolic equation (including $\mathcal{B}_0$ and $\mathcal{B}_1$) are uniformly bounded in the domain $\overline{Q}_{t^0}$ as functions of (x,t) thanks to the corresponding estimates of Theorem 2.1 for $\{u^n(x,t),\rho^n(x,t)\}$ and $\{u(x,t),\rho(x,t)\}$. Hence relations (28) is a linear boundary value problem of the first kind for $\Delta u(x,t)$ and application of the maximum principle allows one to conclude that

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\Delta u(x,t)| \le K_7\Delta t \max_{(x,t)\in\overline{Q}_{t^0}} |\Delta\rho(x,t)| + K_8 \max\big(\max_{0\le t\le t^0}|\Delta v(t)|, \max_{0\le x\le l}|\Delta u(x,t^0)|\big),$$

where K_7, K_8 are positive constants.

Moreover from (29) it follows that

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\Delta\rho(x,t)| \le \Delta t \max_{(x,t,u)\in\overline{D}} |\gamma_u(x,t,u)| \max_{(x,t)\in\overline{Q}_{t^0}} |\Delta u(x,t)| + \max_{0\le x\le l}|\Delta\rho(x,t^0)|.$$

By taking into account these bounds and (27) and choosing the value of Δt from the condition

$$(\Delta t)^2 K_7 \max_{(x,t,u)\in\overline{D}} |\gamma_u(x,t,u)| < 1,$$

we establish that

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\Delta\rho(x,t)| \le K_6 \max_{0\le t\le T} |\Delta v(t)|.$$

Then from the estimate of the maximum principle for $\Delta u(x,t)$ it follows

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\Delta u(x,t)| \le K_5 \max_{0\le t\le T} |\Delta v(t)|.$$

By repeating the analogous arguments for $t \in [t^1, t^1+\Delta t]$ $(t^1 = t^0+\Delta t)$, $t \in [t^2, t^2+\Delta t]$ $(t^2 = t^1+\Delta t)$, etc., we establish estimates (26) on the whole segment $[0,T]$ for $0 \le x \le l$. This allows us to conclude from (25) that

$$|J_g(v^n) - J_g(v)| \le K_9 \max_{0\le t\le T} |\Delta v(t)|, \quad K_9 = \text{const} > 0.$$

Thus, thanks to (24) we obtain the equality $\lim\limits_{n\to\infty} J_g(v^n) = J_g(v)$, which proves the first claim of Theorem 3.2.

To prove the second we note that for any sequence $\{v^n\} \subset V$ (or V_R) weakly convergent in $W_2^2[0,T]$ to an element $v \in V$ (V_R) the following inequalities are valid [7]

$$\|v^n\|_{W_2^2[0,T]} \le K_{10}, \quad \|v\|_{W_2^2[0,T]} \le K_{10}, \quad K_{10} = \text{const} > 0.$$

From the fact that the embedding operator of $W_2^2[0,T]$ into $H^{1+\lambda/2}[0,T]$ $(0 < \lambda < 1)$ is compact [10] and also from the uniqueness of a weak limit, it follows that the sequence $\{v^n(t)\}$ satisfies (24). By repeating the previous arguments, we obtain the equality $\lim\limits_{n\to\infty} J_g(v^n) = J_g(v)$, which proves the weak continuity of the residual functional $J_g(v)$ in $W_2^2[0,T]$ for $v \in V$ (V_R). □

3.3. Regularized approximate solutions for the boundary inverse problem. In what follows we assume that operator equation (21) has the exact solution $v^0 \in V$ for the given g. Such assumption is natural for inverse problems of identification of boundary regimes. This means that the right-hand side of equation (21) $g \in AV$, where $AV \subset G$ is a transform of the set V in G (see (22)).

Now we pass to construction of stable approximations in the class of smooth functions for the considered boundary inverse problem. At first we note that the exact solution of equation (21) is not necessarily unique (an illustrating example will be shown below). Denote the set of such solutions by V^0:

$$V^0 = \{v^0 \in V, \;\; J_g(v^0) = \inf_{v\in V} J_g(v) = 0\}.$$

From the weak continuity of the residual functional $J_g(v)$ in $W_2^2[0,T]$ for $v \in V$ (see Theorem 3.2) and from the weak closure of the set V in $W_2^2[0,T]$ it follows that the set V^0 is weakly closed in $W_2^2[0,T]$ too. Hence there exist elements $v^0_{\min}$ having the minimal norm in $W_2^2[0,T]$:

$$V^0_{\min} = \{v^0_{\min} \in V^0, \;\; \|v^0_{\min}\|_{W_2^2[0,T]} = R^0\} \ne \emptyset, \quad R^0 = \inf_{v^0\in V^0} \|v^0\|_{W_2^2[0,T]}. \tag{30}$$

If on a compact set V_R for the functional $J_g(v)$ the equality $\inf\limits_{v\in V_R} J_g(v) = 0$ holds, then $V_R \cap V^0 \ne \emptyset$ and the quasisolution V_R^* (see (23)) coincides with $V_R \cap V^0$. Thus, the considered inverse problem is reduced to the minimization problem for the residual functional $J_g(v)$ on the compact set V_R which is well-posed in the sense of Tikhonov [11] by Theorem 3.1.

Otherwise, if on a compact set V_R $\inf\limits_{v\in V_R} J_g(v) > 0$, then we consider the quasisolutions V_R^* on the system of extending compact sets V_R for $0 < R < R^0$. Our aim is to show that each element $v_R \in V_R^*$ converges in $W_2^2[0,T]$ to some element of the set $V^0_{\min}$ for $R \to R^0$.

By using the definition of β-convergence of sets [5, 9] this claim is formulated in the following theorem.

Theorem 3.3. *Assume that the input data satisfy the hypotheses of Theorem 2.1. Then the quasisolution V_R^* defined for any R, $0 < R < R^0$, β-converges to the set $V_{\min}^0$ of the exact solutions with the minimal norm for $R \to R^0$:*

$$V_R^* \xrightarrow{\beta} V_{\min}^0 \ (W_2^2[0,T]). \tag{31}$$

Moreover, for $R \to R^0$

$$U_R^* \xrightarrow{\beta} U_{\min}^0 \left(C(\overline{Q})\right), \quad \Re_R^* \xrightarrow{\beta} \Re_{\min}^0 \left(C(\overline{Q})\right), \tag{32}$$

where $\left\{U_R^*, \Re_R^*\right\}$ and $\left\{U_{\min}^0, \Re_{\min}^0\right\}$ are the sets of solutions of nonlinear problem (1)–(4) corresponding to the sets of the boundary functions $v_R \in V_R^*$ and $v_{\min}^0 \in V_{\min}^0$.

Assume, moreover, that for $(x,t,u) \in \overline{D}$ the derivatives of the coefficients of equation (1) $a_{xu}(x,t,u)$, $a_{uu}(x,t,u)$, $b_u(x,t,u)$, $c_u(x,t,u)$, and $d_u(x,t,u)$ are Hölder continuous in x, t, and u with the corresponding exponents λ, $\lambda/2$, λ; moreover, the derivative with respect to u of the function $\gamma(x,t,u)$ in equation (4) satisfies the Hölder condition as a function of x, t with the exponents λ, $\lambda/2$, the derivative $\gamma_{uu}(x,t,u)$ is uniformly bounded.

Then there holds β-convergence in the Hölder spaces for $R \to R^0$:

$$U_R^* \xrightarrow{\beta} U_{\min}^0 \left(H^{2+\lambda,1+\lambda/2}(\overline{Q})\right), \quad \Re_R^* \xrightarrow{\beta} \Re_{\min}^0 \left(H^{\lambda,\lambda/2}(\overline{Q})\right). \tag{33}$$

Proof. At first we note that for $R = R^0$ the quasisolution V_R^* coincides with $V_{\min}^0$. In fact, for all $v_{R^0} \in V_{R^0}^*$ the inequality $\|v_{R^0}\|_{W_2^2[0,T]} \le R^0$ holds. On the other hand, from the definition of R^0 (see (30)) and from the inclusion $V_{R^0}^* \subseteq V^0$ it follows that the inequality $\|v_{R^0}\|_{W_2^2[0,T]} < R^0$ is impossible. Thus, for all $v_{R^0} \in V_{R^0}^*$ $\|v_{R^0}\|_{W_2^2[0,T]} = R^0$, i.e., $V_{R^0}^*$ coincides with $V_{\min}^0$.

By the definition of β-convergence of sets relation (31) means that

$$\sup_{v_R \in V_R^*} \inf_{v_{R^0} \in V_{R^0}^*} \|v_R - v_{R^0}\|_{W_2^2[0,T]} \to 0 \text{ for } R \to R^0. \tag{34}$$

To prove (34) we note that the function $J_g^*(R) = \inf_{v \in V_R} J_g(v)$ is a continuous and nonincreasing function of R for $0 < R \le R^0$ (see [1]), i.e., $J_g^*(R) \to J_g^*(R^0)$ for $R \to R^0$. Hence the sequence $\{v_R\} \subset V_R$ (where v_R is any element of V_R^*) is a minimizing sequence for the functional $J_g(v)$ on the set V_{R^0}. We can therefore conclude by Theorem 3.1 that

$$\inf_{v_{R^0} \in V_{R^0}^*} |v_R - v_{R^0}|_{[0,T]}^{1+\lambda/2} \to 0 \text{ for } R \to R^0.$$

From here and from the arbitrariness of $v_R \in V_R^*$ it follows that

$$\sup_{v_R \in V_R^*} \inf_{v_{R^0} \in V_{R^0}^*} |v_R - v_{R^0}|_{[0,T]}^{1+\lambda/2} \to 0 \text{ for } R \to R^0,$$

i.e., $V_R^* \xrightarrow{\beta} V_{\min}^0 \left(H^{1+\lambda/2}[0,T]\right)$ since $V_{R^0}^*$ coincides with $V_{\min}^0$.

In order to prove more strong claim (31) we note that the set V_{R^0} is weakly compact in $W_2^2[0,T]$ since by its definition, V_{R^0} is convex, closed, and bounded in $W_2^2[0,T]$. Hence we can find a subsequence $\{v_{R_n}\} \subseteq \{v_R\}$ weakly convergent in $W_2^2[0,T]$ to an element v_{R^0} of the set $V_{R^0}^* \subset V_{R^0}$ (see Theorem 3.2 on the weak continuity in $W_2^2[0,T]$ of the functional $J_g(v)$ on the set V_{R^0}).

Note that this subsequence is also strongly convergent in $W_2^2[0,T]$ to v_{R^0}. In fact, from the weak lower semicontinuity of the norm of the element $v_{R^0} \in V_{R^0}^*$ in the Hilbert space $W_2^2[0,T]$ and because it belongs to the boundary of the set V_{R^0} ($\|v_{R^0}\|_{W_2^2[0,T]} = R^0$) it follows that

$$R^0 = \|v_{R^0}\|_{W_2^2[0,T]} \le \underline{\lim}_{n\to\infty} \|v_{R_n}\|_{W_2^2[0,T]} \le \overline{\lim}_{n\to\infty} \|v_{R_n}\|_{W_2^2[0,T]} \le R^0.$$

Thus $\|v_{R_n}\|_{W_2^2[0,T]} \to \|v_{R^0}\|_{W_2^2[0,T]}$ for $n \to \infty$ and hence $v_{R_n} \to v_{R^0}$ strongly in $W_2^2[0,T]$ [7]. Then the arbitrariness of the element $v_R \in V_R^*$ allows one to conclude that claim (34) is valid. This means β-convergence of the set V_R^* to $V_{\min}^0$ in $W_2^2[0,T]$.

The proof of claim (32) on β-convergence of the sets U_R^* and $\Re_R^*$ in $C(\overline{Q})$ is then obvious consequence of the embedding theorems [10] and estimates (26) for

$$\Delta u(x,t) = u_R(x,t) - u_{\min}^0(x,t), \quad \Delta\rho(x,t) = \rho_R(x,t) - \rho_{\min}^0(x,t),$$

where $\{u_R(x,t), \rho_R(x,t)\}$ and $\{u_{\min}^0(x,t), \rho_{\min}^0(x,t)\}$ are the solutions of nonlinear problem (1)–(4) corresponding to the boundary functions $v_R(t) \in V_R^*$ and $v_{\min}^0(t) \in V_{\min}^0$, and where $\Delta v(t) = v_R(t) - v_{\min}^0(t)$.

In order to prove claim (33) on β-convergence of U_R^* and $\Re_R^*$ in Hölder classes it is required to obtain the corresponding stability estimates of the form

$$|\Delta u(x,t)|_{\overline{Q}}^{2+\lambda,1+\lambda/2} \le K_{11}|\Delta v(t)|_{[0,T]}^{1+\lambda/2}, \quad |\Delta\rho(x,t)|_{\overline{Q}}^{\lambda,\lambda/2} \le K_{12}|\Delta v(t)|_{[0,T]}^{1+\lambda/2}, \tag{35}$$

where K_{11}, K_{12} are positive constants. Such estimates are shown step by step (as in the above proof of estimates (26)) for the bounded time intervals $[t^0, t^0+\Delta t]$, $[t^1, t^1+\Delta t]$ ($t^1 = t^0 + \Delta t$), etc., under assumption that for $t \in [0,t^0]$, $0 \le t^0 < T$, they are already established:

$$\begin{gathered} |\Delta u(x,t)|_{0\le x\le l, 0\le t\le t^0}^{2+\lambda,1+\lambda/2} \le K_{11}|\Delta v(t)|_{[0,t^0]}^{1+\lambda/2}, \\ |\Delta\rho(x,t)|_{0\le x\le l, 0\le t\le t^0}^{\lambda,\lambda/2} \le K_{12}|\Delta v(t)|_{[0,t^0]}^{1+\lambda/2}. \end{gathered} \tag{36}$$

In the domain $\overline{Q}_{t^0} = \{0 \le x \le l, t^0 \le t \le t^0 + \Delta t\}$ the following relations hold

$$\begin{gathered} c(x,t,u_R)\rho_R(x,t)\Delta u_t - (a(x,t,u_R)\Delta u_x)_x + \mathcal{D}_0\Delta u_x + \mathcal{D}_1\Delta u \\ = c(x,t,u_{\min}^0)(u_{\min}^0)_t\Delta\rho(x,t), \quad (x,t) \in Q_{t^0}, \\ \Delta u|_{x=0} = 0, \quad \Delta u|_{x=l} = \Delta v(t), \quad t^0 < t \le t^0 + \Delta t, \end{gathered} \tag{37}$$

$$\Delta\rho(x,t) = \int_{t^0}^{t} \gamma_u(x,\tau,u_R)\Delta u(x,\tau)\,d\tau + \Delta\rho(x,t^0), \quad 0 \le x \le l, \tag{38}$$

where the coefficients $\mathcal{D}_0$ and $\mathcal{D}_1$ depend in the corresponding way on the derivatives a_u, a_{xu}, a_{uu}, b_u, c_u, and d_u at the point $(x,t,\sigma u_R + (1-\sigma)u_{\min}^0)$ $(0 < \sigma < 1)$. Moreover, $\mathcal{D}_0$ and $\mathcal{D}_1$ depend on $\rho_R(x,t)$, $u_R(x,t)$ and the derivatives $u_{Rx}(x,t)$, $u_{Rxx}(x,t)$, and $u_{Rt}(x,t)$. The estimates in the Hölder classes of Theorem 2.1 for $u_R(x,t)$ and $\rho_R(x,t)$ and the requirements of Theorem 3.3 to the input data allow one to conclude that all the coefficients of equation (37) are Hölder continuous in x, t with the exponents λ, $\lambda/2$. Thus, $\Delta u(x,t)$ solves the linear boundary value problem of the first kind in $H^{2+\lambda,1+\lambda/2}(\overline{Q}_{t^0})$ and the corresponding estimate in the domain $\overline{Q}_{t^0}$ (see [8]) holds

$$|\Delta u(x,t)|_{\overline{Q}_{t^0}}^{2+\lambda,1+\lambda/2} \le K_{13}\big(|\Delta\rho(x,t)|_{\overline{Q}_{t^0}}^{\lambda,\lambda/2} + |\Delta v(t)|_{[t^0,t^0+\Delta t]}^{1+\lambda/2} + |\Delta u(x,t^0)|_{[0,l]}^{2+\lambda}\big), \tag{39}$$

where K_{13} is a positive constant.

In order to estimate $|\Delta\rho(x,t)|_{\overline{Q}_{t^0}}^{\lambda,\lambda/2}$ in (39) we note that the definition of the norm in the Hölder space $H^{\lambda,\lambda/2}(\overline{Q}_{t^0})$ means that

$$|\Delta\rho(x,t)|_{\overline{Q}_{t^0}}^{\lambda,\lambda/2} = \max_{(x,t)\in\overline{Q}_{t^0}} |\Delta\rho(x,t)| + \langle\Delta\rho(x,t)\rangle_{x,\overline{Q}_{t^0}}^{\lambda} + \langle\Delta\rho(x,t)\rangle_{t,\overline{Q}_{t^0}}^{\lambda/2}.$$

By using relation (38) we can find that

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\Delta\rho(x,t)| \leq \Delta t \max_{(x,t,u)\in\overline{D}} |\gamma_u(x,t,u)| \max_{(x,t)\in\overline{Q}_{t^0}} |\Delta u(x,t)| + \max_{0\leq x\leq l} |\Delta\rho(x,t^0)|,$$

$$\langle\Delta\rho(x,t)\rangle^{\lambda}_{x,\overline{Q}_{t^0}} \leq \Delta t\big\{\langle\gamma_u\rangle^{\lambda}_{x,\overline{D}} \max_{(x,t)\in\overline{Q}_{t^0}} |\Delta u(x,t)| + \max_{(x,t,u)\in\overline{D}} |\gamma_u(x,t,u)| \langle\Delta u(x,t)\rangle^{\lambda}_{x,\overline{Q}_{t^0}}$$

$$+ \max_{(x,t,u)\in\overline{D}} |\gamma_{uu}(x,t,u)| \langle u_R(x,t)\rangle^{\lambda}_{x,\overline{Q}_{t^0}} \max_{(x,t)\in\overline{Q}_{t^0}} |\Delta u(x,t)|\big\} + \langle\Delta\rho(x,t^0)\rangle^{\lambda}_{x,[0,l]},$$

$$\langle\Delta\rho(x,t)\rangle^{\lambda/2}_{t,\overline{Q}_{t^0}} \leq \Delta t^{1-\lambda/2} \max_{(x,t,u)\in\overline{D}} |\gamma_u(x,t,u)| \max_{(x,t)\in\overline{Q}_{t^0}} |\Delta u(x,t)|.$$

Note that

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\Delta u(x,t)| \leq |\Delta u(x,t)|^{2+\lambda,1+\lambda/2}_{\overline{Q}_{t^0}}, \quad \langle\Delta u(x,t)\rangle^{\lambda}_{x,\overline{Q}_{t^0}} \leq |\Delta u(x,t)|^{2+\lambda,1+\lambda/2}_{\overline{Q}_{t^0}}.$$

Hence by (39) the estimates obtained for $|\Delta\rho(x,t)|^{\lambda,\lambda/2}_{\overline{Q}_{t^0}}$ allow one to show that for the sufficiently small but fixed Δt the following inequality holds

$$|\Delta u(x,t)|^{2+\lambda,1+\lambda/2}_{\overline{Q}_{t^0}} \leq K_{14}\big(|\Delta v(t)|^{1+\lambda/2}_{[t^0,t^0+\Delta t]} + |\Delta u(x,t^0)|^{2+\lambda}_{[0,l]} + |\Delta\rho(x,t)|^{\lambda,\lambda/2}_{0\leq x\leq l, 0\leq t\leq t^0}\big),$$

where K_{14} is a positive constant. By our assumption, for $0 \leq x \leq l, 0 \leq t \leq t^0$ estimates (36) are already derived. Hence the above inequality means that analoguos estimates are valid in the domain $\overline{Q}_{t^0}$ including the corresponding estimates for $|\Delta\rho(x,t)|^{\lambda,\lambda/2}_{\overline{Q}_{t^0}}$.

By repeating the similar arguments for the consequent time intervals we can prove estimates (35) for $|\Delta u(x,t)|^{2+\lambda,1+\lambda/2}_{\overline{Q}}$ and $|\Delta\rho(x,t)|^{\lambda,\lambda/2}_{\overline{Q}}$ in the whole domain $\overline{Q}$. The proof of claim (33) is then obvious consequence of estimates (35) and the embedding theorems [10]. □

Remark. By this theorem any element $v_R(t) \in V^*_R$ for $0 < R < R^0$ and the solution $\{u_R(x,t), \rho_R(x,t)\}$ of nonlinear problem (1)–(4) corresponding to this boundary function are approaches in the Hölder spaces to one of the solutions $\{u^0_{\min}(x,t), \rho^0_{\min}(x,t), v^0_{\min}(t)\}$ of the considered boundary inverse problem.

If the set of the exact solutions V^0 consists of the single element v^0, then the claims of Theorem 3.3 mean α-convergence of the corresponding sets for $R \to R^0$:

$$V^*_R \xrightarrow{\alpha} v^0 \ \big(W^2_2[0,T]\big), \quad U^*_R \xrightarrow{\alpha} u^0 \ \big(H^{2+\lambda,1+\lambda/2}(\overline{Q})\big), \quad \Re^*_R \xrightarrow{\beta} \rho^0 \ \big(H^{\lambda,\lambda/2}(\overline{Q})\big),$$

where $\{u^0(x,t), \rho^0(x,t), v^0(t)\}$ is the exact solution of the inverse problem.

In general, without some additional suppositions on the properties of the exact solution of the considered inverse problem it needs not be unique. This is illustrated by the following example.

Example. It is required to find a function $u(x,t)$ in the domain $\overline{Q} = \{0 \leq x \leq 1, 0 \leq t \leq 1\}$, a coefficient $\rho(t)$ and a boundary function $v(t)$ for $0 \leq t \leq 1$ that satisfy the relations

$$\rho(t)u_t - u_{xx} = f(x,t,u), \quad (x,t) \in Q,$$

$$u(x,t)|_{x=0} = 0, \quad u(x,t)|_{x=1} = v(t), \quad 0 < t \leq 1,$$

$$u(x,t)|_{t=0} = x(1-x), \quad 0 \leq x \leq 1,$$

and the additional relation

$$\rho_t(t) = \gamma(x,t,u), \quad 0 < t \leq 1, \quad \rho(t)|_{t=0} = 0.25,$$

with the final condition $\rho(t)|_{t=1} = 1$. Here the functions $f(x,t,u)$ and $\gamma(x,t,u)$ have the form

$$f(x,t,u) = \{2 + 1.5x(1-t) - 1.125x(1-t)^3\}h_1(x,t,u)$$

$$- \{2 + 0.125x(1+t)^3\}h_2(x,t,u),$$

$$\gamma(x,t,u) = 1.5(1-t)h_1(x,t,u) - 0.5(1+t)h_2(x,t,u),$$

where

$$h_1(x,t,u) = \frac{u - x\big(0.75 + 0.25(1+t)^2 - x\big)}{xt(1-t)},$$

$$h_2(x,t,u) = \frac{u - x\big(1.75 - 0.75(1-t)^2 - x\big)}{xt(1-t)}.$$

This inverse problem has two solutions $\{u_1(x,t), \rho_1(t), v_1(t)\}$ and $\{u_2(x,t), \rho_2(t), v_2(t)\}$

$$\begin{cases} u_1(x,t) &= x\{0.75 + 0.25(1+t)^2 - x\}, \\ \rho_1(t) &= 0.25(1+t)^2, \\ v_1(t) &= 0.25t(2+t), \end{cases}$$

$$\begin{cases} u_2(x,t) &= x\{1.75 - 0.75(1-t)^2 - x\}, \\ \rho_2(t) &= 1 - 0.75(1-t)^2, \\ v_2(t) &= 0.75t(2-t). \end{cases}$$

4 Conclusion

The work contains the theoretical studies of new nonlinear parabolic problems with the wide potential applications. Our main aim is to justify such statements in the class of smooth functions for quasilinear parabolic equations with an unknown coefficient in a case of the boundary conditions of the first kind. This justification is an important task since such new statements essentially differ from usual ones. The following results of our analysis can be formulated.

1. Conditions of unique solvability in the Hölder spaces are proved for nonlinear system (1)–(4) which involves a boundary value problem of the first kind for a quasilinear parabolic equation with an unknown coefficient at the time derivative and, moreover, an equation for a time dependence of the sought coefficient. To this end, *a priori* estimates in the corresponding spaces are established for the nonlinear differential-difference system that approximates the original system by the Rothe method. Thanks to these estimates we avoid additional assumptions on the smoothness of the input data (which have usually been imposed by the Rothe method) and determine the faithful character of differential relations for the nonlinear parabolic problem of the considered type. Moreover, these estimates allow one to obtain the error estimates for the Rothe method, i.e., this method provides the approximate solutions for the considered problem.

2. The established results are then used for investigation of the other nonlinear parabolic problem, which is inverse to the statement of system (1)–(4) and consists of determination of a boundary regime at $x = l$ by using the final observation of the sought coefficient $\rho(x,t)$. This inverse problem belongs to a class of ill-posed boundary inverse problems but it has an essential distinction from usual statements since, moreover a boundary function, the coefficient $\rho(x,t)$ must be determined in system (1)–(4). In order to overcome the mentioned difficulty the operator representation of this inverse problem is proposed. This representation is justified by using the results of Section 2 on "natural" function spaces for the input data and the solution in system (1)–(4). For obtaining approximate solutions stable in the chosen spaces the regularization

variational method is developed with application of quasisolutions on the system of extending compact sets. The continuity of the residual functional in the corresponding variational formulation is established by using the estimates in the Hölder spaces for nonlinear system (1)–(4). On the basis of these estimates, results for stability of the regularized approximate solutions are proved in the class of smooth functions.

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References

[1] B.M. Budak, A. Vin'oli, Yu.L. Gaponenko, *A regularization method for a continuous convex functional.* USSR Comput. Math. Math. Phys. 9 (1969), no. 5, 82–95.

[2] N.L. Gol'dman, *Boundary value problems for a quasilinear parabolic equation with an unknown coefficient.* J. Differential Equations 266 (2019), 4925–4952.

[3] N.L. Gol'dman, *Inverse Stefan problems.* Kluwer Academic, Dordrecht, 1997.

[4] A. Friedman, *Partial differential equations of parabolic type.* Prentice Hall, Englewood Cliffs, New York, 1964.

[5] V.K. Ivanov, *On ill-posed problems.* Mat.Sb. 61 (1963), no. 2, 211–223.

[6] V.K. Ivanov, *The solution of operator equations that are not well-posed.* Proc. Steklov Inst. Math. 112 (1971), 241–250.

[7] L.V. Kantorovich, G.P. Akilov, *Functional analysis.* Nauka, Moscow, 1977 (in Russian); Pergamon, New York, 1982.

[8] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva, *Linear and quasilinear equations of parabolic type.* Nauka, Moscow, 1967 (in Russian); Am. Math. Soc., Providence, R.I., 1968.

[9] O.A. Liskovets, *Incorrectly posed problems and stability of quasisolutions.* Sib. Math. J. 10 (1969), no. 2, 266–274.

[10] S.M. Nikol'skii, *Approximation of functions of several variables and embedding theorems.* Nauka, Moscow, 1969 (in Russian); Springer, New York, 1975.

[11] A.N. Tikhonov, *On a solution of ill-posed problems and a regularization method.* Sov. Math. Dokl. 151 (1963), no. 3, 501–504.

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