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SHAVKAT ARIFJANOVICH ALIMOV

(to the 75th birthday)

Shavkat Arifjanovich Alimov was born on March 2, 1945 in the city of Nukus, Uzbekistan. In 1968, he graduated from the Department of Mathematics of Physical Faculty of the M.V. Lomonosov Moscow State University (MSU), receiving a diploma with honors. From 1968 to 1970, he was a post-graduate student in the same department under the supervision of Professor V.A. Il'in. He defended his PhD thesis in 1970. In May 1973, at the age of 28, he defended his doctoral thesis devoted to equations of mathematical physics. In 1973, for research on the spectral theory, he was awarded the highest youth prize of the USSR.

From 1974 to 1984, he worked as a professor in the Department of General Mathematics at the Faculty of Computational Mathematics and Cybernetics. In 1984, Sh.A. Alimov joined the Tashkent State University (TSU) as a professor. From 1985 to 1987 he worked as the Rector of the Samarkand State University, from 1987 to 1990 - the Rector of the TSU, from 1990 to 1992 - the Minister of Higher and Secondary Special Education of the Republic of Uzbekistan. From 1992 to 1994, he headed the Department of Mathematical Physics of the TSU.

After some years of diplomatic work, he continued his academic career as a professor of the Department of Mathematical Physics at the National University of Uzbekistan (NUU). From the first days of the opening of the Tashkent branch of the MSU in 2006, he worked as a professor in the Department of Applied Mathematics. From 2012 to 2017, he headed the Laboratory of Mathematical Modeling of the Malaysian Institute of Microelectronic Systems in Kuala Lumpur. From 2017 to 2019, he worked as a professor at the Department of Differential Equations and Mathematical Physics of the NUU. From 2019 to the present, Sh.A. Alimov is a Scientific Consultant at the Center for Intelligent Software Systems, and an adviser to the Rector of the NUU.

The main scientific activity of Sh.A. Alimov is connected with the spectral theory of partial differential equations and the theory of boundary value problems for equations of mathematical physics. He obtained series of remarkable results in these fields. They cover many important problems of the theory of Schrodinger equations with singular potentials, the theory of boundary control of the heat transfer process, the mathematical problems of peridynamics related to the theory of hypersingular integrals.

In 1984, Sh.A. Alimov was elected a corresponding member and in 2000 an academician of the Academy of Sciences of Uzbekistan. He was awarded several prestigious state prizes.

Sh.A. Alimov has over 150 published scientific and a large number of educational works. Among his pupils there are 10 doctors of sciences and more than 20 candidates of sciences (PhD) working at universities of Uzbekistan, Russia, USA, Finland, and Malaysia.

For about thirty years, Sh.A. Alimov has been actively involved in the reform of mathematical school education.

Sh.A. Alimov meets his 75th birthday in the prime of his life, and the Editorial Board of the Eurasian Mathematical Journal heartily congratulates him on his jubilee and wishes him good health, new successes in scientific and pedagogical activity, family well-being and long years of fruitful life.

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COMPARISON OF MORREY SPACES AND NIKOL'SKII SPACES

V.I. Burenkov, V.S. Guliyev, T.V. Tararykova

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Abstract. We consider two popular function spaces: the Morrey spaces and the Nikol'skii spaces and investigate the relationship between them in the one-dimensional case. In particular, we prove that, under the appropriate assumptions on the numerical parameters, their restrictions to the class of functions f of the form $f(x) = g(|x|)$, where g is a non-negative non-increasing function on $[0, \infty)$, coincide.

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1 Introduction

We shall use the following notation. For a Lebesgue measurable set $G \subset \mathbb{R}^n$ and $0 < p \leq \infty$, $L_p(G)$ is the standard Lebesgue space of all functions f Lebesgue measurable on G for which

$$\|f\|_{L_p(G)} = \left(\int_G |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty$$

if $0 < p < \infty$ and

$$\|f\|_{L_\infty(G)} = \text{ess sup}_{x \in G} |f(x)| < \infty$$

if $p = \infty$. Also, for an open set $G \subset \mathbb{R}^n$, $L_p^{loc}(G)$ is the set of all functions f such that $f \in L_p(K)$ for any compact $K \subset G$. If $G = \mathbb{R}^n$ then, for brevity, we write L_p for $L_p(\mathbb{R}^n)$ and L_p^{loc} for $L_p^{loc}(\mathbb{R}^n)$.

The Morrey spaces M_p^λ, named after C.B. Morrey, were introduced by him in 1938 in [10] and defined as follows. For $\lambda \in \mathbb{R}, 1 \leq p \leq \infty, f \in M_p^\lambda$ if $f \in L_p^{loc}$ and

$$\|f\|_{M_p^\lambda} \equiv \|f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty. \tag{1.1}$$

Here the notation is slightly altered compared with the original definition in [10], namely we write $r^{-\lambda}$ rather than $r^{-\frac{n-\lambda}{p}}$ for the reasons which will be clarified in this paper and explicitly stated in Concluding remark.

If $\lambda = 0$, then clearly

$$M_p^0 = L_p. \tag{1.2}$$

If $\lambda = \frac{n}{p}$, then

$$M_p^{\frac{n}{p}} = L_\infty. \tag{1.3}$$

If $\lambda > \frac{n}{p}$ or $\lambda < 0$, then

$$M_p^\lambda = \Theta,$$

where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. So the admissible range of the parameters is

$$0 < p \le \infty \quad \text{and} \quad 0 \le \lambda \le \frac{n}{p}. \tag{1.4}$$

Discussion of basic properties of the Morrey spaces can be found in [15], [19], [21], [1]. Description of numerous results for various generalizations and variants of the Morrey spaces can be found in survey papers [5–7], [9], [16–18].

Let $\Omega \subset \mathbb{R}^n$ be an open set and, for $\delta > 0$, $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$. The Nikol'skii spaces $H_p^\lambda(\Omega)$ of all functions possessing common smoothness of order λ measured in the L_p metric, named after S.M. Nikol'skii, were introduced by him in 1951 in [11], [12] and defined as follows: $f \in H_p^\lambda(\Omega)$ if $f \in L_p(\Omega)$ and

$$\|f\|_{H_p^\lambda(\Omega)} = \|f\|_{L_p(\Omega)} + \|f\|_{\dot{H}_p^\lambda(\Omega)} < \infty,$$

where

$$\|f\|_{\dot{H}_p^\lambda(\Omega)} = \sup_{h \in \mathbb{R}^n, h \ne 0} |h|^{-\lambda} \|\Delta_h^\sigma f\|_{L_p(\Omega_{\sigma|h|})}$$

and $\Delta_h^\sigma f$ is the difference of f of order $\sigma \in \mathbb{N}$ with step h and $\sigma > \lambda$. (For a wide class of open sets Ω, in particular for $\Omega = \mathbb{R}^n$, for different $\sigma > \lambda$ the definitions are equivalent, see [2], [3], [14].) Detailed exposition of the theory of the Nikol'skii spaces and their generalizations can be found in [2], [14], [20].

We also write $H_p^\lambda \equiv H_p^\lambda(\mathbb{R}^n)$ and say that $f \in (H_p^\lambda)^{loc}$, if $f \in H_p^\lambda(B(0,r))$ for all $r > 0$.

Note that $\|\cdot\|_{M_p^\lambda}$ and $\|\cdot\|_{\dot{H}_p^\lambda}$ have the same behaviour with respect to dilations, that is for any $\varepsilon > 0$ for all admissible values of the parameters p and λ

$$\|f(\varepsilon x)\|_{M_p^\lambda} = \varepsilon^{\lambda - \frac{n}{p}} \|f(x)\|_{M_p^\lambda}, \quad \|f(\varepsilon x)\|_{\dot{H}_p^\lambda} = \varepsilon^{\lambda - \frac{n}{p}} \|f(x)\|_{\dot{H}_p^\lambda}. \tag{1.5}$$

Remark 1. Sometimes the number μ in the equality $\|f(\varepsilon x)\|_Z = \varepsilon^\mu \|f(x)\|_Z$ is called the differential dimension of the space Z. So the differential dimensions of the spaces M_p^λ and H_p^λ coincide and are equal to $\lambda - \frac{n}{p}$.

Example 1. Let $1 \le p \le \infty$, $\lambda > 0$ and $\alpha \in \mathbb{R}$. Then

$$|x|^\alpha \in \left(H_p^\lambda\right)^{loc} \Longleftrightarrow |x|^\alpha \chi_{B(0,1)} \in M_p^\lambda \Longleftrightarrow \alpha \ge \lambda - \frac{n}{p}.$$

2 Main result

Note that the space M_p^λ is not contained in L_p, while clearly the space H_p^λ is contained in L_p. For this reason in order to compare the spaces M_p^λ and H_p^λ it is natural to consider the following variant of the Morrey spaces

$$\widehat{M_p^\lambda} = M_p^\lambda \cap L_p$$

with the quasi-norm (norm if $1 \le p \le \infty$)

$$\|f\|_{\widehat{M_p^\lambda}} \equiv \|f\|_{\widehat{M_p^\lambda}(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)} + \|f\|_{M_p^\lambda(\mathbb{R}^n)}.$$

The space M_p^λ is not monotonic with respect to the parameter λ. Indeed, by Example 1

$$M_p^\mu \not\subset M_p^\lambda$$

for any $0 \le \lambda, \mu \le \frac{n}{p}$, $\mu \ne \lambda$.

In contrast, the space $\widehat{M}_p^\lambda$, similarly to the space H_p^λ, is monotonic with respect to λ: if $0 \le \lambda < \mu \le \frac{n}{p}$, then

$$\widehat{M}_p^\mu \subset \widehat{M}_p^\lambda$$

and

$$\|f\|_{\widehat{M}_p^\lambda} \le 2\|f\|_{\widehat{M}_p^\mu}.$$

Indeed,

$$\begin{aligned}
\|f\|_{\widehat{M}_p^\lambda} &= \sup_{r>0} \sup_{x\in\mathbb{R}^n} r^{-\lambda}\|f\|_{L_p(B(x,r))} + \|f\|_{L_p} \\
&= \max\Big\{ \sup_{0<r\le 1} \sup_{x\in\mathbb{R}^n} r^{-\lambda}\|f\|_{L_p(B(x,r))}, \sup_{r>1} \sup_{x\in\mathbb{R}^n} r^{-\lambda}\|f\|_{L_p(B(x,r))} \Big\} + \|f\|_{L_p} \\
&\le \max\Big\{ \sup_{0<r\le 1} \sup_{x\in\mathbb{R}^n} r^{-\mu}\|f\|_{L_p(B(x,r))}, \|f\|_{L_p} \Big\} + \|f\|_{L_p} \\
&\le \max\Big\{ \|f\|_{M_p^\mu}, \|f\|_{L_p} \Big\} + \|f\|_{L_p} \le 2\|f\|_{\widehat{M}_p^\mu}.
\end{aligned}$$

Let us denote by $(H_p^\lambda)^\downarrow$ and $(\widehat{M}_p^\lambda)^\downarrow$ the subspaces of H_p^λ, $\widehat{M}_p^\lambda$ respectively, consisting of all functions $f \in H_p^\lambda$, $\widehat{M}_p^\lambda$ respectively, of the form $f(x) = g(|x|)$, where g is a non-negative non-increasing function on $[0, \infty)$.

Theorem 2.1. *Let $n = 1$, $1 \le p < \infty$, $0 < \lambda < \frac{1}{p}$. Then*

$$H_p^\lambda \subset \widehat{M}_p^\lambda, \tag{2.1}$$

for any $0 < \varepsilon < \frac{1}{p} - \lambda$

$$H_p^\lambda \not\subset \widehat{M}_p^{\lambda+\varepsilon},$$

for any $0 < \mu \le \lambda$

$$\widehat{M}_p^\lambda \not\subset (H_p^\mu)^{loc},$$

and

$$(\widehat{M}_p^\lambda)^\downarrow = (H_p^\lambda)^\downarrow.$$

Remark 2. For $1 < p < \infty$ the first statement is proved in [8]; for any $0 < p < \infty$ it is proved in [4]. In [13] the n-dimensional case is considered and inclusion (2.1) is proved for $1 \le p < \infty$, $0 < \lambda < \frac{n}{p}$. Here we follow the ideas of proofs developped in [4].

3 Auxiliary statements

First, we need to prove several auxiliary statements.

Lemma 3.1. *Let $1 \le p < \infty$, $h > 0$ and $f \in L_p(0, 2h)$.*

If $p = 1$, then

$$\int_0^h |f|\,dx \le \int_h^{2h} |f|\,dx + \int_0^h |\Delta_h f|\,dx. \tag{3.1}$$

If $1 < p < \infty$, then for all $\varepsilon > 0$

$$\int_0^h |f|^p\,dx \le (1+\varepsilon)\int_h^{2h} |f|^p\,dx + C_p(\varepsilon)\int_0^h |\Delta_h f|^p\,dx, \tag{3.2}$$

where

$$C_p(\varepsilon) = \left(1 - (1+\varepsilon)^{\frac{1}{1-p}}\right)^{1-p}. \tag{3.3}$$

If f is non-negative and non-increasing on $(0, 2h)$, then in (3.1) there is equality, and for $1 < p < \infty$

$$\int_0^h |f|^p dx \ge \int_h^{2h} |f|^p dx + \int_0^h |\Delta_h f|^p dx. \tag{3.4}$$

Proof. We start with the elementary inequality

$$|f(x)| \le |f(x+h)| + |f(x+h) - f(x)|. \tag{3.5}$$

If $1 < p < \infty$, then by applying the inequality

$$(a+b)^p \le (1+\varepsilon)a^p + C_p(\varepsilon)b^p,$$

where $a, b \ge 0$, $\varepsilon > 0$ and $C_p(\varepsilon)$ is defined by (3.3), we get

$$|f(x)|^p \le (1+\varepsilon)|f(x+h)|^p + C_p(\varepsilon)|f(x+h) - f(x)|^p. \tag{3.6}$$

By integrating (3.5) and (3.6) over $(0, h)$ we obtain (3.1), (3.2) respectively.

Let f be non-negative and non-increasing on $(0, 2h)$, then there is equality in (3.5) which imlies, after integration, equality in (3.1). If $1 < p < \infty$, then by (3.5) with the equality sign

$$\begin{aligned} |f(x)|^p &= (|f(x+h)| + |f(x+h) - f(x)|)^p \\ &\ge |f(x+h)|^p + |f(x+h) - f(x)|^p, \end{aligned} \tag{3.7}$$

which implies, after integration, inequality (3.4). □

Lemma 3.2. *Let $k \in \mathbb{N}$, $1 \le p < \infty$, $h > 0$ and $f \in L_p(0, (k+1)h)$.*

If $p = 1$, then

$$\int_0^h |f|\,dx \le \int_{kh}^{(k+1)h} |f|\,dx + \int_0^{kh} |\Delta_h f|\,dx. \tag{3.8}$$

If $1 < p < \infty$, then for all $\varepsilon > 0$

$$\int_0^h |f|^p\,dx \le \int_{kh}^{(k+1)h} |f|^p\,dx + \varepsilon\int_h^{(k+1)h} |f|^p\,dx + C_p(\varepsilon)\int_0^{kh} |\Delta_h f|^p\,dx. \tag{3.9}$$

If f is non-negative and non-increasing on $(0, (k+1)h)$, then there is equality in (3.8), and for $1 < p < \infty$

$$\int_0^h |f|^p dx \ge \int_{kh}^{(k+1)h} |f|^p dx + \int_0^{kh} |\Delta_h f|^p dx. \tag{3.10}$$

Proof. If $p=1$, then from (3.5) it follows that for all $m\in\mathbb{N}$

$$\int_{(m-1)h}^{mh}|f|\,dx\le\int_{mh}^{(m+1)h}|f|\,dx+\int_{(m-1)h}^{mh}|\Delta_h f|\,dx. \tag{3.11}$$

Hence

$$\sum_{m=1}^{k}\int_{(m-1)h}^{mh}|f|\,dx\le\sum_{m=1}^{k}\int_{mh}^{(m+1)h}|f|\,dx+\sum_{m=1}^{k}\int_{(m-1)h}^{mh}|\Delta_h f|\,dx$$

or

$$\int_0^h|f|\,dx+\int_h^{kh}|f|\;dx\le\int_h^{kh}|f|\,dx+\int_{kh}^{(k+1)h}|f|\;dx+\int_0^{kh}|\Delta_h f|dx, \tag{3.12}$$

which implies (3.8).

If $1<p<\infty$, then from (3.6), by a similar argument, it follows that

$$\int_0^h|f|^p\,dx+\int_h^{kh}|f|^p\,dx$$

$$\le(1+\varepsilon)\int_h^{kh}|f|^pdx+(1+\varepsilon)\int_{kh}^{(k+1)h}|f|^p\;dx+C_p(\varepsilon)\int_0^{kh}|\Delta_h f|^p\;dx,$$

which implies (3.8).

Let f be non-negative and non-increasing on $(0,(k+1)h)$, then there is equality in (3.11), hence in (3.12) and (3.8). If $1<p<\infty$, then by inequality (3.7) inequality (3.6) follows with $|f|$ and $|\triangle f|$ replaced by $|f|^p$ and $|\triangle f|^p$ and $\le$ replaced by $\ge$, hence inequality (3.7) follows with $|f|$ and $|\triangle f|$ replaced by $|f|^p$ and $|\triangle f|^p$ and $\ge$, which implies (3.10).

□

Lemma 3.3. *Let $n\in\mathbb{N}$, $1\le p<\infty$, $h>0$ and $f\in L_p(0,nh)$.*

If $p=1$, then

$$\int_0^h|f|\;dx\le\frac{1}{n}\int_0^{nh}|f|\;dx+\int_0^{(n-1)h}|\Delta_h f|\;dx. \tag{3.13}$$

If $1<p<\infty$, then for all $\varepsilon>0$

$$\int_0^h|f|^p\;dx\le\frac{1+\varepsilon}{n}\int_0^{nh}|f|^p\;dx+C_p\Big(\frac{\varepsilon}{n}\Big)\int_0^{(n-1)h}|\Delta_h f|^p\;dx. \tag{3.14}$$

If f is non-negative and non-increasing on $(0,nh)$, then for $1\le p<\infty$

$$\int_0^h|f|^p\;dx\ge\int_0^{(n-1)h}|\Delta_h f|^p\;dx. \tag{3.15}$$

Proof. If $p=1$, then from (3.8) it follows that

$$n\int_0^h|f|\;dx\le\sum_{k=0}^{n-1}\int_{kh}^{(k+1)h}|f|\;dx+\sum_{k=0}^{n-1}\int_0^{kh}|\Delta_h f|\;dx$$

$$\le\int_0^{nh}|f|\;dx+n\int_0^{(n-1)h}|\Delta_h f|\;dx,$$

which implies (3.13).

If $1 < p < \infty$, then from (3.9), by a similar argument, it follows that for all $\gamma > 0$

$$n\int_0^h |f|^p\,dx \le \sum_{k=0}^{n-1}\int_{kh}^{(k+1)h} |f|^p\,dx + \gamma\sum_{k=0}^{n-1}\int_0^{(k+1)h} |f|^p\,dx$$
$$+\, C_p(\gamma)\sum_{k=0}^{n-1}\int_0^{kh} |\Delta_h f|^p\,dx$$
$$\le \int_0^{nh} |f|^p\,dx + \gamma n\int_0^{nh} |f|^p\,dx + C_p(\gamma)n\int_0^{(n-1)h} |\Delta_h f|^p\,dx,$$

which implies (3.14) if to set $\gamma = \frac{\varepsilon}{n}$.

Let f be non-negative and non-increasing on $(0, nh)$, then inequality (3.15) follows for $p = 1$ by (3.8) with $\le$ replaced by $=$ and for $1 < p < \infty$ by for inequality (3.10) with $k = n - 1$. □

4 Proof of main result

Proof of Theorem 1.1.

Step 1*a*. Let $p = 1$, $0 < \lambda < 1$ and $f \in H_1^\lambda$. Given $0 < h < a$, in (3.13) we take $n = \left[\frac{a}{h}\right]$. Then $(n-1)h \le a - h \le nh \le a$. If $0 < h \le \frac{a}{2}$, then $\frac{1}{n} \le \frac{h}{a-h} \le \frac{2h}{a}$. Hence, by (3.13)

$$\int_0^h |f|dx \le \frac{2h}{a}\int_0^a |f|dx + \int_0^{a-h} |\Delta_h f|dx \tag{4.1}$$

If $\frac{a}{2} < h \le a$, then

$$\int_0^h |f|dx \le \frac{2h}{a}\int_0^a |f|dx$$

and again inequalty (4.1) follows.

Inequality (4.1) implies, by passing to the limit as $a \to \infty$, that

$$\int_0^h |f|dx \le \int_0^\infty |\Delta_h f|dx.$$

Also

$$\int_{-h}^0 |f(x)|dx = \int_0^h |f(-x)|dx$$
$$\le \int_0^\infty |f(-x-h) - f(x)|dx = \int_{-\infty}^{-h} |f(x+h) - f(x)|dx.$$

So

$$\int_{-h}^{-h} |f|dx \le \int_{-\infty}^{\infty} |f(x+h) - f(x)|dx.$$

Moreover, for any $x \in \mathbb{R}$

$$\int_{x-h}^{x+h} |f(y)|dy = \int_{-h}^h |f(y+x)|dy$$
$$\le \int_{-\infty}^\infty |f(y+x+h) - f(y+x)|dy = \int_{-\infty}^\infty |f(y+h) - f(y)|dy.$$

Therefore

$$\|f\|_{M_1^\lambda} = \sup_{x\in\mathbb{R}} \sup_{h>0} h^{-\lambda}\|f\|_{L_1(x-h,x+h)} \le \sup_{h>0} h^{-\lambda}\|\Delta_h f\|_{L_1} \le \|f\|_{\dot{H}_1^\lambda}.$$

Step 1*b*. Let $1 < p < \infty$, $0 < \lambda < \frac{1}{p}$ and $f \in H_p^\lambda$. Assume that $0 < h \le \frac{a}{2}$. Then by inequality (3.14) with $n = 2$

$$\|f\|_{L_p(0,h)} \le \Big(\frac{1+\varepsilon}{2}\Big)^{\frac{1}{p}} \|f\|_{L_p(0,2h)} + \Big(C_p\Big(\frac{\varepsilon}{2}\Big)\Big)^{\frac{1}{p}} \|\Delta_h f\|_{L_p(a-h)},$$

hence

$$\begin{aligned} h^{-\lambda}\|f\|_{L_p(0,h)} &\le (1+\varepsilon)^{\frac{1}{p}} 2^{\lambda-\frac{1}{p}} (2h)^{-\lambda}\|f\|_{L_p(0,2h)} \\ &\quad + \Big(C_p\Big(\frac{\varepsilon}{2}\Big)\Big)^{\frac{1}{p}} h^{-\lambda}\|\Delta_h f\|_{L_p(0,a-h)}. \end{aligned}$$

Choose $\varepsilon > 0$ to be such that

$$A = (1+\varepsilon)^{\frac{1}{p}} 2^{\lambda-\frac{1}{p}} < 1,$$

say, $\varepsilon = 2^{-p\lambda} - 2^{-1}$, in which case $A = \Big(\frac{1+2^{p\lambda-1}}{2}\Big)^{\frac{1}{p}}$, and denote

$$\psi(h) = h^{-\lambda}\|f\|_{L_p(0,h)}, \quad B = \Big(C_p\Big(\frac{\varepsilon}{2}\Big)\Big)^{\frac{1}{p}} \sup_{0<\eta\le a} \eta^{-\lambda}\|\Delta_\eta f\|_{L_p(0,a-\eta)}.$$

Then for all $0 < h \le \frac{a}{2}$

$$\psi(h) \le A\psi(2h) + B.$$

Consequently, for all $m \in \mathbb{N}$ such that $2^m h \le a$, we have

$$\begin{aligned} &\psi(h) \le A(A\psi(4h) + B) + B = A^2\psi(4h) + (A+1)B \le \dots \\ &\le A^m\psi(2^m h) + (A^{m-1} + \dots + 1)B \le A^m\psi(2^m h) + \frac{B}{1-A}. \end{aligned}$$

Next we choose m such that $2^m h \le a < 2^{m+1}h$. Then

$$\begin{aligned} h^{-\lambda}\|f\|_{L_p(0,h)} &\le A^m (2^m h)^{-\lambda}\|f\|_{L_p(0,2^m h)} + \frac{B}{1-A} \\ &\le 2^\lambda a^{-\lambda}\|f\|_{L_p(0,a)} + K\|f\|_{\dot{H}_p^\lambda}, \end{aligned} \tag{4.2}$$

where $K = (1-A)^{-1}(C_p(\frac{\varepsilon}{2}))^{\frac{1}{p}}$. If $\frac{a}{2} < h \le a$, then

$$h^{-\lambda}\|f\|_{L_p(0,h)} \le 2^\lambda a^{-\lambda}\|f\|_{L_p(0,a)}.$$

So inequality (4.2) holds for all $0 < h \le a$. By passing in (4.2) to the limit as $a \to \infty$, it follows that for all $h > 0$

$$h^{-\lambda}\|f\|_{L_p(0,h)} \le K\|f\|_{\dot{H}_p^\lambda}.$$

Similarly to Step 1, it also follows that for all $x \in R$

$$h^{-\lambda}\|f\|_{L_p(x-h,x+h)} \le K\|f\|_{\dot{H}_p^\lambda},$$

hence $H_p^\lambda \subset \widehat{M}_p^\lambda$ and

$$\|f\|_{M_p^\lambda} \le K\|f\|_{\dot{H}_p^\lambda}.$$

Step 2. By Example 1

$$|x|^{\lambda-\frac{1}{p}}\chi_{B(0,1)}(x) \in H_p^\lambda, \quad \text{but} \quad |x|^{\lambda-\frac{1}{p}}\chi_{B(0,1)}(x) \notin M_p^{\lambda+\varepsilon}$$

for any $0 < \varepsilon < \frac{1}{p} - \lambda$.

Step 3. Let for $1 \le p < \infty$, $0 < \mu \le \lambda < \frac{1}{p}$

$$f(x) = \sum_{k=2}^{\infty} k^{\gamma(\frac{1}{p}-\lambda)}\chi_k(x),$$

where

$$\chi_k(x) = \chi_{(k^{-\gamma},k^{-\gamma}+\varphi_\gamma(k)]}(x),$$

$$\varphi_\gamma(k) = \frac{1}{2}[(k-1)^{-\gamma} - k^{-\gamma}]$$

and

$$0 < \gamma < \frac{\mu}{\lambda-\mu}. \tag{4.3}$$

Note that for $k \ge 2$

$$\frac{\gamma}{2}k^{-\gamma-1} \le \varphi_\gamma(k) \le \frac{\gamma}{2}(k-1)^{-\gamma-1} \le \gamma 2^\gamma k^{-\gamma-1}. \tag{4.4}$$

So, $f(x) = k^{\gamma(\frac{1}{p}-\lambda)}$, if $k^{-\gamma} < x \le k^{-\gamma} + \varphi_\gamma(k)$; $f(x) = 0$, if $k^{-\gamma} + \varphi_\gamma(k) < x \le (k-1)^{-\gamma}$, $k \in \mathbb{N}$, $k \ge 2$.

Since for $k^{-\gamma} < x < k^{-\gamma} + \varphi_\gamma(k)$, $k \ge 2$, we have $x \le (k-1)^{-\gamma} \le 2^\gamma k^{-\gamma}$, hence $k^\gamma \le 2^\gamma x^{-1}$, it follows that for all $x \in (0,1]$

$$0 \le f(x) \le 2^{\gamma(\frac{1}{p}-\lambda)}x^{\lambda-\frac{1}{p}}.$$

Therefore, by Example 1, $f \in \widehat{M}_p^\lambda$.

Let $h = \varphi_\gamma(m)$, $m \in \mathbb{N}$, $m \ge 2$, then

$$\begin{aligned}\|\Delta_{\varphi_\gamma(m)}f\|_{L_p((-2,2)_{\varphi_\gamma(m)})} &\ge \int_0^1 |f(x+\varphi_\gamma(m)) - f(x)|^p dx \\ &\ge \sum_{k=2}^{m} \int_{k^{-\gamma}+\varphi_\gamma(k)-\varphi_\gamma(m)}^{k^{-\gamma}+\varphi_\gamma(k)} |f(x+\varphi_\gamma(m)) - f(x)|^p dx.\end{aligned}$$

Since $\varphi_\gamma(m) \le \varphi_\gamma(k)$, for $k^{-\gamma} + \varphi_\gamma(k) - \varphi_\gamma(m) < x < k^{-\gamma} + \varphi_\gamma(k)$, we have

$$k^{-\gamma} < x < k^{-\gamma} + \varphi_\gamma(k) \quad \text{and} \quad k^{-\gamma} + \varphi_\gamma(k) < x + \varphi_\gamma(m) < k^{-\gamma} + 2\varphi_\gamma(k) = (k-1)^{-\gamma},$$

therefore $f(x)=k^{\gamma(\frac{1}{p}-\lambda)}$ and $f(x+\varphi_\gamma(m))=0$. Hence, by (4.4)

$$\|\Delta_{\varphi_\gamma(m)}f\|^p_{L_p((-2,2)_{\varphi_\gamma(m)})} \geq \sum_{k=2}^m k^{\gamma(\frac{1}{p}-\lambda)p}\varphi_\gamma(m) \geq \Big(\sum_{k\geq\frac{m}{2}}^m k^{\gamma(1-p\lambda)}\Big)\varphi_\gamma(m)$$
$$\geq \frac{m}{2}\Big(\frac{m}{2}\Big)^{\gamma(1-p\lambda)}\frac{\gamma}{2}m^{-\gamma-1} = \gamma 2^{\gamma(pl-1)-2}m^{-\gamma p\lambda}.$$

Therefore, by (4.3) and (4.4)

$$\|f\|_{H_p^\mu(-2,2)} \geq \sup_{h>0} h^{-\mu}\|\Delta_h f\|_{L_p(-2+h,2-h)}$$
$$\geq (\gamma 2^{\gamma(pl-1)-2})^{\frac{1}{p}} \sup_{m\in\mathbb{N},m\geq 2}\varphi_\gamma(m)^{-\mu}m^{-\gamma\lambda}$$
$$\geq (\gamma 2^\gamma)^{-\mu}\Big(\gamma 2^{\gamma(pl-1)-3}\Big)^{\frac{1}{p}} \sup_{m\in\mathbb{N},m\geq 2} m^{\gamma(\mu-\lambda)+\mu} = \infty.$$

So $f\notin (H_p^\mu)^{loc}$.

Step 4. Let $1\leq p<\infty$, $0<\lambda<\frac{1}{p}$ and $f\in(\widehat{M_p^\lambda})^{\downarrow}$. By passing in (3.15) to the limit as $n\to\infty$, we get that for any $h>0$

$$\int_0^\infty |\Delta_h f|^p dx \leq \int_0^h |f|^p dx.$$

Similarly to Step 1 we get that also

$$\int_{-\infty}^{-h} |\Delta_h f|^p dx \leq \int_{-h}^0 |f|^p dx.$$

Hence

$$\|\Delta_h f\|_{L_p} = \Big(\int_{-\infty}^\infty |\Delta_h f|^p dx\Big)^{\frac{1}{p}} \leq \Big(\int_{-h}^h |f|^p dx + \int_{-h}^0 |f(x+h)-f(x)|^p dx\Big)^{\frac{1}{p}}$$
$$\leq \Big(\int_{-h}^h |f|^p dx + 2^{p-1}\Big(\int_{-h}^0 |f(x+h)|^p dx + \int_{-h}^0 |f(x)|^p dx\Big)\Big)^{\frac{1}{p}}$$
$$= (1+2^{p-1})^{\frac{1}{p}}\Big(\int_{-h}^h |f|^p dx\Big)^{\frac{1}{p}} \leq 2\|f\|_{L_p(-h,h)} \leq 2h^\lambda\|f\|_{M_p^\lambda}.$$

If $h<0$, then

$$\|\Delta_h f\|_{L_p} = \|\Delta_{-h} f\|_{L_p} \leq 2|h|^\lambda\|f\|_{M_p^\lambda}.$$

So $(\widehat{M_p^\lambda})^{\downarrow}\subset(H_p^\lambda)^{\downarrow}$,

$$\|f\|_{\dot{H}_p^\lambda} \leq 2\|f\|_{M_p^\lambda}$$

and

$$\|f\|_{H_p^\lambda} \leq 2\|f\|_{\widehat{M_p^\lambda}}.$$

5 Concluding remark

By statements of Sections 2.1 and 2.2 it follows that the Morrey space M_p^λ is not a space of functions possessing any kind of common smoothness of order μ measured in L_p-metrics for any $0<\mu\leq\lambda$, but the expressions $\|f\|_{L_p(B(x,r))}$ behave like the ones for functions f possessing certain smoothness of order λ measured in L_p-metrics.

Moreover, for functions of the form $f(x) = g(|x|)$, where g is a non-negative non-increasing function λ plays a role of the smoothness parameter. This is the reason why the initial notation for the space Morrey space was altered.

It appears that, in many situations in real analysis and especially in applications to the theory of partial differential equations, of primary importance is the behaviour of the expressions $\|f\|_{L_p(B(x,r))}$ rather than smoothness properties of f. In such cases the usage of the Morrey spaces is natural and effective.

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