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ORDER ESTIMATES FOR THE KOLMOGOROV WIDTHS OF WEIGHTED SOBOLEV CLASSES WITH RESTRICTIONS ON DERIVATIVES

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Abstract. In this paper order estimates for the Kolmogorov widths of weighted Sobolev classes with restrictions on the derivatives of order r and 0 are obtained. The functions are defined on a John domain or on $\mathbb{R}^d$.

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1 Introduction

The problem of estimating the Kolmogorov widths or approximation numbers for embeddings of weighted Sobolev classes on a multi-dimensional domain with restrictions only on the derivatives of order r into a weighted Lebesgue space was considered by H. Triebel [19] and T. Mieth [12, 13]; see also [20]. In [1, 2, 4, 11, 14, 18] the same problem was studied for weighted Sobolev classes with restrictions on derivatives of different orders. First we give some definitions.

Let $\Omega \subset \mathbb{R}^d$ be a domain, $g, v : \Omega \to (0, \infty)$ be positive measurable functions, and $1 \le p, q \le \infty$.

For $\overline{\beta} = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d := (\mathbb{N} \cup \{0\})^d$ we set $|\overline{\beta}| = \beta_1 + \dots + \beta_d$. Given a distribution f on Ω, we write $\nabla^r f = \left(\partial^r f / \partial x^{\overline{\beta}}\right)_{|\overline{\beta}|=r}$, and denote by $l_{r,d}$ the number of components of the vector-valued distribution $\nabla^r f$. We write

$$W^r_{p,g}(\Omega) = \left\{ f: \ \Omega \to \mathbb{R} \big| \ \exists \psi: \ \Omega \to \mathbb{R}^{l_{r,d}}: \ \|\psi\|_{L_p(\Omega)} \le 1, \ \nabla^r f = g \cdot \psi \right\},$$

$\nabla^r f / g := \psi$ (the function ψ is measurable, and $g \cdot \psi \in L_1^{\rm loc}(\Omega)$). We call the set $W^r_{p,g}(\Omega)$ a weighted Sobolev class (with restrictions only on derivatives of order r).

Given a measurable function f, we write $\|f\|_{L_{q,v}(\Omega)} = \|f\|_{q,v} = \|fv\|_{L_q(\Omega)}$. The weighted Lebesgue space $L_{q,v}(\Omega)$ is defined as follows:

$$L_{q,v}(\Omega) = \left\{ f : \Omega \to \mathbb{R} | \ \ \|f\|_{q,v} < \infty \right\}.$$

Let $1 \le p_0, p_1 \le \infty$. We denote

$$M = \left\{ f : \Omega \to \mathbb{R} : \ \left\| \frac{\nabla^r f}{g} \right\|_{L_{p_1}(\Omega)} \le 1, \quad \|wf\|_{L_{p_0}(\Omega)} \le 1 \right\}. \tag{1.1}$$

So, M is the intersection of $W^r_{p_1,g}(\Omega)$ and the unit ball in $L_{p_0,w}(\Omega)$.

Recall that the Kolmogorov width of order $n \in \mathbb{Z}_+$ for a subset C of a normed space X is defined as follows (see, e.g., [17]):

$$d_n(C, X) = \inf_{L\in\mathcal{L}_n(X)} \sup_{x\in C} \inf_{y\in L} \|x-y\|;$$

here $\mathcal{L}_n(X)$ is the family of all subspaces in X of dimension at most n.

Triebel [18, §3.8.3] obtained upper estimates for the Kolmogorov widths of the set M for $p_1 = p_0 \le q$, where Ω has a smooth boundary and the weights are powers of the distance to $\partial\Omega$. The parameters are such that the estimates depend only on the restriction on the derivatives of order r. Lizorkin and Otelbaev [11], Mynbaev and Otelbaev [14], Aitenova and Kussainova [1, 2] studied the problem of estimating the approximation numbers of embeddings of the set M into a weighted Lebesgue space with $p_0 = p_1$ and general weights; for $q \le 2$ or $p_1 \ge 2$ under some conditions on the weights, order estimates for the approximation numbers were obtained. Additionally, in [14] the special case of weights $(1+|x|)^{\beta_j}$ on $\mathbb{R}^d$ was considered (again for $p_0 = p_1$). Boykov [4] obtained order estimates for the Kolmogorov widths of intersections of weighted Sobolev classes $\cap_{0\le k\le r} W^k_{p_k,g_k}(K)$, where $p_k = p$ for $l+1 \le k \le r$, $p_k = \infty$ for $0 \le k \le l$, K is a cube, g_k are powers of the distance to the boundary of K; again the parameters are such that the estimates depend only on the restriction on the derivatives of order r. Also note that the problem about embeddings of intersections of weighted Sobolev classes were studied by many authors (see, e.g., [6, 3, 8, 9, 10, 15, 16, 18]).

In this paper we obtain order estimates for the Kolmogorov widths of the set M in $L_{q,v}(\Omega)$; here p_0 and p_1 may differ. In Examples 1 and 2 the functions are defined on a John domain and the weights are functions of the distance from an h-subset of the boundary (the definitions are given below). In Example 3 the functions are defined on $\mathbb{R}^d$ and the weights have the form $(1+|x|)^{\beta_j}$ (as in [14]).

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $a > 0$. We say that $\Omega \in FC(a)$ if there is $x_* \in \Omega$ such that for any $x \in \Omega$ there is a number $T(x) > 0$ and a curve $\gamma_x : [0, T(x)] \to \Omega$ with the following properties: 1) γ_x has the natural parametrization with respect to the standard Euclid norm on $\mathbb{R}^d$, 2) $\gamma_x(0) = x$, $\gamma_x(T(x)) = x_*$, 3) $B_{at}(\gamma_x(t)) \subset \Omega$ for all $t \in [0, T(x)]$.

We say that Ω is a John domain if $\Omega \in FC(a)$ for some $a > 0$.

The h-sets were introduced in [5].

Let $\Gamma \subset \mathbb{R}^d$ be a nonempty compact set, and let $h : (0, 1] \to (0, \infty)$ be a nondecreasing function. We say that Γ is an h-set if there are a number $c_* \ge 1$ and a finite countably additive measure μ on $\mathbb{R}^d$ such that $\operatorname{supp}\mu = \Gamma$ and

$$c_*^{-1}h(t) \le \mu(B_t(x)) \le c_* h(t)$$

for any $x \in \Gamma$ and $t \in (0, 1]$.

2 Main results

In order to formulate the main results of the paper, we need the following definition.

Definition 1. Given s_*, $\tilde\theta$, $\hat\theta \in \mathbb{R}$, we define the numbers $j_0 \in \mathbb{N}$ and $\theta_j \in \mathbb{R}$ $(1 \le j \le j_0)$ as follows.

1. For $p_0 \ge q$, $p_1 \ge q$: $j_0 = 2$, $\theta_1 = s_*$, $\theta_2 = \tilde\theta$.
2. For $p_0 > q$, $p_1 < q \le 2$: $j_0 = 3$, $\theta_1 = s_* + \frac{1}{q} - \frac{1}{p_1}$, $\theta_2 = \tilde\theta$, $\theta_3 = \hat\theta$.
3. For $p_0 > q$, $2 \le p_1 < q$: $j_0 = 4$, $\theta_1 = s_*$, $\theta_2 = \frac{q(s_*+1/q-1/p_1)}{2}$, $\theta_3 = \tilde\theta$, $\theta_4 = \frac{q\hat\theta}{2}$.
4. For $p_0 > q$, $p_1 < 2 < q$: $j_0 = 5$, $\theta_1 = s_* + \frac{1}{2} - \frac{1}{p_1}$, $\theta_2 = \frac{q(s_*+1/q-1/p_1)}{2}$, $\theta_3 = \tilde\theta$, $\theta_4 = \hat\theta + \frac{1}{2} - \frac{1}{q}$, $\theta_5 = \frac{q\hat\theta}{2}$.

5. For $p_0 \le q$, $p_1 \le q \le 2$: $j_0 = 2$, $\theta_1 = s_* + \frac{1}{q} - \frac{1}{p_1}$, $\theta_2 = \hat{\theta}$.

6. For $p_0 < q \le 2$, $p_1 > q$: $j_0 = 3$, $\theta_1 = s_*$, $\theta_2 = \tilde{\theta}$, $\theta_3 = \hat{\theta}$.

7. For $p_0 < q$, $q > 2$, $\max\{p_0, p_1\} \le 2$: $j_0 = 4$, $\theta_1 = s_* + \frac{1}{2} - \frac{1}{p_1}$, $\theta_2 = \frac{q(s_*+1/q-1/p_1)}{2}$, $\theta_3 = \hat{\theta} + \frac{1}{2} - \frac{1}{q}$, $\theta_4 = \frac{q\hat{\theta}}{2}$.

8. For $p_0 < q$, $q > 2$, $\min\{p_0, p_1\} \ge 2$: $j_0 = 4$, $\theta_1 = s_*$, $\theta_2 = \frac{q(s_*+1/q-1/p_1)}{2}$, $\theta_3 = \tilde{\theta}$, $\theta_4 = \frac{q\hat{\theta}}{2}$.

9. For $p_0 < q$, $q > 2$, $\min\{p_0, p_1\} < 2 < \max\{p_0, p_1\}$: $j_0 = 5$, $\theta_1 = s_* + \min\left\{\frac{1}{2} - \frac{1}{p_1}, 0\right\}$, $\theta_2 = \frac{q(s_*+1/q-1/p_1)}{2}$, $\theta_3 = \tilde{\theta}$, $\theta_4 = \hat{\theta} + \frac{1}{2} - \frac{1}{q}$, $\theta_5 = \frac{q\hat{\theta}}{2}$.

Let X, Y be sets, $f_1, f_2 : X \times Y \to \mathbb{R}_+$. We write $f_1(x, y) \underset{y}{\asymp} f_2(x, y)$ if, for any $y \in Y$, there exists $c(y) \ge 1$ such that $\frac{1}{c(y)} f_2(x, y) \le f_1(x, y) \le c(y) f_2(x, y)$ for each $x \in X$.

Given $d \in \mathbb{N}$, $r \in \mathbb{N}$, we set

$$s_* = \frac{r}{d}. \tag{2.1}$$

Example 1. Let $\Omega \subset \mathbb{R}^d$, $\Omega \in FC(a)$, let $\Gamma \subset \partial\Omega$ be an h-set,

$$h(t) = t^\theta, \tag{2.2}$$

$0 \le \theta < d$, $r \in \mathbb{N}$, $1 < p_0, p_1 \le \infty$, $1 < q < \infty$, $\beta, \lambda, \sigma \in \mathbb{R}$,

$$g(x) = \operatorname{dist}^{-\beta}(x, \Gamma), \quad w(x) = \operatorname{dist}^{-\sigma}(x, \Gamma), \quad v(x) = \operatorname{dist}^{-\lambda}(x, \Gamma). \tag{2.3}$$

Denote $\mathfrak{Z}_1 = (r, d, p_0, p_1, q, a, c_*, \theta, \beta, \lambda, \sigma, R)$, where $R = \operatorname{diam}\Omega$.

We set

$$\tilde{\theta} = \frac{r}{d} \cdot \frac{\sigma - \lambda + \frac{d-\theta}{q} - \frac{d-\theta}{p_0}}{\beta + \sigma - \left(r + \frac{d}{p_0} - \frac{d}{p_1}\right)\left(1 - \frac{\theta}{d}\right)}, \tag{2.4}$$

$$\hat{\theta} = \frac{\sigma\left(\frac{r}{d} + \frac{1}{q} - \frac{1}{p_1}\right) + \beta\left(\frac{1}{q} - \frac{1}{p_0}\right) - \lambda\left(\frac{r}{d} + \frac{1}{p_0} - \frac{1}{p_1}\right)}{\beta + \sigma - \left(r + \frac{d}{p_0} - \frac{d}{p_1}\right)\left(1 - \frac{\theta}{d}\right)}. \tag{2.5}$$

Theorem 2.1. *Let* (2.1), (2.2), (2.3), (2.4), (2.5) *hold, and let* $\frac{r}{d} + \min\left\{\frac{1}{q}, \frac{1}{p_0}\right\} - \frac{1}{p_1} > 0$, $\min\{\beta + \sigma - r - \frac{d-\theta}{p_0} + \frac{d-\theta}{p_1}, \beta + \sigma - r - \frac{d}{p_0} + \frac{d}{p_1}\} > 0$*; suppose that* $\tilde{\theta} > 0$ *for* $p_0 \ge q$, $\hat{\theta} > 0$ *for* $p_0 < q$. *Let the set* M *be given by formula* (1.1). *Let* $j_0 \in \mathbb{N}$ *and* θ_j *be as in Definition* 1. *Suppose that there is* $j_* \in \{1, \ldots, j_0\}$ *such that* $\theta_{j_*} < \min_{j \ne j_*} \theta_j$. *Then*

$$d_n(M, L_{q,v}(\Omega)) \underset{\mathfrak{Z}_1}{\asymp} n^{-\theta_{j_*}}.$$

Example 2. Let $\Omega \subset \mathbb{R}^d$, $\Omega \in FC(a)$, $\Gamma \subset \partial\Omega$ be an h-set,

$$g(x) = \varphi_g(\operatorname{dist}(x, \Gamma)), \quad w(x) = \varphi_w(\operatorname{dist}(x, \Gamma)), \quad v(x) = \varphi_v(\operatorname{dist}(x, \Gamma)); \tag{2.6}$$

$\varphi_g, \varphi_w, \varphi_v : (0, \infty) \to (0, \infty)$ are continuous functions; in a neighborhood of zero

$$h(t) = |\log t|^{-\gamma}, \quad \gamma \ge 0, \tag{2.7}$$

$$\varphi_g(t) = t^{-\beta}|\log t|^{\mu}, \quad \varphi_w(t) = t^{-\sigma}|\log t|^{\alpha}, \quad \varphi_v(t) = t^{-\lambda}|\log t|^{\nu}, \tag{2.8}$$

$$\beta + \lambda = r + \frac{d}{q} - \frac{d}{p_1}, \quad \sigma - \lambda = \frac{d}{p_0} - \frac{d}{q}. \tag{2.9}$$

We set

$$\tilde{\theta} = \frac{r}{d} \cdot \frac{\alpha - \nu + (\gamma+1)\left(\frac{1}{p_0} - \frac{1}{q}\right)}{\mu + \alpha + (\gamma+1)\left(\frac{r}{d} + \frac{1}{p_0} - \frac{1}{p_1}\right)}, \tag{2.10}$$

$$\hat{\theta} = \frac{\alpha\left(\frac{r}{d} + \frac{1}{q} - \frac{1}{p_1}\right) + \mu\left(\frac{1}{q} - \frac{1}{p_0}\right) - \nu\left(\frac{r}{d} + \frac{1}{p_0} - \frac{1}{p_1}\right)}{\mu + \alpha + (\gamma+1)\left(\frac{r}{d} + \frac{1}{p_0} - \frac{1}{p_1}\right)}. \tag{2.11}$$

Denote $\mathfrak{Z}_2 = (r,\, d,\, p_0,\, p_1,\, q,\, a,\, c_*, h,\, \varphi_g,\, \varphi_w,\, \varphi_v,\, R)$, where $R = \operatorname{diam}\Omega$.

Theorem 2.2. *Let* $\frac{r}{d} + \min\left\{\frac{1}{q},\, \frac{1}{p_0}\right\} - \frac{1}{p_1} > 0$, *and let* (2.1), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11) *hold; suppose that* $\min\{\mu + \alpha + (\gamma+1)(1/p_0 - 1/p_1),\, \mu + \alpha\} > 0$. *Let* $\tilde{\theta} > 0$ *for* $p_0 \ge q$, $\hat{\theta} > 0$ *for* $p_0 < q$. *Let* M *be defined by* (1.1). *Let* $j_0 \in \mathbb{N}$ *and* θ_j *be as in Definition* 1. *Suppose that there is* $j_* \in \{1, \ldots, j_0\}$ *such that* $\theta_{j_*} < \min_{j \ne j_*} \theta_j$. *Then*

$$d_n(M,\, L_{q,v}(\Omega)) \underset{\mathfrak{Z}_2}{\asymp} n^{-\theta_{j_*}}.$$

Example 3. Let $\Omega = \mathbb{R}^d$,

$$g(x) = (1+|x|)^{\beta}, \quad w(x) = (1+|x|)^{\sigma}, \quad v(x) = (1+|x|)^{\lambda}. \tag{2.12}$$

We set

$$\tilde{\theta} = \frac{r}{d} \cdot \frac{\sigma - \lambda + \frac{d}{p_0} - \frac{d}{q}}{\beta + \sigma + r + \frac{d}{p_0} - \frac{d}{p_1}}, \tag{2.13}$$

$$\hat{\theta} = \frac{\sigma\left(\frac{r}{d} + \frac{1}{q} - \frac{1}{p_1}\right) + \beta(\frac{1}{q} - \frac{1}{p_0}) - \lambda\left(\frac{r}{d} + \frac{1}{p_0} - \frac{1}{p_1}\right)}{\beta + \sigma + r + \frac{d}{p_0} - \frac{d}{p_1}}. \tag{2.14}$$

Denote $\mathfrak{Z}_3 = (r,\, d,\, p_0,\, p_1,\, q,\, \beta,\, \lambda,\, \sigma)$.

Theorem 2.3. *Let* (2.1), (2.12), (2.13), (2.14) *hold, and let*

$$\frac{r}{d} + \min\left\{\frac{1}{q},\, \frac{1}{p_0}\right\} - \frac{1}{p_1} > 0, \quad \beta + \sigma + r + d/p_0 - d/p_1 > 0.$$

Suppose that $\tilde{\theta} > 0$ *for* $p_0 \ge q$, $\hat{\theta} > 0$ *for* $p_0 < q$. *Let the set* M *be given by formula* (1.1). *Let* $j_0 \in \mathbb{N}$ *and* θ_j *be as in Definition* 1. *Suppose that there exists* $j_* \in \{1, \ldots, j_0\}$ *such that* $\theta_{j_*} < \min_{j \ne j_*} \theta_j$. *Then*

$$d_n(M,\, L_{q,v}(\mathbb{R}^d)) \underset{\mathfrak{Z}_3}{\asymp} n^{-\theta_{j_*}}.$$

In order to prove these results, we obtain upper and lower estimates of widths for some abstract function classes. To this end, we prove embedding theorems for these classes and use the inclusion $k_1 B^{\nu}_{p_1} \cap k_0 B^{\nu}_{p_0} \subset k_1^{1-t} k_0^t B^{\nu}_s$, where $\frac{1}{s} = \frac{1-t}{p_1} + \frac{t}{p_0}$, $t \in [0, 1]$, $B^{\nu}_{\alpha} = \{(x_1, \ldots, x_{\nu}) \in \mathbb{R}^{\nu} :\ \sum\limits_{j=1}^{\nu} |x_j|^{\alpha} \le 1\}$ (this is a particular case of Galeev's result [7]).

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