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The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 562
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

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INTERPOLATION THEOREMS FOR NONLINEAR URYSOHN INTEGRAL OPERATORS IN GENERAL MORREY-TYPE SPACES

V.I. Burenkov, E.D. Nursultanov

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Abstract. In this paper, we present new interpolation theorems for nonlinear Urysohn integral operators. In particular, interpolation theorems of Marcinkiewicz–Calderon type and Stein–Weiss–Peetre type are obtained.

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1 Introduction

Let (U,μ) be a space with measure μ, Ω some set. By $G_\Omega=\{G_{t,y}\}_{t>0,\, y\in\Omega}$ we denote a two-parameter family of μ -measurable sets satisfying the following condition:

$$G_{t,y}\subset G_{s,y} \quad \text{for} \quad 0<t<s, \quad y\in\Omega.$$

This family of sets will be called a net. If the sets $G_{t,y},\ t>0$ do not depend on the parameter y, then such nets will be denoted by $G=\{G_t\}_{t>0}$. For $y\in\Omega$, set $G_{\{y\}}=\{G_{t,y}\}_{t>0}$. We will say that these nets are generated by the net G_Ω.

Let $0<p,q\le\infty, 0<\lambda<\infty$. We denote by $M^\lambda_{p,q}(G_\Omega,\mu)$ the space of all μ-measurable functions $f:U\to\mathbb{R}$ such that for $q<\infty$

$$M^\lambda_{p,q}(G_\Omega,\mu)=\left\{f:\left(\int_0^\infty\left(t^{-\lambda}\sup_{y\in\Omega}\left(\int_{G_{t,y}}|f(x)|^p d\mu\right)^{1/p}\right)^q\frac{dt}{t}\right)^{1/q}<\infty\right\},$$

and for $q=\infty$

$$M^\lambda_{p,\infty}(G_\Omega,\mu)=\left\{f:\sup_{t>0,\, y\in\Omega}t^{-\lambda}\left(\int_{G_{t,y}}|f(x)|^p d\mu\right)^{1/p}<\infty\right\}.$$

If $U=\mathbb{R}^n$, μ is the Lebesgue measure, $G_{t,y}=B(y,t)$ (the ball centered at point $y\in\mathbb{R}^n$ of radius $t>0$), then this space will be denoted by $M^\lambda_{p,q,\Omega}$. In particular, for $q=\infty$ and $\Omega=\mathbb{R}^n$ this is the classical Morrey space M^λ_p.

If $U=\mathbb{R}^n$, μ is the Lebesgue measure, $\Omega=\{0\}$ and $G_t=G_{t,0}=B(0,t)$, then the space $M^\lambda_{p,q}(G_\Omega,\mu)$ is the local Morrey-type space $LM^\lambda_{p,q}$, introduced and used to study the properties maximal and fractional maximal operator in the works [5, 6, 7, 8]. If $\Omega = \{y\}$ and $G_t = G_{t,y} = B(y,t)$, then the corresponding local Morrey-type space will be denoted by $LM_{p,q,y}$.

The interpolation problem for the real method for the Morrey spaces was considered in the papers [25, 15, 22, 24, 25, 4, 18]. It follows from the results of [22] that for $1 \le p < \infty$

$$(M_p^{\lambda_0}, M_p^{\lambda_1})_{\theta,\infty} \subset M_p^{\lambda}, \quad \text{where} \quad \lambda = (1-\theta)\lambda_0 + \theta\lambda_1 \quad 0 < \theta < 1,$$

where $(\cdot,\cdot)$ denotes the real interpolation functor. In the works [24, 4] it was established that this inclusion is strict.

In [19], it was proved that the embedding

$$(M_{p_0}^{\lambda_0}, M_{p_1}^{\lambda_1})_{\theta,\infty} \subset M_p^{\lambda},$$

where

$$1 \le p_1, p_2 < \infty, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \lambda = (1-\theta)\lambda_0 + \theta\lambda_1, \quad 0 < \theta < 1,$$

holds if and only if $p_0 = p_1$.

In the papers [9, 10, 11, 12] it was established that, in contrast to the scale of the Morrey spaces M_p^{λ}, the scale of the local Morrey-type spaces $LM_{p,q}^{\lambda}$ with fixed p is closed with respect to the interpolation procedure, namely, it was proved that if $0 < p, q_0, q_1, q \le \infty$, $0 < \theta < 1$, $\lambda_0 \ne \lambda_1$ and $0 < \lambda_0, \lambda_1 < \infty$, then

$$\left(LM_{p,q_0}^{\lambda_0}, \ LM_{p,q_1}^{\lambda_1}\right)_{\theta,q} = LM_{p,q}^{\lambda}, \quad \text{where} \quad \lambda = (1-\theta)\lambda_0 + \theta\lambda_1, \quad 0 < \theta < 1.$$

In the paper [10], a description of interpolation spaces is obtained also for spaces that are substantially more general than $LM_{p,q}^{\lambda}$.

In the papers [12, 13], the following interpolation theorem for quasi-additive operators was proved.

Theorem 1.1. *Let* $\Omega \subset \mathbb{R}^n$, $0 < p, q, \sigma, \tau \le \infty$, $0 \le \alpha_0, \alpha_1 < \infty$, $\alpha_0, \alpha_1 > 0$, *if* $\sigma < \infty$, $\alpha_0 \ne \alpha_1$, $0 \le \beta_0, \beta_1 < \infty$, $\beta_0 \ne \beta_1$, $0 < \theta < 1$ *and*

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \ \beta = (1-\theta)\beta_0 + \theta\beta_1.$$

Let an operator T *be quasi-additive*[1] *on* $\bigcup_{y\in\Omega} \left(LM_{p,\sigma,y}^{\alpha_0} + LM_{p,\sigma,y}^{\alpha_1}\right)$ *with a quasi-additivity constant* A.

If for some $M_0, M_1 > 0$ *the inequalities*

$$\|Tf\|_{LM_{q,\infty,y}^{\beta_i}} \le M_i \|f\|_{LM_{p,\sigma,y}^{\alpha_i}} \tag{1.1}$$

hold for all $y \in \Omega$ *and for all functions* $f \in LM_{p,\sigma,y}^{\alpha_i}$, $i = 0, 1$, *then the inequality*

$$\|Tf\|_{M_{q,\tau,\Omega}^{\beta}} \le cAM_0^{1-\theta} M_1^{\theta} \|f\|_{M_{p,\tau,\Omega}^{\alpha}} \tag{1.2}$$

holds for all functions $f \in M_{p,\tau,\Omega}^{\alpha}$, *where* $c > 0$ *depends only on* $\alpha_0, \alpha_1, \beta_0, \beta_1, q, \sigma, \tau$ *and* θ.

---

[1] That is,

$$|T(f_0 + f_1)(x)| \le A(|Tf_0(x)| + |Tf_1(x)|)$$

for almost all $x \in \mathbb{R}^n$ for all $y \in \Omega$ and for all $f_0 \in LM_{p,\sigma,y}^{\alpha_0}$, $f_1 \in LM_{p,\sigma,y}^{\alpha_1}$.

In this theorem, in a certain sense strong estimate (1.2) is derived from in a certain sense weak estimates (1.1).

In this paper, we consider the interpolation properties of nonlinear Urysohn integral operators

$$(Tf)(y) = \int_U K(f(x), x, y) d\mu(x) \ , \ \ y \in V, \tag{1.3}$$

where $f : U \to \mathbb{R}, K : f(U) \times U \times V \to \mathbb{R}$, in much more general classes of Morrey-type spaces, which allow one to obtain analogues of the Marcinkiewicz–Calderon and Stein–Weiss–Peetre interpolation theorems for a wide class of operators of form (1.3). Note that the well-known interpolation theorems of Marcinkiewicz–Calderon [14], Stein–Weiss–Peetre [26, 23] do not cover this class of operators.

For the properties of Urysohn operators and, in particular, for the conditions on K under which the integral in (1.3) exists and is finite for almost all $y \in V$, see the book [17] and the articles [20], [21].

In particular, if K is Borel measurable on $F(U) \times U \times V$, $f \in L_p(U, \mu)$, where $0 < p < \infty$, for some $0 \leq \alpha \leq p$ and $c > 0$

$$|K(z, x, y)| \leq c|z|^\alpha |K(1, x, y)|$$

for all $x \in U, y \in V, z \in f(U)$, and $\|K(1, \cdot, y)\|_{L_r(U,\mu)} < \infty$ for almost all $y \in V$, where $\frac{1}{r} = 1 - \frac{\alpha}{p}$, then the integral in (1.3) exists and is finite for almost all $y \in V$.

2 Interpolation theorems for Morrey-type spaces

Let (U, μ) be a space with measure μ, Ω some set, $G_\Omega = \{G_{t,y}\}_{t>0,\, y\in\Omega}$ be a net, $s > 0$. Let us define the nets

$$\check{G}^s_\Omega = \left\{\check{G}^s_{t,y}\right\}_{t>0,\, y\in\Omega}, \ \hat{G}^s_\Omega = \left\{\hat{G}^s_{t,y}\right\}_{t>0,\, y\in\Omega} \tag{2.1}$$

and, for $y \in \Omega$, the nets generated by them

$$\check{G}^s_{\{y\}} = \left\{\check{G}^s_{t,y}\right\}_{t>0}, \ \ \hat{G}^s_{\{y\}} = \left\{\hat{G}^s_{t,y}\right\}_{t>0}, \tag{2.2}$$

where

$$\check{G}^s_{t,y} = \begin{cases} G_{t,y}, & \text{if } \ s \geq t, \\ G_{s,y}, & \text{if } \ s < t \ , \end{cases} \tag{2.3}$$

$$\hat{G}^s_{t,y} = \begin{cases} \varnothing, & \text{if } \ s \geq t, \\ G_{t,y}, & \text{if } \ s < t. \end{cases} \tag{2.4}$$

Remark 1. If $G = \{G_t\}_{t>0}$ is a filtering, i.e. G is a system of expanding $\sigma-$algebras of measurable sets, then in the theory of stochastic processes the procedure defined by relation (2.3) is called a stop corresponding to the moment s, and procedure (2.3) defines the beginning, corresponding to the moment s. These transformations play an important role in the construction of interpolation methods for stochastic processes [1, 2, 3]. In this section, we present the main results of this work, where these transformations also play an essential role.

Theorem 2.1. *Let (U, μ), (V, ν) be spaces with measures μ, ν, $\Omega \subset \mathbb{R}^n$.*

Let G_Ω, $\check{G}^s_\Omega$, $\hat{G}^s_\Omega$, $\check{G}^s_{\{y\}}$, $\hat{G}^s_{\{y\}}$, where $s > 0, y \in \Omega$, be the nets in U, defined by (2.1) – (2.4).

Let $F_\Omega = \{F_{t,y}\}_{t>0,\, y\in\Omega}$ be a net in V and $F_{\{y\}}$, where $y \in \Omega$, be the nets in V generated by F.

Let $0 < p, q, \sigma, \tau \leq \infty$, $0 \leq \alpha_0, \alpha_1 < \infty$, $\alpha_0, \alpha_1 > 0$, if $\sigma < \infty$, $\alpha_0 \neq \alpha_1$, $0 \leq \beta_0, \beta_1 < \infty$, $\beta_0 \neq \beta_1$, $0 < \theta < 1$ and

$$\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1, \quad \beta = (1 - \theta)\beta_0 + \theta\beta_1.$$

Let $f \in M^{\alpha}_{p,\tau}(G_\Omega,\mu)$ *and* T *be an Urysohn integral operator* (1.3).
If for some $M_0, M_1 > 0$ *the inequalities*

$$\|Tf\chi_{G_{s,y}}\|_{M^{\beta_0}_{q,\infty}(F_{\{y\}},\nu)} \le M_0\|f\|_{M^{\alpha_0}_{p,\sigma}(\check{G}^s_{\{y\}},\mu)}, \tag{2.5}$$

$$\|Tf(1-\chi_{G_{s,y}})\|_{M^{\beta_1}_{q,\infty}(F_{\{y\}},\nu)} \le M_1\|f\|_{M^{\alpha_1}_{p,\sigma}(\hat{G}^s_{\{y\}},\mu)}, \tag{2.6}$$

hold for all $y \in \Omega,\ s > 0$, *then the inequality*

$$\|Tf\|_{M^{\beta}_{q,\tau}(F_\Omega,\nu)} \le cM_0^{1-\theta}M_1^{\theta}\|f\|_{M^{\alpha}_{p,\tau}(G_\Omega,\mu)}, \tag{2.7}$$

holds, where $c > 0$ *depends only on* $\alpha_0, \alpha_1, \beta_0, \beta_1, q, \sigma, \tau$ *and* θ.

Remark 2. Conditions (2.5), (2.6) of the theorem are formulated correctly since if $f \in M^{\alpha}_{p,\tau}(G_\Omega,\mu)$, then for arbitrary $y \in \Omega$ and $s > 0$

$$\|f\|_{M^{\alpha_0}_{p,\sigma}(\check{G}^s_{\{y\}},\mu)} < \infty,$$

$$\|f(1-\chi_{G_{s,y}})\|_{M^{\alpha_1}_{p,\sigma}(G_{\{y\}},\mu)} \le \|f\|_{M^{\alpha_1}_{p,\sigma}(\hat{G}^s_{\{y\}},\mu)} < \infty.$$

Remark 3. This theorem, although similar in form to classical interpolation theorems, has an essential distinction. The point is that the conditions and statement of the theorem are formulated for a fixed function $f \in M^{\alpha}_{p,q}(G_\Omega,\mu)$. We can say that here we are not talking about the interpolation of an operator, but about the interpolation of inequalities for a fixed function. This fact makes the statement more universal for application. In particular, the sets $G_{s,y}$ can be chosen to depend on f.

Let $U = V = \mathbb{R}^n$, and $\mu = \nu$ be the Lebesgue measure, $\Omega \subset \mathbb{R}^n$, $0 < p, q \le \infty$, $0 \le \lambda < \infty$, $\lambda > 0$ if $q < \infty$. Let v be a positive locally absolutely continuous strictly increasing function defined on $(0,\infty)$. We define the spaces $M^{\lambda}_{p,q,\Omega}(v)$: for $0 < q < \infty$

$$M^{\lambda}_{p,q,\Omega}(v) = \Bigg\{ f \in L^{loc}_p(\mathbb{R})^n) :$$

$$\|f\|_{M^{\lambda}_{p,q,\Omega}(v)} = \left(\int\limits_0^\infty \left((v(r))^{-\lambda} \sup_{y\in\Omega} \|f\|_{L_p(B(y,r))} \right)^q \frac{dv(r))}{v(r)} \right)^{1/q} < \infty \Bigg\}$$

and

$$M^{\lambda}_{p,\infty,\Omega}(v) = \left\{ f :\ \|f\|_{M^{\lambda}_{p,\infty,\Omega}(v)} = \sup_{r>0, y\in\Omega} (v(r))^{-\lambda}\|f\|_{L_p(B(y,r))} < \infty \right\}.$$

Corollary 2.1. *Let* $\Omega \subset \mathbb{R}^n$, $0 < p, q, \sigma, \tau \le \infty$, $0 < \alpha_0, \alpha_1 < \infty$, $\alpha_0, \alpha_1 > 0$, *if* $\sigma < \infty$, $\alpha_0 \ne \alpha_1$, $0 \le \beta_0, \beta_1 < \infty$, $\beta_0 \ne \beta_1$, $0 < \theta < 1$ *and*

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \quad \beta = (1-\theta)\beta_0 + \theta\beta_1.$$

Let functions v, w *satisfy the conditions listed above and let* T *be an Urysohn integral operator* (1.3).
If for some $M_0, M_1 > 0$ *the inequalities*

$$\|Tf\|_{M^{\beta_i}_{q,\infty,y}(v)} \le M_i\|f\|_{M^{\alpha_i}_{p,\sigma,y}(w)}$$

hold for all $y \in \Omega$ *and for all functions* $f \in LM^{\alpha_i}_{p,\sigma,y}(w)$, $i = 0, 1$, *then the inequality*

$$\|Tf\|_{M^{\beta}_{q,\tau,\Omega}(v)} \le cM_0^{1-\theta}M_1^{\theta}\|f\|_{M^{\alpha}_{p,\tau,\Omega}(w)}$$

holds for all functions $f \in LM^{\alpha}_{p,\tau,\Omega}(w)$, *where* $c > 0$ *depends only on* $\alpha_0, \alpha_1, \beta_0, \beta_1, q, \sigma, \tau$ *and* θ.

3 An interpolation theorem of Marcinkiewicz–Calderon type

Recall that, given a space (U, μ) with measure μ and a μ-measurable function f defined on U, the function

$$f^*(t) = \inf\{\sigma \geq 0 : \mu(\{x \in U : |f(x)| > \sigma\}) \leq t\}, \quad t \geq 0,$$

is called the non-increasing rearrangement of f. Moreover, for $0 < r < \infty$, $0 < q \leq \infty$. the Lorentz space $L_{r,q}(U, \mu)$ is the space of all μ-measurable functions f defined on U for which

$$\|f\|_{L_{r,q}(U,\mu)} = \left(\int_0^\infty \left(t^{\frac{1}{r}} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q} < \infty.$$

We say that a measure μ satisfies the regularity condition if for every μ-measurable set e, and $\alpha \in (0, \frac{\mu(e)}{2}]$ there is a μ-measurable subset $w \subset e$ such that

$$\alpha \leq \mu(w) \leq 2\alpha. \tag{3.1}$$

Theorem 3.1. *Let (U, μ), (V, ν) be spaces with measures μ, ν satisfying regularity condition* (3.1).
Let $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0, q_1 < \infty$, $q_0 \neq q_1$, $0 < \sigma, \tau \leq \infty$, $0 < \theta < 1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let T be an Urysohn integral operator (1.3).
If for some $M_0, M_1 > 0$ the inequalities

$$\|Tf\|_{L_{q_i,\infty}(V,\nu)} \leq M_i \|f\|_{L_{p_i,\sigma}(U,\mu)} \tag{3.2}$$

hold for all functions $f \in L_{p_i,\sigma}(U, \mu)$, $i = 0, 1$, then the inequality

$$\|Tf\|_{L_{q,\tau}(V,\nu)} \leq c M_0^{1-\theta} M_1^\theta \|f\|_{L_{p,\tau}(U,\mu)} \tag{3.3}$$

holds for all functions $f \in L_{p,\tau}(U, \mu)$, where $c > 0$ depends only on $p_0, p_1, q_0, q_1, \sigma, \tau$ and θ.

3 An interpolation theorem of Stein–Weiss–Peetre type

Let μ be a measure on U satisfying regularity condition (3.1) and w a positive μ-measurable function on U (weight function).

By $L_p(U, w, \mu)$, where $0 < p \leq \infty$ we denote the space of all μ-measurable functions on U for which

$$\|f\|_{L_p(U,w,\mu)} = \left(\int_U (w(x)|f(x)|)^p d\mu \right)^{\frac{1}{p}} < \infty.$$

If $w \equiv 1$, then $L_p(U, 1, \mu) \equiv L_p(U, \mu)$; if μ is the Lebesgue measure, then $L_p(U, w, \mu) \equiv L_p(U, w)$.

Theorem 3.1. *Let $0 < p \leq q < \infty$, $0 < \theta < 1$. Let w_0, w_1 be positive μ-measurable functions on U and T be an Urysohn integral operator* (1.3).
If for some $M_0, M_1 > 0$ the inequalities

$$\|Tf\|_{L_q(U,w_i,\mu)} \leq M_i \|f\|_{L_p(U,w_i,\mu)}$$

hold for all functions $f \in L_p(U, w_i, \mu)$, $i = 0, 1$, then the inequality

$$\|Tf\|_{L_q(U,w_0^{1-\theta} w_1^\theta,\mu)} \leq c M_0^{1-\theta} M_1^\theta \|f\|_{L_p(U,w_0^{1-\theta} w_1^\theta,\mu)} \tag{3.1}$$

holds for all functions $f \in L_p(U, w_0^{1-\theta} w_1^\theta, \mu)$, where $c > 0$ depends only on $q, \alpha_0, \alpha_1, \lambda_0, \lambda_1$ and θ.

Remark 4. Theorem 3.1, in the case when T is a linear operator and $p_0 = p_1 = p$, was proved by Stein and Weiss [26]. In this case, the constant c in inequality (3.1) is equal to 1. In the case when T is a quasi-additive operator, this theorem was proved by Peetre [23].

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Victor Ivanovich Burenkov
S.M. Nikol'skii Institute of Mathematics
RUDN University
6 Miklukho Maklay St
117198 Moscow, Russia
and
V.A. Steklov Institute of Mathematics
Russian Academy of Sciences
42 Vavilov St
117966 Moscow, Russia
E-mail: burenkov@cardiff.ac.uk.

Erlan Dautbekovich Nursultanov
Lomonosov Moscow State University (Kazakhstan branch)
11 Kazhimukan St
010010 Astana, Kazakhstan
and
L.N. Gumilyov Eurasian National University
2 Mirzoyan St
010008 Astana, Kazakhstan
E-mail: er-nurs@yandex.ru

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