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SOME INTEGRAL INEQUALITIES FOR QUASIMONOTONE FUNCTIONS IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACE WITH $0 < p(x) < 1$

A. Senouci, A. Zanou

Communicated by T.V. Tararykova

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Abstract. The aim of this paper is to obtain some weighted Hardy's inequalities for quasi-monotone functions in weighted variable exponent Lebesgue space with $0 < p(x) < 1$.

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1 Introduction

For the first time the variable exponent Lebesgue space appeared in the literature already in the thirties of the last century, being introduced by W. Orlicz. At the beginning these spaces had theoretical interest. Later on the end of the last century, their first use beyond the function spaces theory itself, was in variational problems and studies of $p(x)$ - Laplacian, in Zhikov [9], [10], which in its turn gave an essential impulse for the development of this theory. The extensive investigation of these spaces was also widely stimulated by applications to various problems of Applied Mathematics, e.g., in modelling of electrorheological fluids [7]. Variable Lebesgue space appeared as a special case of the Musielak-Orlicz spaces introduced by H. Nakano and developed by J. Musielak and W. Orlicz.

The variable exponent Lebesgue spaces $L_{p(x)}$ for $p(x) \geq 1$ appeared in the literature for the first time in [6]. Further developement of this theory was connected with the theory of modular functions.

Many investigations are devoted to the problem of boundedness of the Hardy operator in the Lebesgue spaces $L_{p(x)}$ for $p(x) \geq 1$ (see for example [1] and [6]). But the investigations of the Hardy inequality in variable exponent Lebesgue spaces $L_{p(x)}$ for $0 < p(x) < 1$ are much less known.

It is well known that for L_p-spaces with $0 < p < 1$, the Hardy inequalities are not satisfied for arbitrary non-negative measurable function, but are satisfied for non-negative quasi-monotone functions with sharp constants (see [3] for more details). The object of this work is to obtain weighted inequalities for the Hardy operators acting from one weighted variable exponent Lebesgue space to another weighted variable exponent Lebesgue space for $0 < p(x) < 1$, for the functions defined on $(0, \infty)$ and satisfying conditions of quasi-monotonicity. Some results obtained in [2] are generalized. Moreover, some new weighted integral inequalities are obtained.

2 Preliminaries

In this section, we state the following Definitions, Lemmas, Corollaries and Theorems that are useful in the proofs of main results.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, x_2, \dots, x_n)$, Ω be a Lebesgue measurable subset of $\mathbb{R}^n$. Suppose that $p(x)$ is a Lebesgue measurable function on Ω such that $0 < \underline{p} \le p(x) \le \overline{p} < 1$, $\underline{p} = \operatorname{ess\,inf}_{x\in\Omega} p(x)$, $\overline{p} = \operatorname{ess\,sup}_{x\in\Omega} p(x)$ and ω is a weight function, that is a positive Lebesgue measurable function on Ω.

Definition 1. By $L_{p(x),\omega(x)}(\Omega)$ we denote the set of all Lebesgue measurable function f on Ω such that

$$\rho_{p(x),\omega(x)}(f) = \int_\Omega (|f(x)|\omega(x))^{p(x)} dx < \infty. \tag{2.1}$$

Note that the expression

$$\|f\|_{L_{p(x),\omega(x)}(\Omega)} = \inf\{\lambda > 0;\ \int_\Omega \Big(\frac{|f(x)|\omega(x)}{\lambda}\Big)^{p(x)} dx \le 1\} \tag{2.2}$$

defines a quasi-norm on $L_{p(x),\omega(x)}(\Omega)$. $L_{p(x),\omega(x)}(\Omega)$ is a quasi-Banach space equipped with this quasi-norm (see [8]).

In [2] the following statement was proved.

Corollary 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and p, q be Lebesgue measurable functions on Ω, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < \infty$ and $r(x) = \frac{p(x)q(x)}{q(x)-p(x)}$. Suppose that ω_1 and ω_2 are weight functions in Ω satisfying the condition:*

$$\Big\|\frac{\omega_1}{\omega_2}\Big\|_{L_{r(x)}(\Omega)} < \infty.$$

Then the inequality

$$\|f\|_{L_{p(x),\omega_1}(\Omega)} \le (A_1 + B_1 + \|\chi_{\Omega_2}\|_{L^\infty(\Omega)})^{\frac{1}{\underline{p}}} \Big\|\frac{\omega_1}{\omega_2}\Big\|_{L_{r(x)}(\Omega)} \|f\|_{L_{q(x)\omega_2(x)}(\Omega)} \tag{2.3}$$

holds for every $f \in L_{q(x),\omega_2(x)}(\Omega)$, where

$$\Omega_1 = \{x \in \Omega:\ p(x) < q(x)\}, \qquad \Omega_2 = \{x \in \Omega:\ p(x) = q(x)\},$$

$$A_1 = \sup_{x\in\Omega_1} \frac{p(x)}{q(x)},\ B_1 = \sup_{x\in\Omega_1} \frac{q(x)-p(x)}{q(x)}.$$

The following statement is known (see [1]).

Lemma 2.1. *Let $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ be measurable sets, p be a Lebesgue measurable function on Ω_1 and q be a Lebesgue measurable function on Ω_2, $1 \le \underline{p} \le p(x) \le q(y) \le \overline{q} < \infty$ for all $x \in \Omega_1$ and $y \in \Omega_2$. If $p \in C(\Omega_1)$, $q \in C(\Omega_2)$, then the inequality*

$$\Big\|\|f\|_{L_{p(x)}(\Omega_1)}\Big\|_{L_{q(x)}(\Omega_2)} \le C_{p,q}\Big\|\|f\|_{L_{q(x)}(\Omega_2)}\Big\|_{L_{p(x)}(\Omega_1)} \tag{2.4}$$

is valid, where

$$C_{p,q} = \Big(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + \frac{\overline{p}}{\underline{q}} + \frac{\underline{p}}{\overline{q}}\Big)(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty), \tag{2.5}$$

$$\underline{q} = \operatorname*{ess\,inf}_{\Omega_2} q(x), \quad \overline{q} = \operatorname*{ess\,sup}_{\Omega_2} q(x),$$

$$\Delta_1 = \{(x,y) \in \Omega_1 \times \Omega_2;\ p(x) = q(y)\}, \qquad \Delta_2 = (\Omega_1 \times \Omega_2)\backslash\Delta_1,$$

$C(\Omega_1)$, $C(\Omega_2)$ are the spaces of continuous functions in Ω_1, Ω_2 and $f : \Omega_1\times\Omega_2 \to \mathbb{R}$ is any measurable function such that $\Big\|\|f\|_{L_{q(x)}(\Omega_2)}\Big\|_{L_{p(x)}(\Omega_1)} < \infty$.

The following theorems were proved in [2].

Theorem 2.1. *Let p, q be Lebesgue measurable functions on $(0,\infty)$, $0<\underline{p}\le p(x)\le q(x)\le \overline{q}<1$, $r(x)=\frac{\underline{p}p(x)}{p(x)-\underline{p}}$, $x\in(0,\infty)$ and f be a non-negative and non-increasing function defined on $(0,\infty)$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

Then the inequality

$$\|Hf\|_{L_{q(x),\omega_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}} C_{p,q} d_p \left\| \frac{t^{\frac{1}{\overline{p}'}} \|\frac{\omega_2(x)}{x}\|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{2.6}$$

holds, where

$$C_{p,q} = \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \underline{p}\left(\frac{1}{\underline{q}} - \frac{1}{\overline{q}}\right)\right) \left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)}\right),$$

$S_1=\{x\in(0,\infty):\ p(x)=\underline{p}\}$, $S_2=(0,\infty)\backslash S_1$ and

$$d_p = \left(1+\frac{\overline{p}-\underline{p}}{\overline{p}} + \|\chi_{S_1}\|_{L_\infty(0,\infty)}\right)^{\frac{1}{\underline{p}}}.$$

Theorem 2.2. *Let p, q be Lebesgue measurable functions on $(0,1)$, $0<\underline{p}\le p(x)\le q(x)\le \overline{q}<1$, $r(x)=\frac{\underline{p}p(x)}{p(x)-\underline{p}}$, $x\in(0,1)$ and f be a non-negative and non-decreasing function defined on $(0,1)$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,1)$.*

Then the inequality

$$\|Hf\|_{L_{q(x),\omega_2(x)}(0,1)} \le \underline{p}^{\frac{1}{\underline{p}}} C_{p,q} d_p \left\| \frac{\left\|\frac{(x-t)^{1/\overline{p}'}\omega_2(x)}{x}\right\|_{L_{q(x)}(t,1)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,1)} \|f\|_{L_{p(x),\omega_1(x)}(0,1)} \tag{2.7}$$

holds, where $C_{p,q}$ and d_p are the constants in Theorem 2.1.

The following definition was introduced in [3].

Definition 2. We say that a non-negative function f is quasimonotone on $]0,\infty[$, if for some real number α, $x^\alpha f(x)$ is a decreasing or an increasing function of x. More precisely, given $\beta\in\mathbb{R}$, we say that $f\in Q_\beta$ if $x^{-\beta}f(x)$ is non-increasing and $f\in Q^\beta$ if $x^{-\beta}f(x)$ is non-decreasing.

The following proposition was proved in [3].

Proposition 2.1. (*a*) *Let $-\infty<\beta<\infty$, $f\in Q_\beta$, $0\le a<\infty$ for $\beta>-1$ and $0<a<b\le\infty$ for $\beta\le -1$. If $0<p\le 1$ and $\beta\ne -1$, then*

$$\left(\int_a^b f(y)dy\right)^p \le p|\beta+1|^{1-p}\int_a^b \left(\frac{|y^{\beta+1}-a^{\beta+1}|}{y^\beta}\right)^{p-1} f^p(y)dy. \tag{2.8}$$

If $0<p\le 1$ and $\beta=-1$, then

$$\left(\int_a^b f(y)dy\right)^p \le p\int_a^b \left(y\ \ln\frac{y}{a}\right)^{p-1} f^p(y)dy. \tag{2.9}$$

The inequalities hold in the reversed direction if $1 \le p < \infty$.

(*b*) *Let* $-\infty < \beta < \infty$, $f \in Q^{\beta}$ *and* $0 \le a < b \le \infty$ *for* $\beta < -1$ *and* $0 \le a < b < \infty$ *for* $\beta \ge -1$. *If* $0 < p \le 1$ *and* $\beta \ne -1$, *then*

$$\left(\int_a^b f(y)dy\right)^p \le p|\beta+1|^{1-p} \int_a^b \left(\frac{|y^{\beta+1} - b^{\beta+1}|}{y^{\beta}}\right)^{p-1} f^p(y)dy. \tag{2.10}$$

If $0 < p \le 1$ *and* $\beta = -1$, *then*

$$\left(\int_a^b f(y)dy\right)^p \le p \int_a^b \left(y \ln\frac{b}{y}\right)^{p-1} f^p(y)dy. \tag{2.11}$$

The inequalities hold in the reversed direction if $1 \le p < \infty$.

(*c*) *The constants in these inequalities are the best possible in all cases.*

If in (2.8), we set $a = 0$ and in (2.10) we put $b = \infty$, we get the following special cases of Proposition which are useful in the proofs of main results.

Corollary 2.2. *Let* $0 < p \le 1$.

(*a*) *If* $\beta > -1$, $f \in Q_{\beta}$ *and* $0 < b \le \infty$, *then*

$$\left(\int_0^b f(y)dy\right)^p \le p(\beta+1)^{1-p} \int_0^b y^{p-1} f^p(y)dy. \tag{2.12}$$

(*b*) *If* $\beta < -1$, $f \in Q^{\beta}$ *and* $0 \le a < \infty$, *then*

$$\left(\int_a^{\infty} f(y)dy\right)^p \le p|\beta+1|^{1-p} \int_a^{\infty} y^{p-1} f^p(y)dy. \tag{2.13}$$

If in (2.10), we take $a = 0$, $b = x$, $\beta > -1$, we have the following statement.

Corollary 2.3. *Let* $0 < p \le 1$, $\beta > -1$, $f \in Q^{\beta}$ *and* $0 \le x < \infty$, *then*

$$\left(\int_0^x f(y)dy\right)^p \le p(\beta+1)^{1-p} \int_0^x \left[y^{-\beta}\left(x^{\beta+1} - y^{\beta+1}\right)\right]^{p-1} f^p(y)dy. \tag{2.14}$$

3 Main results

Let us consider the Hardy operators

$$(H_1 f)(x) = \frac{1}{x}\int_0^x f(t)dt, \quad (H_2 f)(x) = \frac{1}{x}\int_x^{\infty} f(t)dt,$$

where f is a non-negative Lebesgue measurable function on $(0, \infty)$.

Theorem 3.1. *Let* p, q *be Lebesgue measurable functions on* $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, *for* $x \in (0, \infty)$, $\beta > -1$ *and* $f \in Q_{\beta}$. *Suppose that* ω_1 *and* ω_2 *are weight functions defined on* $(0, \infty)$.

Then the inequality

$$\|H_1 f\|_{L_{q(x),\omega_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{p'}} C_{p,q} d_p \left\| \frac{t^{\frac{1}{p'}} \left\|\frac{\omega_2(x)}{x}\right\|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.1}$$

holds, where

$$C_{p,q} = \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \underline{p}\left(\frac{1}{\underline{q}} - \frac{1}{\overline{q}}\right)\right) \left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)} \right),$$

$S_1 = \{x \in (0,\infty) : \ p(x) = \underline{p}\}$, $S_2 = (0,\infty)\backslash S_1$ *and*

$$d_p = \left(1 + \frac{\overline{p} - \underline{p}}{\overline{p}} + \|\chi_{S_1}\|_{L_\infty(0,\infty)}\right)^{\frac{1}{\underline{p}}}.$$

Proof. By applying Corollary 2.2 (a) with $p = \underline{p}$, we obtain

$$\|H_1 f\|_{L_{q(x),\omega_2(x)}(0,\infty)} = \|\omega_2(x) H_1 f\|_{L_{q(x)}(0,\infty)} = \left\| \frac{\omega_2(x)}{x} \int_0^x f(t)dt \right\|_{L_{q(x)}(0,\infty)}$$

$$\leq \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{p'}} \left\| \frac{\omega_2(x)}{x} \Big(\int_0^x f^{\underline{p}}(t) t^{\underline{p}-1} dt \Big)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)}.$$

Let

$$J_1 = \left\| \frac{\omega_2(x)}{x} \Big(\int_0^x f^{\underline{p}}(t) t^{\underline{p}-1} dt \Big)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)},$$

then

$$J_1 = \left\| \Big(\int_0^\infty f^{\underline{p}}(t) \chi_{(0,x)}(t) \Big[\frac{\omega_2(x)}{x}\Big]^{\underline{p}} t^{\underline{p}-1} dt \Big)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)}$$

$$= \left\| \Big(\int_0^\infty f^{\underline{p}}(t) \chi_{(0,x)}(t) \Big[\frac{\omega_2(x)}{x}\Big]^{\underline{p}} t^{\underline{p}-1} dt \Big) \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}$$

$$= \left\| \left\| f^{\underline{p}}(t) \chi_{(0,x)}(t) \Big[\frac{\omega_2(x)}{x}\Big]^{\underline{p}} t^{\underline{p}-1} \right\|_{L_1(0,\infty)} \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}.$$

Next, by applying Lemma 2.2, we get

$$J_1 \leq C_{p,q} \Big(\int_0^\infty \left\| [f^{\underline{p}}(t)] \chi_{(0,x)}(t) \Big[\frac{\omega_2(x)}{x}\Big]^{\underline{p}} t^{\underline{p}-1} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} dt \Big)^{\frac{1}{\underline{p}}}$$

$$= C_{p,q} \Big(\int_0^\infty f^{\underline{p}}(t) t^{\underline{p}-1} \left\| \chi_{(0,x)}(t) \Big[\frac{\omega_2(x)}{x}\Big]^{\underline{p}} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} dt \Big)^{\frac{1}{\underline{p}}}$$

$$= C_{p,q} \Big(\int_0^\infty f^{\underline{p}}(t) t^{\underline{p}-1} \left\| \frac{\omega_2(x)}{x} \right\|^{\underline{p}}_{L_{q(x)}(t,\infty)} dt \Big)^{\frac{1}{\underline{p}}}$$

$$= C_{p,q} \left\| f(t) t^{\frac{1}{p'}} \left\| \frac{\omega_2(x)}{x} \right\|_{L_{q(x)}(t,\infty)} \right\|_{L_{\underline{p}}(0,\infty)}.$$

Let $J_2 = \left\| f(t) t^{\frac{1}{p'}} \left\| \frac{\omega_2(x)}{x} \right\|_{L_{q(x)}(t,\infty)} \right\|_{L_{\underline{p}}(0,\infty)}$, then by applying Corollary 2.1, we obtain

$$J_2 \leq d_p \left\| \frac{t^{\frac{1}{p'}} \|\frac{\omega_2(x)}{x}\|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)},$$

consequently

$$\|H_1 f\|_{L_{q(x),\omega_2(x)}(0,\infty)} \leq \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{p'}} C_{p,q} d_p \left\| \frac{t^{\frac{1}{p'}} \|\frac{\omega_2(x)}{x}\|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)}.$$

□

Remark 1. If in inequality (3.1) we put $\beta = 0$, we get inequality (2.6) of Theorem 2.1.

By using Corollary 2.2 (b) with $a = 0$, the following theorem is proved similarly.

Theorem 3.2. *Let p, q Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta < -1$ and $f \in Q^{\beta}$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$. Then the inequality*

$$\|H_1 f\|_{L_{q(x),\omega_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}}|\beta+1|^{-\frac{1}{p'}} C_{p,q} d_p \left\| \frac{t^{\frac{1}{p'}}\|\frac{\omega_2(x)}{x}\|_{L_{q(x)}(0,t)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.2}$$

holds, where $C_{p,q}$ and d_p are the constants in Theorem 3.1.

Theorem 3.3. *Let p, q Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta > -1$ and $f \in Q^{\beta}$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

Then the inequality

$$\|H_1 f\|_{L_{q(x),\omega_2}(0,\infty)}$$
$$\le \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{p'}} C_{p,q} d_p \left\| \frac{\|[t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\frac{1}{p'}}\frac{\omega_2(x)}{x}\|_{L_{q(x)}(t,\infty)}}{\omega_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \tag{3.3}$$

holds, where $C_{p,q}$ and d_p are the constants in Theorem 3.1.

Proof. By applying Corollary 2.3 with $p = \underline{p}$, we have

$$\|H_1 f\|_{L_{q(x),\omega_2(x)}(0,\infty)} = \|\omega_2(x) H_1 f\|_{L_{q(x)}(0,\infty)} = \left\| \frac{\omega_2(x)}{x}\int_0^x f(t)dt \right\|_{L_{q(x)}(0,\infty)}$$
$$\le \underline{p}^{\frac{1}{\underline{p}}}|\beta+1|^{-\frac{1}{p'}} \left\| \frac{\omega_2(x)}{x}\left(\int_0^x [t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\underline{p}-1} f^p(t)dt\right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)}.$$

Let $K_1 = \left\| \frac{\omega_2(x)}{x}\left(\int_0^x [t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{p-1} f^p(t)dt\right)^{\frac{1}{p}} \right\|_{L_{q(x)}(0,\infty)}$, then

$$K_1 = \left\| \left(\int_0^x f^{\underline{p}}(t)\chi_{(0,x)}(t)\left[\frac{\omega_2(x)}{x}\right]^{\underline{p}}[t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\underline{p}-1}dt\right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)}$$
$$= \left\| \left(\int_0^1 f^{\underline{p}}(t)\chi_{(0,x)}(t)\left[\frac{\omega_2(x)}{x}\right]^{\underline{p}}[t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\underline{p}-1}dt\right) \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}.$$

Next, by using Lemma 2.2, we get

$$I_1 \le C_{p,q}\left(\int_0^1 \left\| [f^{\underline{p}}(t)]\chi_{(0,x)}(t)\left[\frac{\omega_2(x)}{x}\right]^{\underline{p}}[t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\underline{p}-1} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} dt\right)^{\frac{1}{\underline{p}}}$$
$$= C_{p,q}\left(\int_0^1 f^{\underline{p}}(t)\left\| \chi_{(0,x)}(t)\left[\frac{[t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\frac{1}{p'}}\omega_2(x)}{x}\right]^{\underline{p}} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} dt\right)^{\frac{1}{\underline{p}}}$$
$$= C_{p,q}\left(\int_0^1 f^{\underline{p}}(t)\left\| \frac{[t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\frac{1}{p'}}\omega_2(x)}{x} \right\|^{\underline{p}}_{L_{q(x)}(t,\infty)} dt\right)^{\frac{1}{\underline{p}}}$$

$$= C_{p,q}\Big\|f(t)\Big\|\frac{[t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\frac{1}{p'}}\omega_2(x)}{x}\Big\|_{L_{q(x)}(t,1)}\Big\|_{L_{\underline{p}}(0,\infty)}.$$

Let

$$K_2 = \Big\|f(t)[t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\frac{1}{p'}}\Big\|\frac{\omega_2(x)}{x}\Big\|_{L_{q(x)}(t,\infty)}\Big\|_{L_{\underline{p}}(0,\infty)},$$

then by applying Corollary 2.1, we obtain

$$K_2 \le d_p\Big\|\frac{[t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\frac{1}{p'}}\|\frac{\omega_2(x)}{x}\|_{L_{q(x)}(t,\infty)}}{\omega_1(x)}\Big\|_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),\omega_1(x)}(0,\infty)},$$

consequently

$$\|H_1 f\|_{L_{q(x),\omega_2(x)}(0,\infty)}$$

$$\le \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{p'}}C_{p,q}d_p\Big\|\frac{\|[t^{-\beta}(x^{\beta+1}-t^{\beta+1})]^{\frac{1}{p'}}\frac{\omega_2(x)}{x}\|_{L_{q(x)}(t,\infty)}}{\omega_1(x)}\Big\|_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),\omega_1(x)}(0,\infty)}.$$

□

Remark 2. If in inequality (3.3) we put $\beta = 0$, we get inequality (2.7) of Theorem 2.2.

Remark 3. For constant $p(x) = q(x) = p$ and $\omega_1(x) = \omega_2(x) = x^{\alpha}$, inequalities (3.1), (3.2) and (3.3) with sharp constants, were proved in [3] and inequalities (3.1) and (3.3) for $\beta = 0$, were earlier proved in [4] and [5].

Now we consider the case $\beta = -1$.

Theorem 3.4. *Let p, q Lebesgue measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ for $x \in (0,\infty)$ and $\beta = -1$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$.*

1) If $f \in Q_{-1}$, then the inequality

$$\|H_2 f\|_{L_{q(x),\omega_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}}C_{p,q}d_p\Big\|\frac{t^{\frac{1}{p'}}(\ln\frac{t}{x})^{\frac{1}{p'}}\|\frac{\omega_2(x)}{x}\|_{L_{q(x)}(0,t)}}{\omega_1(x)}\Big\|_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \quad (3.4)$$

holds.

2) If $f \in Q^{-1}$, then the inequality

$$\|H_1 f\|_{L_{q(x),\omega_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}}C_{p,q}d_p\Big\|\frac{t^{\frac{1}{p'}}(\ln\frac{x}{t})^{\frac{1}{p'}}\|\frac{\omega_2(x)}{x}\|_{L_{q(x)}(t,\infty)}}{\omega_1(x)}\Big\|_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),\omega_1(x)}(0,\infty)} \quad (3.5)$$

holds.

Proof. 1) Let $a = x$ and $b = +\infty$ in (2.9), the

$$\left(\int_x^{\infty} f(t)dt\right) \le p^{\frac{1}{p}}\left(\int_x^{\infty}(t\ \ln\frac{t}{x})^{p-1}f^p(t)dt\right)^{\frac{1}{p}}.$$

We apply this inequality with $p = \underline{p}$ and the rest is similar to the proof of Theorem 3.1.

2) Let $a = 0$ and $b = x$ in $(\overline{2.11})$. The rest is similar to the proof of Theorem 3.1. □

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