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COMPLEXES IN RELATIVE ELLIPTIC THEORY

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Abstract. Given a pair (M, X), where X is a smooth submanifold in a smooth manifold M, we consider complexes of operators associated with this pair. We describe the notion of ellipticity in this situation and prove the Fredholm property for elliptic complexes. As applications, we consider the relative de Rham complex and Dolbeault complex.

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1 Introduction

Relative elliptic theory is a theory of elliptic operators associated with pairs (M, X), where X is a submanifold in an ambient manifold M. It was introduced by Sternin [20, 22] as a theory of boundary value problems with conditions on submanifolds of arbitrary dimension (for the first time such a problem was considered for the polyharmonic equation by Sobolev [18]). More precisely, in relative theory one studies matrix operators (morphisms) of the form

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} : \begin{matrix} H(M) \\ \oplus \\ H(X) \end{matrix} \longrightarrow \begin{matrix} H(M) \\ \oplus \\ H(X) \end{matrix}, \tag{1.1}$$

where

- $H(M)$ and $H(X)$ are some Sobolev spaces on M and X;
- A and D are pseudodifferential operators on M and X;
- B is a boundary operator equal to the composition of pseudodifferential operators on M and X, and the operator of restriction $u \mapsto u|_X$ of functions to the submanifold, while C is a coboundary operator equal to the composition of pseudodifferential operators on M and X, and the operator of extension of functions from the submanifold to the ambient manifold.

It turns out that compositions (and also almost inverses) of operators of form (1.1) contain an additional summand in the left upper corner of the matrix, the so-called Green operator. We mention the analogy between operators (1.1) and Green operators for them with matrix operators in the theory of pseudodifferential boundary value problems in [26, 8, 5, 15, 16]).

Relative elliptic theory was studied by many authors. For instance, ellipticity condition and index formulas for operators (1.1) were obtianed by Sternin [22]; Novikov and Sternin [12, 13] obtained a Riemann-Roch type theorem in elliptic theory for the embedding $X \subset M$; Sternin [21] studied relative theory for submanifolds with singularities and introduced a new class of operators called translators, which act between different submanifolds; Sternin and Shatalov [19], Nazaikinskii and Sternin [11] studied the algebra of general morphisms using the theory of Fourier integral operators; Bohlen and Schulz [3] studied a generalization of relative theory to Lie groupoids, etc.

Meanwhile, complexes of operators often arise in applications. The fundamental examples are given by the de Rham complex and Dolbeault complex on a complex manifold. One of the most important results in the theory of elliptic complexes is the Atiyah–Bott–Lefschetz formula [1, 2], which computes the Lefschetz numbers of endomorphisms of elliptic complexes on compact closed manifolds in terms of invariants of fixed points. We refer the reader to the papers [7, 6, 14, 15, 17] for further developments of the theory of elliptic complexes on closed manifolds and for boundary value problems on manifolds with boundary.

The aim of this paper is to develop the theory of complexes in relative elliptic theory. We consider complexes, whose differentials are given by operators of form (1.1). We describe an ellipticity condition, which guarantees the Fredholm property of a complex in Sobolev spaces. Note that our proof of the Fredholm property is more complicated than that in the classical elliptic theory on compact closed manifolds. Indeed, for a complex of pseudodifferential operators one defines a single operator (as the sum of the original complex and the complex of adjoint operators) such that the original complex has the Fredholm property if and only if the operator has the Fredholm property. It turns out that this construction does not define a bounded operator in relative elliptic theory, since the boundary operators in the original complex act in Sobolev spaces of sufficiently smooth functions, while the coboundary operators in the adjoint complex act in spaces of distributions. Hence, the sum of the original and the adjoint complexes does not define a bounded operator in Sobolev spaces. We overcome this difficulty by reducing our complex to the complex of zero-order operators in L^2-spaces (in this case the original and the adjoint complex can be added). Two examples are considered: the relative de Rham complex and the relative Dolbeault complex.

2 Complexes in relative theory

In the relative theory, we deal with pairs (M, X), where M is a closed smooth manifold and X is its submanifold of codimension ν, $i : X \hookrightarrow M$ denotes the corresponding embedding. We choose a Riemannian metric on M, while X is endowed with the induced Riemannian metric.

Given a complex vector bundle E on M, we define the elementary boundary (restriction) operator

$$i^* : H^s(M, E) \longrightarrow H^{s-\nu/2}(X, E|_X), \quad i^* : u \longmapsto u|_X, \qquad s > \nu/2.$$

We fix a Hermitian metric on E. Then the dual elementary coboundary operator is defined (cf. [20])

$$i_* : H^{-s+\nu/2}(X, E|_X) \longrightarrow H^{-s}(M, E), \qquad s > \nu/2.$$

In this paper we study complexes of bounded operators acting in Sobolev spaces

$$0 \to \begin{matrix} H^{s_0}(M, E_0) \\ \oplus \\ H^{t_0}(X, F_0) \end{matrix} \xrightarrow{d_0} \begin{matrix} H^{s_1}(M, E_1) \\ \oplus \\ H^{t_1}(X, F_1) \end{matrix} \xrightarrow{d_1} \begin{matrix} H^{s_2}(M, E_2) \\ \oplus \\ H^{t_2}(X, F_2) \end{matrix} \xrightarrow{d_2} \dots \xrightarrow{d_{m-1}} \begin{matrix} H^{s_m}(M, E_m) \\ \oplus \\ H^{t_m}(X, F_m) \end{matrix} \to 0 \tag{2.1}$$

where

- E_j, F_j are complex vector bundles on M and X respectively, H^s are Sobolev spaces of vector bundle sections;

- the operators d_j are morphisms in the sense of [21]

$$d_j=\begin{pmatrix} A_j & C_j\\ B_j & D_j\end{pmatrix}.$$

More precisely, this means that A_j and D_j are pseudodifferential operators (ψDOs in what follows) on M and X, respectively, of orders $\operatorname{ord} A_j=s_j-s_{j+1}$, $\operatorname{ord} D_j=t_j-t_{j+1}$, while the boundary and coboundary operators B_j and C_j are equal to

$$B_j=D''_{X,j}i^*D''_{M,j},\qquad C_j=D'_{M,j}i_*D'_{X,j} \tag{2.2}$$

for some ψDOs $D'_{M,j},D''_{M,j}$ and $D'_{X,j},D''_{X,j}$ on M and X respectively. The orders of the operators should also satisfy the conditions

$$\operatorname{ord} D''_{X,j}+\nu/2+\operatorname{ord} D''_{M,j}=s_j-t_{j+1},\quad \operatorname{ord} D'_{M,j}+\nu/2+\operatorname{ord} D'_{X,j}=t_j-s_{j+1},$$

$$s_j-\operatorname{ord} D''_{M,j}-\nu/2>0,\quad t_j-\operatorname{ord} D'_{X,j}<0.$$

- finally, we suppose that (2.1) is a complex: $d_{j+1}d_j=0$ for all j.

Let us note that we can also consider complexes with more general morphisms d_j, in which the right hand sides in (2.2) contain finite sums of operators. Below we treat only boundary operators as in (2.2) for short.

3 Ellipticity condition and main theorem

Let us recall that a complex

$$0\to H_0\xrightarrow{A_0}H_1\xrightarrow{A_1}H_2\xrightarrow{A_2}\ldots\xrightarrow{A_{m-1}}H_m\to 0,$$

where H_j are Hilbert spaces and A_j are bounded operators, has the Fredholm property if all its cohomology spaces $\ker A_j/\mathrm{Im} A_{j-1}$ are finite dimensional.

Our aim is to obtain the ellipticity conditions, which guarantee that complex (2.1) has the Fredholm property. To obtain these conditions, we use the method of frozen coefficients. More precisely, by locality, the ellipticity condition is obtained at each point in M. There are two types of points: points in $M\setminus X$ and points in X.

First, given a point in $M\setminus X$, the components B_j,C_j,D_j are smoothing in a neighborhood of this point. Hence, in this neighborhood our complex reduces to the sequence

$$0\to H^{s_0}(M,E_0)\xrightarrow{A_0}H^{s_1}(M,E_1)\xrightarrow{A_1}H^{s_2}(M,E_2)\xrightarrow{A_2}\ldots\xrightarrow{A_{m-1}}H^{s_m}(M,E_m)\to 0. \tag{3.1}$$

Moreover, this sequence is a complex modulo lower order operators, i.e., $A_{j+1}A_j$ is equal to zero modulo lower order operators. Hence, we require the usual ellipticity condition for this (almost) complex: the symbol complex

$$0\to\pi^*E_0\xrightarrow{\sigma(A_0)}\pi^*E_1\xrightarrow{\sigma(A_1)}\pi^*E_2\xrightarrow{\sigma(A_2)}\ldots\to\pi^*E_m\to 0 \tag{3.2}$$

should be exact on $T^*M\setminus 0$, where $\pi:T^*M\to M$ is the natural projection and $\sigma(A_j)$ stands for the principal symbol of A_j.

Secondly, we consider a point in $X\subset M$. In a neighborhood of this point we choose local coordinates $(x,t)\in\mathbb{R}^k\times\mathbb{R}^\nu$, $k=\dim X$, such that X is given by the equation $t=0$. Moreover, we choose coordinates such that the volume forms on M and X are equal to $dxdt$ and dx respectively. We also denote by (ξ,τ) the dual coordinates in the fibers of T^*M. We freeze the coefficients of the operators in complex (2.1) and make Fourier transform in x (i.e., along the submanifold). This gives us the complex

$$0 \to \begin{matrix} H^{s_0}(\mathbb{R}^\nu, E_{0,x}) \\ \oplus \\ F_{0,x} \end{matrix} \xrightarrow{\sigma(d_0)(x,\xi)} \begin{matrix} H^{s_1}(\mathbb{R}^\nu, E_{1,x}) \\ \oplus \\ F_{1,x} \end{matrix} \xrightarrow{\sigma(d_1)(x,\xi)} \begin{matrix} H^{s_2}(\mathbb{R}^\nu, E_{2,x}) \\ \oplus \\ F_{2,x} \end{matrix} \xrightarrow{\sigma(d_2)(x,\xi)} \ldots$$

$$\ldots \xrightarrow{\sigma(d_{m-1})(x,\xi)} \begin{matrix} H^{s_m}(\mathbb{R}^\nu, E_{m,x}) \\ \oplus \\ F_{m,x} \end{matrix} \to 0. \quad (3.3)$$

Here

- $E_{j,x}$ and $F_{j,x}$ stand for the fibers of the vector bundles E_j and F_j over x;
- $H^s(\mathbb{R}^\nu)$ stand for the Sobolev spaces in the transverse directions to X;
- the operator-valued symbol of the morphism d_j is equal to

$$\sigma(d_j)(x,\xi) = \begin{pmatrix} \sigma(A_j) & \sigma(C_j) \\ \sigma(B_j) & \sigma(D_j) \end{pmatrix}(x,\xi), \qquad \text{where}$$

$$\begin{aligned} &\sigma(A_j)(x,\xi)u(t) = A_j(x,\xi,0,-i\partial)u(t), \\ &\sigma(B_j)(x,\xi)u(t) = D''_{X,j}(x,\xi)j^* D''_{M,j}(x,\xi,0,-i\partial)u(t), \quad \text{here} \quad j^* : u(t) \mapsto u(0), \\ &\sigma(C_j)(x,\xi)q = D'_{M,j}(x,\xi,0,-i\partial)j_* D'_{X,j}(x,\xi)q, \quad \text{here} \quad j_* : q \mapsto q\delta(t), \\ &\sigma(D_j)(x,\xi)q = D_j(x,\xi)q. \end{aligned}$$

Here $\partial = \partial/\partial t$, and for a ψDO $D\left(x, -i\frac{\partial}{\partial x}, t, -i\frac{\partial}{\partial t}\right)$ we denote its principal symbol by $D(x,\xi,t,\tau)$.

Definition 1. Complex (2.1) is *elliptic* if the following conditions are satisfied:

1) symbol complex (3.2) is exact on $T^*M \setminus 0$;

2) symbol complex (3.3) is exact on $T^*X \setminus 0$.

Theorem 3.1. *If complex* (2.1) *is elliptic, then it has the Fredholm property.*

Remark 1. We will show in the proof that the condition of exactness of complex (3.3) can be reduced to a finite-dimensional condition.

Remark 2. Theorem 3.1 and its proof can be generalized to the case of complexes (2.1), where the differentials include the so-called Green operators and are taken from the algebras of morphisms studied in [11]. We did not consider this case explicitly in the present paper because, first, the computations in the proof become even more cumbersome and, secondly, the ellipticity condition in this more general case cannot be reduced to a finite-dimensional condition.

Proof. 1. Let us reduce complex (2.1) to a complex of zero order operators. We consider the commutative diagram

$$\begin{array}{ccc} H^{s_j}(M,E_j) & & H^{s_{j+1}}(M,E_{j+1}) \\ \oplus & \xrightarrow{d_j} & \oplus \\ H^{t_j}(X,F_j) & & H^{t_{j+1}}(X,F_{j+1}) \\ {\scriptstyle I_j}\downarrow & & \downarrow{\scriptstyle I_{j+1}} \\ L^2(M,E_j) & & L^2(M,E_{j+1}) \\ \oplus & \xrightarrow{\tilde d_j} & \oplus \\ L^2(X,F_j) & & L^2(X,F_{j+1}) \end{array}, \quad \text{where} \quad I_j=\begin{pmatrix}\Delta_j^{s_j/2} & 0\\ 0 & \Delta_j^{t_j/2}\end{pmatrix}.$$

Here Δ_j are nonnegative Laplacians on respective vector bundles. So, we obtain the complex

$$0\to \begin{array}{c}L^2(M,E_0)\\ \oplus\\ L^2(X,F_0)\end{array} \xrightarrow{\tilde d_0} \begin{array}{c}L^2(M,E_1)\\ \oplus\\ L^2(X,F_1)\end{array} \xrightarrow{\tilde d_1} \begin{array}{c}L^2(M,E_2)\\ \oplus\\ L^2(X,F_2)\end{array} \xrightarrow{\tilde d_2}\dots\xrightarrow{\tilde d_{m-1}} \begin{array}{c}L^2(M,E_m)\\ \oplus\\ L^2(X,F_m)\end{array}\to 0, \tag{3.4}$$

$$\text{where}\quad \tilde d_j=I_{j+1}d_jI_j^{-1}=\begin{pmatrix}\Delta_{j+1}^{s_{j+1}/2} & 0\\ 0 & \Delta_{j+1}^{t_{j+1}/2}\end{pmatrix}\begin{pmatrix}A_j & C_j\\ B_j & D_j\end{pmatrix}\begin{pmatrix}\Delta_j^{-s_j/2} & 0\\ 0 & \Delta_j^{-t_j/2}\end{pmatrix}=$$
$$=\begin{pmatrix}\Delta_{j+1}^{s_{j+1}/2}A_j\Delta_j^{-s_j/2} & \Delta_{j+1}^{s_{j+1}/2}C_j\Delta_j^{-t_j/2}\\ \Delta_{j+1}^{t_{j+1}/2}B_j\Delta_j^{-s_j/2} & \Delta_{j+1}^{t_{j+1}/2}D_j\Delta_j^{-t_j/2}\end{pmatrix}=\begin{pmatrix}\tilde A_j & \tilde C_j\\ \tilde B_j & \tilde D_j\end{pmatrix}.$$

Clearly, complexes (2.1) and (3.4) are isomorphic.

2. It is well known (see e.g. [15]) that a complex is Fredholm if and only if all its Laplacians are Fredholm operators. The j-th Laplacian associated with complex (3.4) is equal to

$$L_j=\tilde d_{j-1}\tilde d_{j-1}^*+\tilde d_j^*\tilde d_j : \begin{array}{c}L^2(M,E_j)\\ \oplus\\ L^2(X,F_j)\end{array}\longrightarrow \begin{array}{c}L^2(M,E_j)\\ \oplus\\ L^2(X,F_j)\end{array},$$

where we take adjoint operators with respect to inner products in L^2-spaces associated with the metrics on the manifolds and in the vector bundles. We obtain

$$L_j=$$
$$\begin{pmatrix}\tilde A_{j-1}\tilde A_{j-1}^*+\tilde A_j^*\tilde A_j+\tilde B_j^*\tilde B_j+\tilde C_{j-1}\tilde C_{j-1}^* & \tilde A_{j-1}\tilde B_{j-1}^*+\tilde C_{j-1}\tilde C_{j-1}^*+\tilde A_j^*\tilde C_j+\tilde B_j^*\tilde D_j\\ \tilde B_{j-1}\tilde A_{j-1}^*+\tilde D_{j-1}\tilde C_{j-1}^*+\tilde C_j^*\tilde A_j+\tilde D_j^*\tilde B_j & \tilde C_j^*\tilde C_j+\tilde D_j^*\tilde D_j+\tilde B_{j-1}\tilde B_{j-1}^*+\tilde D_{j-1}\tilde D_{j-1}^*\end{pmatrix}.$$

Denote the upper left corner of L_j by $L_{j_{11}}$ and decompose this operator as

$$L_{j_{11}}=L_{M_j}+G_j,$$

where $L_{M_j}=\tilde A_{j-1}\tilde A_{j-1}^*+\tilde A_j^*\tilde A_j$ is a ψDO on M and $G_j=\tilde B_j^*\tilde B_j+\tilde C_{j-1}\tilde C_{j-1}^*$ is the so-called Green operator, see [19, 11, 10].

3. The operator L_{M_j} is elliptic (as a Laplacian of the complex of zero-order operators associated with complex (3.1)). We denote its inverse modulo lower order operators by $L_{M_j}^{-1}$ and consider the product

$$L_j'=\begin{pmatrix}L_{M_j}^{-1} & 0\\ 0 & 1\end{pmatrix}L_j\equiv\begin{pmatrix}1+G_j' & C_j'\\ B_j' & D_j'\end{pmatrix}: \begin{array}{c}L^2(M,E_j)\\ \oplus\\ L^2(X,F_j)\end{array}\longrightarrow \begin{array}{c}L^2(M,E_j)\\ \oplus\\ L^2(X,F_j)\end{array}, \tag{3.5}$$

where $G'_j = L^{-1}_{M_j} G_j, B'_j = L_{j_{21}}, C'_j = L^{-1}_{M_j} L_{j_{12}}$ and $D'_j = L_{j_{22}}$.

4. By [11, 10] the symbol of Green operator (3.5) is the operator-function

$$\sigma(L'_j)(x,\xi) = \begin{pmatrix} 1+\sigma(G'_j) & \sigma(C'_j) \\ \sigma(B'_j) & \sigma(D'_j) \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}^\nu, E_{j,x}) \\ \oplus \\ F_{j,x} \end{matrix} \quad (x,\xi) \longrightarrow \begin{matrix} L^2(\mathbb{R}^\nu, E_{j,x}) \\ \oplus \\ F_{j,x} \end{matrix} \tag{3.6}$$

on the cotangent bundle $T^*X \setminus 0$. Its components are equal to

$$\begin{aligned}\sigma(G'_j)(x,\xi) = \Big[&\Delta_j^{s_j/2} A_{j-1} \Delta_{j-1}^{-s_{j-1}} A^*_{j-1} \Delta_j^{s_j/2}(x,\xi,0,-i\partial) + \Delta_j^{-s_j/2} A^*_j \Delta_{j+1}^{s_{j+1}} A_j \Delta_j^{-s_j/2}(x,\xi,0,-i\partial)\Big]^{-1} \\ &\times \Big[\Delta_j^{-s_j/2}(x,\xi,0,-i\partial) B^*_j \Delta_{j+1}^{t_{j+1}}(x,\xi) B_j \Delta_j^{-s_j/2}(x,\xi,0,-i\partial) \\ &+ \Delta_j^{s_j/2}(x,\xi,0,-i\partial) C_{j-1} \Delta_{j-1}^{-t_{j-1}}(x,\xi) C^*_{j-1} \Delta_j^{s_j/2}(x,\xi,0,-i\partial)\Big],\end{aligned}$$

$$\begin{aligned}\sigma(B'_j)(x,\xi) = &\Delta_j^{t_j/2}(x,\xi) B_{j-1} \Delta_{j-1}^{s_{j-1}} A^*_{j-1} \Delta_j^{s_j/2}(x,\xi,0,-i\partial) \\ &+ \Delta_j^{t_j/2} D_{j-1} \Delta_{j-1}^{-t_{j-1}}(x,\xi) C^*_{j-1} \Delta_j^{s_j/2}(x,\xi,0,-i\partial) \\ &+ \Delta_j^{-t_j/2}(x,\xi) C^*_j \Delta_{j+1}^{s_{j+1}} A_j \Delta_j^{-s_j/2}(x,\xi,0,-i\partial) + \Delta_j^{-t_j/2} D^*_j \Delta_{j+1}^{t_{j+1}}(x,\xi) B_j \Delta_j^{-s_j/2}(x,\xi,0,-i\partial),\end{aligned}$$

$$\begin{aligned}\sigma(C_j)(x,\xi) = \Big[&\Delta_j^{s_j/2} A_{j-1} \Delta_{j-1}^{-s_{j-1}} A^*_{j-1} \Delta_j^{s_j/2}(x,\xi,0,-i\partial) + \Delta_j^{-s_j/2} A^*_j \Delta_{j+1}^{s_{j+1}} A_j \Delta_j^{-s_j/2}(x,\xi,0,-i\partial)\Big]^{-1} \\ &\times \Big[\Delta_j^{s_j/2} A_{j-1} \Delta_{j-1}^{-s_{j-1}}(x,\xi,0,-i\partial) B^*_{j-1} \Delta_j^{t_j/2}(x,\xi) \\ &+ \Delta_j^{s_j/2}(x,\xi,0,-i\partial) C_{j-1} \Delta_{j-1}^{-t_{j-1}}(x,\xi) C^*_{j-1} \Delta_j^{s_j/2}(x,\xi,0,-i\partial_t) \\ &+ \Delta_j^{-s_j/2} A^*_j \Delta_{j+1}^{s_{j+1}}(x,\xi,0,-i\partial) C_j \Delta_j^{-t_j/2}(x,\xi) + \Delta_j^{-s_j/2}(x,\xi,0,-i\partial) B^*_j \Delta_{j+1}^{t_{j+1}} D_j \Delta_j^{-t_j/2}(x,\xi)\Big],\end{aligned}$$

$$\begin{aligned}\sigma(D'_j)(x,\xi) = &\Delta_j^{-t_j/2}(x,\xi) C^*_j \Delta_{j+1}^{s_{j+1}}(x,\xi,0,-i\partial) C_j \Delta_j^{-t_j/2}(x,\xi) + \Delta_j^{-t_j/2} D^*_j \Delta_{j+1}^{t_{j+1}} D_j \Delta_j^{-t_j/2}(x,\xi) \\ &+ \Delta_j^{t_j/2}(x,\xi) B_{j-1} \Delta_{j-1}^{-s_{j-1}}(x,\xi,0,-i\partial) B^*_{j-1} \Delta_j^{t_j/2}(x,\xi) + \Delta_j^{t_j/2} D_{j-1} \Delta_{j-1}^{-t_{j-1}} D^*_{j-1} \Delta_j^{t_j/2}(x,\xi).\end{aligned}$$

By [11, 10] the Fredholm property of (3.5) is equivalent to the invertibility of symbol (3.6) for all $(x,\xi) \in T^*X \setminus 0$.

5. Now, let us make steps 1-3 for symbol complex (3.3). First, we reduce (3.3) to a complex of operators of order zero

$$\begin{matrix} L^2(\mathbb{R}^\nu, E_{0,x}) \\ \oplus \\ F_{0,x} \end{matrix} \xrightarrow{\tilde\sigma(d_0)(x,\xi)} \begin{matrix} L^2(\mathbb{R}^\nu, E_{1,x}) \\ \oplus \\ F_{1,x} \end{matrix} \xrightarrow{\tilde\sigma(d_1)(x,\xi)} \begin{matrix} L^2(\mathbb{R}^\nu, E_{2,x}) \\ \oplus \\ F_{2,x} \end{matrix} \xrightarrow{\tilde\sigma(d_2)(x,\xi)} \ldots \xrightarrow{\tilde\sigma(d_{m-1})(x,\xi)} \begin{matrix} L^2(\mathbb{R}^\nu, E_{m,x}) \\ \oplus \\ F_{m,x} \end{matrix} \tag{3.7}$$

where

$$\tilde\sigma(d_j) = \begin{pmatrix} (\xi^2-\partial^2)^{s_{j+1}/2}\sigma(A_j)(\xi^2-\partial^2)^{-s_j/2} & (\xi^2-\partial^2)^{s_{j+1}/2}\sigma(C_j)|\xi|^{-t_j} \\ |\xi|^{t_{j+1}}\sigma(B_j)(\xi^2-\partial^2)^{-s_j/2} & |\xi|^{t_{j+1}}\sigma(D_j)|\xi|^{-t_j} \end{pmatrix} \equiv \begin{pmatrix} \tilde\sigma(A_j) & \tilde\sigma(C_j) \\ \tilde\sigma(B_j) & \tilde\sigma(D_j) \end{pmatrix}. \tag{3.8}$$

6. The Laplacians $\mathcal{L}_j$ for complex (3.7) are equal to

$$\mathcal{L}_j = \begin{pmatrix} \begin{matrix}\tilde\sigma(A_{j-1})\tilde\sigma^*(A_{j-1}) + \tilde\sigma^*(A_j)\tilde\sigma(A_j) + \\ +\tilde\sigma^*(B_j)\tilde\sigma(B_j) + \tilde\sigma(C_{j-1})\tilde\sigma^*(C_{j-1})\end{matrix} & \begin{matrix}\tilde\sigma(A_{j-1})\tilde\sigma^*(B_{j-1}) + \tilde\sigma(C_{j-1})\tilde\sigma^*(C_{j-1}) + \\ +\tilde\sigma^*(A_j)\tilde\sigma(C_j) + \tilde\sigma^*(B_j)\tilde\sigma(D_j)\end{matrix} \\ \begin{matrix}\tilde\sigma(B_{j-1})\tilde\sigma^*(A_{j-1}) + \tilde\sigma(D_{j-1})\tilde\sigma^*(C_{j-1}) + \\ +\tilde\sigma^*(C_j)\tilde\sigma(A_j) + \tilde\sigma^*(D_j)\tilde\sigma(B_j)\end{matrix} & \begin{matrix}\tilde\sigma^*(C_j)\tilde\sigma(C_j) + \tilde\sigma^*(D_j)\tilde\sigma(D_j) + \\ +\tilde\sigma(B_{j-1})\tilde\sigma^*(B_{j-1}) + \tilde\sigma(D_{j-1})\tilde\sigma^*(D_{j-1})\end{matrix} \end{pmatrix} :$$

$$\begin{matrix} L^2(\mathbb{R}^\nu, E_{j,x}) \\ \oplus \\ F_{j,x} \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}^\nu, E_{j,x}) \\ \oplus \\ F_{j,x} \end{matrix}. \tag{3.9}$$

The upper left corner of $\mathcal{L}_j$ has the following decomposition

$$\mathcal{L}_{j_{11}} = \mathcal{L}_{M_j} + \mathcal{G}_j, \qquad \begin{aligned} \mathcal{L}_{M_j} &= \tilde{\sigma}(A_{j-1})\tilde{\sigma}^*(A_{j-1}) + \tilde{\sigma}^*(A_j)\tilde{\sigma}(A_j), \\ \mathcal{G}_j &= \tilde{\sigma}^*(B_j)\tilde{\sigma}(B_j) + \tilde{\sigma}(C_{j-1})\tilde{\sigma}^*(C_{j-1}). \end{aligned} \tag{3.10}$$

7. The operator $\mathcal{L}_{M_j}$ is invertible (it is the Laplacian for the exact complex with the differentials $\tilde{\sigma}(A_{j-1})$). We now define the operator-function

$$\mathcal{L}'_j = \begin{pmatrix} \mathcal{L}_{M_j}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mathcal{L}_j \equiv \begin{pmatrix} 1+\mathcal{G}'_j & \mathcal{C}_j \\ \mathcal{B}_j & \mathcal{D}_j \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}^\nu, E_{j,x}) \\ \oplus \\ F_{j,x} \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}^\nu, E_{j,x}) \\ \oplus \\ F_{j,x} \end{matrix}, \tag{3.11}$$

where $\mathcal{G}'_j = \mathcal{L}_{M_j}^{-1}\mathcal{G}_j$, $\mathcal{C}_j = \mathcal{L}_{M_j}^{-1}\mathcal{L}_{j_{12}}$ and $\mathcal{B}_j, \mathcal{D}_j$ are lower left and lower right corners of operator (3.9) respectively. Here (we skip arguments for brevity)

$$\begin{aligned} \mathcal{G}'_j = {} & [(\xi^2-\partial^2)^{s_j/2}\sigma(A_{j-1})(\xi^2-\partial^2)^{-s_{j-1}}\sigma^*(A_{j-1})(\xi^2-\partial^2)^{s_j/2} \\ & + (\xi^2-\partial^2)^{-s_j/2}\sigma^*(A_j)(\xi^2-\partial^2)^{s_{j+1}}\sigma(A_j)(\xi^2-\partial^2)^{-s_j/2}]^{-1} \\ & \times [(\xi^2-\partial^2)^{-s_j/2}\sigma^*(B_j)|\xi|^{2t_{j+1}}\sigma(B_j)(\xi^2-\partial^2)^{-s_j/2} + (\xi^2-\partial^2)^{s_j/2}\sigma(C_{j-1})|\xi|^{-2t_{j-1}}\sigma^*(C_{j-1})(\xi^2-\partial^2)^{s_j/2}], \end{aligned}$$

$$\begin{aligned} \mathcal{B}_j = {} & |\xi|^{t_j}\sigma(B_{j-1})(\xi^2-\partial^2)^{s_{j-1}/2}\sigma^*(A_{j-1})(\xi^2-\partial^2)^{s_j/2} + |\xi|^{t_j}\sigma(D_{j-1})|\xi|^{-2t_{j-1}}\sigma^*(C_{j-1})(\xi^2-\partial^2)^{s_j/2} \\ & + |\xi|^{-t_j}\sigma^*(C_j)(\xi^2-\partial^2)^{s_{j+1}}\sigma(A_j)(\xi^2-\partial^2)^{-s_j/2} + |\xi|^{-t_j}\sigma^*(D_j)|\xi|^{2t_{j+1}}\sigma(B_j)(\xi^2-\partial^2)^{-s_j/2}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}_j = {} & [(\xi^2-\partial^2)^{s_j/2}\sigma(A_{j-1})(\xi^2-\partial^2)^{-s_{j-1}}\sigma^*(A_{j-1})(\xi^2-\partial^2)^{s_j/2} \\ & + (\xi^2-\partial^2)^{-s_j/2}\sigma^*(A_j)(\xi^2-\partial^2)^{s_{j+1}}\sigma(A_j)(\xi^2-\partial^2)^{-s_j/2}]^{-1} \\ & \times [(\xi^2-\partial^2)^{s_j/2}\sigma(A_{j-1})(\xi^2-\partial^2)^{-s_{j-1}}\sigma^*(B_{j-1})|\xi|^{t_j} + (\xi^2-\partial^2)^{s_j/2}\sigma(C_{j-1})|\xi|^{-2t_{j-1}}\sigma^*(C_{j-1})(\xi^2-\partial^2)^{s_j/2} \\ & + (\xi^2-\partial^2)^{-s_j/2}\sigma^*(A_j)(\xi^2-\partial^2)^{s_{j+1}}\sigma(C_j)|\xi|^{-t_j} + (\xi^2-\partial^2)^{-s_j/2}\sigma^*(B_j)|\xi|^{2t_{j+1}}\sigma(D_j)|\xi|^{-t_j}], \end{aligned}$$

$$\begin{aligned} \mathcal{D}_j = {} & |\xi|^{-t_j}\sigma^*(C_j)(\xi^2-\partial^2)^{s_{j+1}}\sigma(C_j)|\xi|^{-t_j} + |\xi|^{-t_j}\sigma^*(D_j)|\xi|^{2t_{j+1}}\sigma(D_j)|\xi|^{-t_j} \\ & + |\xi|^{-t_j}\sigma(B_{j-1})(\xi^2-\partial^2)^{-s_{j-1}}\sigma^*(B_{j-1})|\xi|^{t_j} + |\xi|^{t_j}\sigma(D_{j-1})|\xi|^{-2t_{j-1}}\sigma^*(D_{j-1})|\xi|^{t_j}. \end{aligned}$$

8. On one hand, by Steps 1-4 above, the Fredholm property of original complex (2.1) is equivalent to the invertibility of symbol (3.6). On the other hand, the exactness of symbol complex (3.3) is equivalent to the invertibility of symbol (3.11). However, we can see that operators (3.11) and (3.6) coincide. □

Remark 3. It follows from the proof of Theorem 3.1 that the exactness of symbol complex (3.3) is equivalent to the invertibility of operator (3.11). However, the latter operator is actually equal to the identity plus a finite rank operator. Moreover, if we apply Fourier transform in $t \in \mathbb{R}^\nu$ in (3.11) we reduce this operator to the identity plus an integral operator with a degenerate kernel. This shows that the ellipticity condition can be checked explicitly.

Remark 4. Note that complexes (2.1) in relative elliptic theory are well defined typically only for a bounded interval $I \subset \mathbb{R}$ of Sobolev smoothness exponents s_0 (all other smoothness exponents in the complex are uniquely determined by s_0) because the boundary operator i^* acts between spaces of sufficiently smooth functions, while the coboundary operator i_* acts in spaces of distributions (see [11]). It is expected that the ellipticity condition and the cohomology spaces actually do not depend on the Sobolev smoothness exponent $s_0 \in I$, but this problem requires further study, which we intend to carry out elsewhere.

4 Mapping cones

Let us present examples of elliptic complexes. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{s_0}(M,E_0) & \xrightarrow{A_0} & H^{s_1}(M,E_1) & \xrightarrow{A_1} \dots \xrightarrow{A_{m-1}} & H^{s_m}(M,E_m) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle B_0} & & \downarrow{\scriptstyle B_1} & & \downarrow{\scriptstyle B_m} & & \\ 0 & \longrightarrow & H^{t_0}(X,F_0) & \xrightarrow{D_0} & H^{t_1}(X,F_1) & \xrightarrow{D_1} \dots \xrightarrow{D_{m-1}} & H^{t_m}(X,F_m) & \longrightarrow & 0 \end{array} \tag{4.1}$$

where the rows are elliptic complexes of pseudodifferential operators A_j and D_j on M and X, while the B_j's are boundary operators as in (2.2). We suppose that the diagram is commutative, in other words, the vertical mappings define a morphism of complexes. Let us consider the mapping cone for this morphism:

$$0 \to \begin{array}{c} H^{s_0}(M,E_0) \\ \oplus \\ \{0\} \end{array} \xrightarrow{d_0} \begin{array}{c} H^{s_1}(M,E_1) \\ \oplus \\ H^{t_0}(X,F_0) \end{array} \xrightarrow{d_1} \begin{array}{c} H^{s_2}(M,E_2) \\ \oplus \\ H^{t_1}(X,F_1) \end{array} \xrightarrow{d_2} \dots \xrightarrow{d_{m+1}} \begin{array}{c} \{0\} \\ \oplus \\ H^{t_m}(X,F_m) \end{array} \to 0, \tag{4.2}$$

where

$$d_j = \begin{pmatrix} -A_j & 0 \\ B_j & D_{j-1} \end{pmatrix}.$$

The symbol complex for (4.2) at a point $(x,\xi) \in T^*X \setminus 0$ is equal to

$$0 \to \begin{array}{c} H^{s_0}(\mathbb{R}^\nu, E_{0,x}) \\ \oplus \\ \{0\} \end{array} \xrightarrow{\sigma(d_0)(x,\xi)} \begin{array}{c} H^{s_1}(\mathbb{R}^\nu, E_{1,x}) \\ \oplus \\ F_{0,x} \end{array} \xrightarrow{\sigma(d_1)(x,\xi)} \begin{array}{c} H^{s_2}(\mathbb{R}^\nu, E_{2,x}) \\ \oplus \\ F_{1,x} \end{array} \xrightarrow{\sigma(d_2)(x,\xi)} \dots$$

$$\dots \xrightarrow{\sigma(d_{m+1})(x,\xi)} \begin{array}{c} \{0\} \\ \oplus \\ F_{m,x} \end{array} \to 0, \tag{4.3}$$

where

$$\sigma(d_j) = \begin{pmatrix} -\sigma(A_j) & 0 \\ \sigma(B_j) & \sigma(D_{j-1}) \end{pmatrix}.$$

Theorem 4.1. *If the rows in* (4.1) *are elliptic, then mapping cone* (4.2) *is elliptic and the cohomology groups of* (4.2) *and the subcomplex of smooth sections of vector bundles are isomorphic.*

Proof. 1. Let us prove the ellipticity using Theorem 3.1. Since (4.3) is a complex, we have $\sigma(d_{j+1})\sigma(d_j) = 0$. Consequently, $\sigma(A_{j+1})\sigma(A_j) = 0$, $\sigma(D_{j+1})\sigma(D_j) = 0$ and $\sigma(B_{j+1})\sigma(A_j) = \sigma(D_j)\sigma(B_j)$.

Since the rows in (4.3) are elliptic by the assumption, their symbol complexes are exact: $\mathrm{Im}(\sigma(A_j)) = \ker(\sigma(A_{j+1}))$ and $\mathrm{Im}(\sigma(D_j)) = \ker(\sigma(D_{j+1}))$. Let us show that (4.3) is exact. Let us find its kernel:

$$\ker(\sigma(d_j)) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \begin{array}{c} H^{s_j}(\mathbb{R}^\nu, E_{j,x}) \\ \oplus \\ F_{j+1,x} \end{array} \middle| \begin{pmatrix} -\sigma(A_j) & 0 \\ \sigma(B_j) & \sigma(D_{j-1}) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

It corresponds to the system (we use ellipticity of the rows in (4.2))

$$\left\{ \begin{array}{r} -\sigma(A_j)u = 0 \\ \sigma(B_j)u + \sigma(D_{j-1})v = 0 \end{array} \right. \iff \left\{ \begin{array}{r} u = -\sigma(A_{j-1})u_0 \\ -\sigma(B_j)\sigma(A_{j-1})u_0 + \sigma(D_{j-1})v = 0 \end{array} \right. \iff$$

$$\Longleftrightarrow \begin{cases} u = -\sigma(A_{j-1})u_0 \\ v = \sigma(B_{j-1})u_0 + \sigma(D_{j-2})v_0. \end{cases}$$

This is equivalent to $(u,v) \in \mathrm{Im}(\sigma(d_{j-1}))$. As $\mathrm{Im}(\sigma(d_{j-1})) = \ker(\sigma(d_j))$, complex (4.3) is exact and mapping cone (4.2) is elliptic.

2. Let us now give a direct proof of the Fredholm property for complex (4.2). Since its rows are elliptic by the assumption, there exist parametrices $\{P_k\}$, $\{R_k\}$ modulo smoothing operators for the rows. This means (e.g., see [15]) that

$$A_{k-1}P_{k-1} + P_kA_k = 1 + C_k, \quad D_{k-1}R_{k-1} + R_kD_k = 1 + C'_k, \qquad \text{for all } k,$$

where C_k and C'_k are integral operators with smooth kernels. It is straightforward that a parametrix modulo smoothing operators for complex (4.2) is equal to

$$p_k = \begin{pmatrix} -P_k & 0 \\ R_{k-1}B_kP_k & R_{k-1} \end{pmatrix}.$$

Indeed, we have

$$\begin{aligned} &d_{k-1}p_{k-1} + p_kd_k = \\ &= \begin{pmatrix} A_{k-1}P_{k-1} + P_kA_k & 0 \\ -B_{k-1}P_k + D_{k-2}R_{k-2}B_{k-1}P_{k-1} - R_{k-1}B_kP_kA_k + R_{k-1}B_k & D_{k-1}R_{k-1} + R_kD_k \end{pmatrix} = \\ &\begin{pmatrix} 1 + C_k & 0 \\ -B_{k-1}P_k + (1 - R_{k-1}D_{k-1} + C'_{k-1})B_{k-1}P_{k-1} - R_{k-1}B_k(1 - A_{k-1}P_{k-1} + C_{k-1}) + R_{k-1}B_k & 1 + C'_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 + C_k & 0 \\ C''_k & 1 + C'_{k-1} \end{pmatrix}. \end{aligned} \tag{4.4}$$

Since $\{p_k\}$ is a parametrix for (4.2), the latter complex has the Fredholm property.

3. Denote the subspace of smooth sections in the kernel by

$$\ker_{C^\infty} d_k \subset \ker d_k \subset H^{s_k}(M, E_k) \oplus H^{t_{k-1}}(X, F_{k-1})$$

and the range of d_{k-1} on smooth sections by

$$\mathrm{Im}_{C^\infty} d_{k-1} \subset \mathrm{Im} d_{k-1} \subset H^{s_k}(M, E_k) \oplus H^{t_{k-1}}(X, F_{k-1}).$$

We have to show that the identity mapping $u \mapsto u$ defines an isomorphism of cohomology groups

$$\alpha : \ker_{C^\infty} d_k / \mathrm{Im}_{C^\infty} d_{k-1} \xrightarrow{\simeq} \ker d_k / \mathrm{Im} d_{k-1}.$$

Let us show that α is surjective. Indeed, given $u \in \ker d_k$, we use (4.4) and obtain

$$d_{k-1}p_{k-1}u + p_kd_ku = u + Cu,$$

where C is a smoothing operator. Since $d_ku = 0$, we obtain $u = -Cu - d_{k-1}p_{k-1}u$. This means that u is cohomologous to the smooth section $-Cu$. This proves the surjectivity of α.

Let us show that α is injective. Indeed, given a smooth section u such that $[u] \in \ker \alpha$ is equivalent to $u \in \mathrm{Im} d_k$, or $u = dv$, where v is in a suitable Sobolev space. We obtain

$$d_{k-1}p_{k-1}v + p_kd_kv = v + Cv.$$

Hence, $v - d_{k-1}p_{k-1}v = p_kd_kv - Cv = p_ku - Cv$ is a smooth section. Therefore, $u = d_k(v - d_{k-1}p_{k-1}v)$. This implies that $[u] = 0$ and completes the proof of the injectivity of α. □

Example 1. Let us define the relative de Rham complex. To this end, we denote by $\Omega^*(M)$ and $\Omega^*(X)$ the spaces of all differential forms on M and X. Then we consider the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^n(M) & \longrightarrow & 0 \\ & & {\scriptstyle i^*}\downarrow & & {\scriptstyle i^*}\downarrow & & & & {\scriptstyle i^*}\downarrow & & \\ 0 & \longrightarrow & \Omega^0(X) & \xrightarrow{d} & \Omega^1(X) & \xrightarrow{d} & \dots & \xrightarrow{d} & 0 & \longrightarrow & 0 \end{array} \qquad n = \dim M. \tag{4.5}$$

Its rows are the de Rham complexes on M and X and the vertical mappings are induced by the embedding $i : X \to M$. The commutativity $di^* = i^*d$ follows from the naturality of the exterior differential d. Hence, the cone of the morphism i^* is defined

$$0 \to \begin{array}{c} \Omega^0_s(M) \\ \oplus \\ \{0\} \end{array} \xrightarrow{d_0} \begin{array}{c} \Omega^1_{s-1}(M) \\ \oplus \\ \Omega^0_{s-\nu/2}(X) \end{array} \xrightarrow{d_1} \begin{array}{c} \Omega^2_{s-2}(M) \\ \oplus \\ \Omega^1_{s-\nu/2-1}(X) \end{array} \xrightarrow{d_2} \dots \to 0, \tag{4.6}$$

where Ω^*_s stand for differential forms with coefficients in H^s, $s > \dim X + \nu/2$ and

$$d_j = \begin{pmatrix} -d & 0 \\ i^* & d \end{pmatrix}.$$

We can apply Theorem 4.1 and obtain that cone (4.6) is a Fredholm complex and its cohomology coincides with the cohomology of the complex of smooth differential forms.

Denote by $H^*_{dR}(M, X)$ the cohomology spaces of the subcomplex in (4.6) of all smooth differential forms. Since the restriction mapping i^* is surjective, it follows (cf. [17]) that this cohomology group is isomorphic to that of the subcomplex

$$\Omega^*(M, X) = \{\omega \in \Omega^*(M) \mid i^*\omega = 0\}. \tag{4.7}$$

It is well known that $H^*_{dR}(M, X)$ is isomorphic to the singular cohomology $H^*(M, X)$ of the pair (M, X) (e.g., see [4, 9]). Moreover, an explicit isomorphism is defined by the mapping:

$$\omega \in \Omega^*(M, X) \longmapsto I_{dR}\omega \in C^*(M, X),$$

where $C^*(M, X) = \mathrm{Hom}(C_*(M)/C_*(X), \mathbb{R})$ is the space of all singular cochains with real coefficients, $C_*(M), C_*(X)$ are spaces of all singular chains on M and X respectively, while

$$(I_{dR}\omega)(\gamma) = \int_\gamma \omega, \qquad \text{for } \gamma \in C_*(M).$$

Example 2. Let us define the relative Dolbeault complex (cf. [24, 23, 25]). To this end, we suppose that M is a complex manifold and X is its complex submanifold of complex codimension ν. Denote by $\Omega^{0,*}(M)$ and $\Omega^{0,*}(X)$ the spaces of all antiholomorphic differential forms on M and X respectively. In local coordinates $z_1, ..., z_n$ antiholomorphic differential forms are equal to

$$\omega = \sum_I \omega_I(z) d\overline{z}_I, \text{ where } I = (i_1, ..., i_k), \quad d\overline{z}_I = d\overline{z}_{i_1} \wedge \dots \wedge d\overline{z}_{i_k}.$$

Then we consider the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Omega^{0,0}(M) & \xrightarrow{\overline{\partial}} & \Omega^{0,1}(M) & \xrightarrow{\overline{\partial}} & \dots & \xrightarrow{\overline{\partial}} & \Omega^{0,n}(M) & \longrightarrow & 0 \\ & & {\scriptstyle i^*}\downarrow & & {\scriptstyle i^*}\downarrow & & & & {\scriptstyle i^*}\downarrow & & \\ 0 & \longrightarrow & \Omega^{0,0}(X) & \xrightarrow{\overline{\partial}} & \Omega^{1,0}(X) & \xrightarrow{\overline{\partial}} & \dots & \xrightarrow{\overline{\partial}} & 0 & \longrightarrow & 0 \end{array} \tag{4.8}$$

Its rows are the $\overline{\partial}$-complexes on M and X. Recall the definition of the differential:

$$\overline{\partial}\left(\sum_I \omega_I(z)d\overline{z}_I\right) = \sum_I \sum_j \frac{\partial \omega_I}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_I.$$

The vertical mappings in (4.8) are induced by the embedding $i : X \to M$. The commutativity $\overline{\partial} i^* = i^* \overline{\partial}$ follows from the naturality of the $\overline{\partial}$-operator. Hence, the cone of the morphism i^* is defined

$$0 \to \begin{matrix} \Omega^{0,0}_s(M) \\ \oplus \\ \{0\} \end{matrix} \xrightarrow{d_0} \begin{matrix} \Omega^{0,1}_{s-1}(M) \\ \oplus \\ \Omega^{0,0}_{s-\nu}(X) \end{matrix} \xrightarrow{d_1} \begin{matrix} \Omega^{0,2}_{s-2}(M) \\ \oplus \\ \Omega^{0,1}_{s-\nu-1}(X) \end{matrix} \xrightarrow{d_2} \cdots \to 0, \tag{4.9}$$

where $\Omega^{0,*}_s$ stand for differential forms with coefficients in H^s and

$$d_j = \begin{pmatrix} -\overline{\partial} & 0 \\ i^* & \overline{\partial} \end{pmatrix}.$$

We can apply Theorem 4.1 and obtain that cone (4.9) is a Fredholm complex and its cohomology coincides with the cohomology of the subcomplex of all smooth differential forms.

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