ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2020, Volume 11, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Nur-Sultan, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

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The Eurasian Mathematical Journal (EMJ)
The Nur-Sultan Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Nur-Sultan, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 562
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 11, Number 4 (2020), 35 – 44

SOME CLASSES OF OPERATORS IN GENERAL MORREY-TYPE SPACES

M.L. Goldman, E.G. Bakhtigareeva

Communicated by E.D. Nursultanov

Key words: ideal spaces, rearrangement invariant space, local and global Morrey spaces, Morrey-type spaces, symmetrization, embedding, classes of operators.

AMS Mathematics Subject Classification: 46E30, 47A15, 47A30.

Abstract. In this paper we continue the study of general Morrey spaces using a general rearrangement invariant space as a basic space, and a general ideal space as an outer space. Here we consider some classes of positively homogeneous monotone operators from general rearrangement invariant spaces to general Morrey spaces and obtain the estimates for their norms. This approach covers many operators of analysis, such as the embedding and symmetrization operators, Hardy-Littlewood maximal operators, generalized Riesz potentials, Hardy-type operators.

DOI: https://doi.org/10.32523/2077-9879-2020-11-4-35-44

1 Introduction

In this paper we preserve the notation and definitions of our paper [14]. We recall some conclusions from [14]. Let $L_0 = L_0(\mathbb{R}^n)$ be the set of all Lebesgue-measurable functions $f : \mathbb{R}^n \longrightarrow \mathbb{C}$; $\dot{L}_0 = \dot{L}_0(\mathbb{R}^n)$ be the subspace of all functions having the decreasing rearrangement f^* not identical to infinity. Here, f^* is the *decreasing rearrangement* of the function $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ with respect to the $n-$dimensional Lebesgue measure μ_n; namely,

$$f^*(\tau) = \inf \{y > 0 : \lambda_f(y) \le \tau\}, \tau \in \mathbb{R}_+, \tag{1.1}$$

where $\lambda_f(y)$ is the Lebesgue distribution function:

$$\lambda_f(y) = \mu_n \{x \in R^n : |f(x)| > y\}. \tag{1.2}$$

For $f \in \dot{L}_0$ we also consider the symmetric rearrangements

$$f^{\#}(x) = f_0(|x|), x \in \mathbb{R}^n;\ f_0(\rho) = f^*(v_n \rho^n), \rho \in \mathbb{R}_+. \tag{1.3}$$

Here, v_n is the volume of a unit ball in $\mathbb{R}^n$. We will use here the concepts of *an ideal space* (shortly: IS) $F = F(\mathbb{R}_+)$, and *a generalized rearrangement invariant space* (shortly: GRIS) $E = E(\mathbb{R}^n)$. For definitions and properties of these spaces we refer to paper [14]; for more detailed discussion see [1]. The reader who does not need such level of generality may assume that F is *a Banach function space* (shortly: BFS), and E is *a rearrangement invariant space* (shortly: RIS) in the system of axioms developed by C. Bennett and R. Sharpley [2]. Many results in this direction may be found in the book by S.G. Krein et al. [17].

For a basic GRIS $E = E(\mathbb{R}^n)$ and an outer IS $F = F(\mathbb{R}_+)$ we consider the following variants of general Morrey spaces.

1. The local Morrey space:

$$LM_{EF} = \{f \in L_0 : ||f||_{LM_{EF}} = || \, ||f\chi_{B_t}||_E||_F < \infty\} . \tag{1.4}$$

2. The global Morrey space:

$$GM_{EF} = \left\{ f \in L_0 : ||f||_{GM_{EF}} = \sup_{x\in\mathbb{R}^n} || \, ||f\chi_{B(x,t)}||_E||_F < \infty \right\} . \tag{1.5}$$

3. They are closely related to the general variant of Lorentz-type space:

$$M_{EF} = \left\{ f \in \dot{L}_0 : ||f||_{M_{EF}} = || \, ||f^{\#}\chi_{B_t}||_E||_F < \infty \right\} . \tag{1.6}$$

Here,

$$B(x,t) = \{y \in \mathbb{R}^n : |y-x| < t\} , \quad B_t = B(0,t); \quad \chi_\Omega(y) = \begin{cases} 1, & y \in \Omega \\ 0 & y \notin \Omega \end{cases} \tag{1.7}$$

The inner (quasi)norm in (1.4)-(1.6) is calculated with respect to y in $E(\mathbb{R}^n)$; the outer (quasi)norm is calculated with respect to t in $F(\mathbb{R}_+)$.

The classical Morrey spaces LM_p^λ and GM_p^λ we obtain by setting

$$E = L_p, \, 1 \le p \le \infty; \, ||g||_F = \sup_{t>0} \left[t^{-\lambda}|g(t)|\right] , \, 0 \le \lambda \le n/p.$$

The following assumptions give the guarantee of the nontriviality of these spaces

$$||1||_F = \infty; \, ||\chi_{[t_0,\infty)}||_F < \infty, \, \forall t_0 \in \mathbb{R}_+; \tag{1.8}$$

$$\exists t_1 \in \mathbb{R}_+ : ||\varphi_E(|B_t|)||_{F(0,t_1)} < \infty; \, \varphi_E(\tau) = ||\chi_\Omega||_E, \, |\Omega| := \mu_n(\Omega) = \tau \in \mathbb{R}_+. \tag{1.9}$$

Here φ_E is the so-called fundamental function of GRIS E.

Namely, under these assumptions we have $M_{EF} \neq \{0\} , M_{EF} \neq E$, and

$$M_{EF} \subset GM_{EF} \subset LM_{EF} \subset E^{loc} \tag{1.10}$$

(all embeddings are strict). Moreover,

$$||f||_{LM_{EF}} \le ||f||_{GM_{EF}} \le ||f||_{M_{EF}}, \, \forall f \in M_{EF}. \tag{1.11}$$

All the above results were established in [14].

Many examples and descriptions for classical Morrey spaces and their various generalizations can be found in [7]-[13], [15, 16]. We refer to the surveys of V.I. Burenkov [4, 5] where many results and references in this field are contained. This paper is organized as follows. In Section 2 some classes of operators acting from a GRIS to a general Morrey spaces are considered and comparisons of their norms for different variants of Morrey spaces are carried out. In Section 3 we apply this approach to concrete operators, such as the embedding and symmetrization operators, Hardy-Littlewood maximal function, generalized Riesz potential, Hardy-type operators.

2 Classes of operators from a GRIS to a general Morrey-type spaces

We assume that all notation and definitions of Section 1 are preserved. Let $E_1 = E_1(\mathbb{R}^n)$, $E = E(\mathbb{R}^n)$ be GRISs.

Let $F(\mathbb{R}_+)$ be an IS with additional properties:

$$a)\quad ||1||_F = \infty; \tag{2.1}$$

$$b)\quad ||\chi_{[t_0,\infty]}||_F < \infty,\ \forall t_0 \in \mathbb{R}_+. \tag{2.2}$$

Moreover, we assume that

$$\exists t_1 \in \mathbb{R}_+ : ||\varphi_E(|B_t|)||_{F(0,t_1)} < \infty. \tag{2.3}$$

We shall use the notation of spaces LM_{EF}, GM_{EF}, M_{EF}, see formulas (1.4)-(1.6).

For a positively homogeneous operator $A : E_1(\mathbb{R}^n) \to L_0(\mathbb{R}^n)$ we introduce the following norms

$$||A||_G = \sup\left\{||Af||_{GM_{EF}} :\ f \in E_1;\ ||f||_{E_1} \le 1\right\}; \tag{2.4}$$

$$||A||_L = \sup\left\{||Af||_{LM_{EF}} :\ f \in E_1;\ ||f||_{E_1} \le 1\right\}; \tag{2.5}$$

$$||A||_* = \sup\left\{||Af||_{M_{EF}} :\ f \in E_1;\ ||f||_{E_1} \le 1\right\}; \tag{2.6}$$

$$||A||_{*,K} = \sup\left\{||Af||_{M_{EF}} :\ f \in E_1 \cap K;\ ||f||_{E_1} \le 1\right\}. \tag{2.7}$$

Let us note that formulas (2.6), (2.7) make sense if $A : E_1(\mathbb{R}^n) \to \dot{L}_0(\mathbb{R}^n)$. Here, we use the following notation:

$$\tilde{K} = \{h : \mathbb{R}_+ \to [0,\infty) : h \downarrow, h(\rho + 0) = h(\rho), \rho \in \mathbb{R}_+\}; \tag{2.8}$$

$$K = \left\{f(x) = f_0(|x|), x \in \mathbb{R}^n, f_0 \in \tilde{K}\right\}. \tag{2.9}$$

According to (1.11)

$$||f||_{LM_{EF}} \le ||f||_{GM_{EF}} \le ||f||_{M_{EF}}. \tag{2.10}$$

Therefore, for the norms of the operator A we obtain

$$||A||_L \le ||A||_G \le ||A||_*, \tag{2.11}$$

and also, obviously,

$$||A||_{*,K} \le ||A||_*. \tag{2.12}$$

Theorem 2.1. *Let E_1, E be GRISs, and the operator $A : E_1 \to \dot{L}_0$ satisfy the following condition:*

$$\exists c_0 \in [1,\infty);\ (Af)^* \le c_0(Af^\#)^* \Leftrightarrow (Af)^\# \le c_0(Af^\#)^\#. \tag{2.13}$$

Then the two-sided estimate holds

$$||A||_{*,K} \le ||A||_* \le c_0||A||_{*,K}. \tag{2.14}$$

Proof. Let $f \in E_1$. Then $f^\# \in E_1 \cap K$, $||f^\#||_{E_1} = ||f||_{E_1}$ because E_1 is a GRIS. Moreover, by (2.13) we have

$$(Af)^\# \chi_{B_t} \le c_0 (Af^\#)^\# \chi_{B_t} \Rightarrow ||(Af)^\#||_{E(B_t)} \le c_0 ||(Af^\#)^\#||_{E(B_t)}. \tag{2.15}$$

By taking into account the monotonicity of $||\cdot||_F$ we obtain

$$||Af||_{M_{EF}} = || \, ||(Af)^\#||_{E(B_t)}||_F \le c_0 \left[|| \, ||(Af^\#)^\#||_{E(B_t)}||_F \right] = c_0 ||Af^\#||_{M_{EF}}.$$

Thus,

$$||A||_* = \sup\left\{||Af||_{M_{EF}} : f \in E_1;\ ||f||_{E_1} \le 1\right\} \le c_0 \sup\left\{||Af^\#||_{M_{EF}} : f \in E_1;\ ||f||_{E_1} \le 1\right\}.$$

Also,

$$f \in E_1;\ ||f||_{E_1} \le 1 \Rightarrow h = f^\# \in E_1 \cap K; ||h||_{E_1} \le 1.$$

Consequently,

$$\sup\left\{||Af^\#||_{M_{EF}} : f \in E_1;\ ||f||_{E_1} \le 1\right\} \le \sup\left\{||Ah||_{M_{EF}} : h \in E_1 \cap K;\ ||h||_{E_1} \le 1\right\} = ||A||_{*,K},$$

and we obtain (2.14). □

Theorem 2.2. *Let E_1, E be GRISs, and the operator $A : E_1 \to \dot{L}_0$ satisfy the following condition:*

$$\exists c_1 \in [1, \infty);\ (Ah)^\# \chi_{B_t} \le c_1 ((Ah)\chi_{B_t})^\#,\ \forall h \in E_1 \cap K,\ \forall t \in \mathbb{R}_+. \tag{2.16}$$

Then

$$||A||_{*,K} \le c_1 ||A||_L. \tag{2.17}$$

Proof. By (2.16), we have, for $h \in E_1 \cap K$,

$$||(Ah)^\#||_{E(B_t)} = ||(Ah)^\# \chi_{B_t}||_E \le c_1 ||((Ah)\chi_{B_t})^\#||_E = c_1 ||(Ah)\chi_{B_t}||_E.$$

Therefore, for $h \in E_1 \cap K$,

$$||Ah||_{M_{EF}} = || \, ||(Ah)^\#||_{E(B_t)}||_F \le c_1 || \, ||(Ah)\chi_{B_t}||_E||_F = c_1 ||Ah||_{LM_{EF}}.$$

Finally,

$$\begin{aligned} ||A||_{*,K} &= \sup\left\{||Ah||_{M_{EF}} : h \in E_1 \cap K;\ ||h||_{E_1} \le 1\right\} \\ &\le c_1 \sup\left\{||Ah||_{LM_{EF}} : h \in E_1 \cap K;\ ||h||_{E_1} \le 1\right\} \le c_1 ||A||_L. \end{aligned}$$

□

Remark 1. Let us note that the following inequality holds (see [14])

$$(Af)^\# \chi_{B_t} \ge ((Af)\chi_{B_t})^\#,\ f \in E_1. \tag{2.18}$$

In Theorem 2.2 we select the class of operators having inverse estimate (2.16) on the cone $E_1 \cap K$.

We also introduce cones of functions on $\mathbb{R}$ with monotonicity properties. Let us fix a constant $c \in [1, \infty)$. Denote

$$\tilde{K}_c = \{g : \mathbb{R} \to [0, \infty),\ g(-x) = g(x),\ x \in \mathbb{R};\ 0 < x_1 < x_2 \Rightarrow g(x_2) \le c g(x_1)\} \tag{2.19}$$

$$K_c = \left\{ f : \mathbb{R}^n \to [0, \infty),\ f(x) = f_0(|x|),\ x \in \mathbb{R}^n,\ f_0 \in \tilde{K}_c \right\}. \tag{2.20}$$

Note that $g \in \tilde{K}_1 \Rightarrow g|_{\mathbb{R}_+} \in \tilde{K}$; $\quad g \in \tilde{K}, \tilde{g}(x) = g(|x|), x \in \mathbb{R} \Rightarrow \tilde{g} \in \tilde{K}_1$, (see (2.8), (2.9)).

Theorem 2.3. *The following assertions hold*

$$f \in K_1 \Rightarrow f^{\#}\chi_{B_t} = (f\chi_{B_t})^{\#}; \tag{2.21}$$

$$f \in K_c,\ c \in (1,\infty) \Rightarrow (f\chi_{B_t})^{\#} \le f^{\#}\chi_{B_t} \le c(f\chi_{B_t})^{\#}. \tag{2.22}$$

Proof. First, we have implications $f \in K_1 \Rightarrow f\chi_{B_t} \in K_1 \Rightarrow (f\chi_{B_t})^{\#} = f\chi_{B_t} = f^{\#}\chi_{B_t}$, and (2.21) follows. Now we prove (2.22). The left-hand-side inequality was already proved (in [14]). Let us prove the right-hand-side inequality. Let $f \in K_c$. Then, $f(x) = f_0(|x|), x \in \mathbb{R}^n$, $f_0 \in \tilde{K}_c$. For $f_0 \in \tilde{K}_c$ we define

$$\tilde{f}_0(t) = \sup_{\tau \ge t} f_0(\tau), t \ge 0; \tilde{f}_0(y) = \tilde{f}_0(|y|), y \in \mathbb{R}.$$

Then, $\tilde{f}_0 \in \tilde{K}_1; f_0 \le \tilde{f}_0 \le cf_0$. Thus, for $\tilde{f}(x) = \tilde{f}_0(|x|) \in K_1, x \in \mathbb{R}^n$, we have the following:

$$f \le \tilde{f} \le cf; \quad \tilde{f} \in K_1 \Rightarrow \tilde{f}^{\#}\chi_{B_t} = \left(\tilde{f}\chi_{B_t}\right)^{\#}$$

(see (2.21)), and we obtain

$$f \le \tilde{f} \Rightarrow f^{\#} \le \tilde{f}^{\#} \Rightarrow f^{\#}\chi_{B_t} \le \tilde{f}^{\#}\chi_{B_t},$$

$$\tilde{f} \le cf \Rightarrow \tilde{f}^{\#}\chi_{B_t} = \left(\tilde{f}\chi_{B_t}\right)^{\#} \le c\left(f\chi_{B_t}\right)^{\#}.$$

As a result, we come to inequality (2.22). □

Corollary 2.1. *Let the operator $A : E_1 \to \dot{L}_0$ be such that $A : E_1 \cap K_1 \to \dot{L}_0 \cap K_c$, $c \in [1,\infty)$. Then, the following inequality holds for $h \in E_1 \cap K_1$*

$$(\,(Ah)\,\chi_{B_t})^{\#} \le (Ah)^{\#}\,\chi_{B_t} \le c\,(\,(Ah)\,\chi_{B_t})^{\#}. \tag{2.23}$$

Here, if $c = 1$, we obtain the equality

$$(\,(Ah)\,\chi_{B_t})^{\#} = (Ah)^{\#}\,\chi_{B_t},\ \forall h \in E_1 \cap K, \forall t \in \mathbb{R}_+. \tag{2.24}$$

In particular, we have (2.16) with $c_1 = 1$.

3 Application to concrete operators of analysis

Let us consider some examples of operators.

Example 1. Identy and symmetrization operators

Let us consider the following embedding:

$$E_1(\mathbb{R}^n) \subset \dot{L}_0(\mathbb{R}^n). \tag{3.1}$$

Let $A = I$ (the embedding operator). Then,

$$(Af)^{\#} = f^{\#} = (f^{\#})^{\#} = \left(Af^{\#}\right)^{\#}, \tag{3.2}$$

and (2.13) holds with $c_0 = 1$. It gives the equality in (2.14)

$$||A||_{*,K} = ||A||_{*}. \tag{3.3}$$

Moreover, Corollary 2.1 is applicable here, and (2.24) is also true. Thus, we obtain (2.16), (2.17) with $c_1 = 1$. As a result, we have for the identical operator $A = I$:

$$||A||_L = ||A||_G = ||A||_* = ||A||_{*,K}. \tag{3.4}$$

The same is true for the symmetrization operator $A : Af = f^{\#}$.

Example 2. The Hardy-Littlewood maximal operator

Let us consider the maximal operator $M : E_1(\mathbb{R}^n) \to \dot{L}_0(\mathbb{R}^n)$, where:

$$(Mf)(x) = \sup_{t>0} \left\{ |B_t|^{-1} \int_{B(x,t)} |f(y)| d\mu_n(y) \right\}, x \in \mathbb{R}^n. \tag{3.5}$$

Then,

$$(Mf)^{\#} \le c_0 (Mf^{\#})^{\#}, \tag{3.6}$$

where $c_0 = d_2 d_1^{-1}$, and $0 < d_1 \le d_2 < \infty$ are constants in the well- known inequality (see [2])

$$d_1 (Mf)^*(t) \le t^{-1} \int_0^t f^*(\tau) d\tau \le d_2 (Mf)^*(t),\ t \in \mathbb{R}_+. \tag{3.7}$$

Indeed, we have (see details below)

$$(Mf)^*(t) \le d_1^{-1} t^{-1} \int_0^t f^*(\tau) d\tau = d_1^{-1} t^{-1} \int_0^t (f^{\#})^*(\tau) d\tau \le d_1^{-1} d_2 (Mf^{\#})^*(t).$$

At the first step we apply the the left-hand-side inequality of (3.7), then we use the equality $f^* = (f^{\#})^*$. The final step is based on the right -hand-side inequality in (3.7) for the function $f^{\#}$ instead of f. As a result, we obtain (3.6). Inequality (3.6) implies

$$||M||_{*,K} \le ||M||_* \le c_0 ||M||_{*,K}, \tag{3.8}$$

according to $(2.13), (2.14)$. Now, let us note that

$$h \in E_1 \cap K \Rightarrow Mh \in \dot{L}_0 \cap K, \tag{3.9}$$

(see, for example, the corresponding conclusion in [1, (22)-(26)]. Thus, Corollary 2.1 is applicable here, and (2.24) is also true for $A = M$. Thus, we obtain $(2.16), (2.17)$ with $c_1 = 1$. As a result, we have for the maximal operator $A = M : E_1 \to \dot{L}_0$ the following chain of inequalities:

$$||A||_{*,K} \le ||A||_L \le ||A||_G \le ||A||_* \le c_0 ||A||_{*,K}. \tag{3.10}$$

Remark 2. In [13, Section 3] it was shown that in fact we have the equality $\|M\|_L = \|M\|_G$ because the operator M commutes with the shift operator.

Example 3. Generalized Riesz potential

Let us consider the convolution operator

$$A : E_1(\mathbb{R}^n) \to \dot{L}_0(\mathbb{R}^n). \tag{3.11}$$

Here,

$$Af(x) = (G * f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} G(x-y) f(y) dy;\ f \in E_1(\mathbb{R}^n); \tag{3.12}$$

The kernel G satisfies the following conditions

$$G(x) \cong \Phi(|x|), x \in \mathbb{R}^n, \tag{3.13}$$

where $\Phi \in J_n$: that is (see $(2.8), (2.9)$)

$$\Phi \in K;\ \exists c \in \mathbb{R}_+ : \int_0^r \Phi(\rho) \rho^{n-1} d\rho \le c \Phi(r) r^n, r \in R_+.$$

We obtain the classical Riesz potential if $\Phi(\rho) = \rho^{\alpha-n}$, $0 < \alpha < n$.

We introduce

$$\varphi(\tau) = \Phi\left((\tau/v_n)^{1/n}\right) \in J_1, \tau \in \mathbb{R}_+. \tag{3.14}$$

Note that

$$\varphi \in \tilde{K};\ \varphi(t) \leq \frac{1}{t}\int_0^t \varphi(\tau)d\tau \leq \tilde{c}\varphi(t), t \in \mathbb{R}_+. \tag{3.15}$$

By (3.13) we have

$$G^{\#}(x) \cong \Phi(|x|),\ x \in \mathbb{R}^n;\ G^*(\tau) \cong \varphi(\tau), \tau \in \mathbb{R}_+. \tag{3.16}$$

Spaces of potentials $A(E_1)$ were studied in papers [3, 11, 12], where some criteria were found for the boundedness of operator (3.11)-(3.12) and optimal RISs were described for this problem. Here, the following inequalities are of special interest. For
$f \in E_1(\mathbb{R}^n)$, $t \in \mathbb{R}_+$,

$$(Af)^*(t) = (G * f)^*(t) \leq c_3\left(\varphi(t)\int_0^t f^*(\tau)d\tau + \int_t^\infty \varphi(\tau)f^*(\tau)d\tau\right). \tag{3.17}$$

(see [11, (3.8)-(3.11)]). In the case $f = f^{\#}$ the inverse inequality takes place

$$(Af^{\#})^*(t) \geq c_4\left(\varphi(t)\int_0^t f^*(\tau)d\tau + \int_t^\infty \varphi(\tau)f^*(\tau)d\tau\right). \tag{3.18}$$

(see [11, (3.14)-(3.26)]). Therefore, we have inequality (2.13) with $c_0 = c_3c_4^{-1}$.

Moreover, if $h \in K \cap E_1$, then $Ah = (G*h) \in \dot{L}_0$ is the convolution of two functions $h, G \in K$. Note that for such functions, the convolution preserves the properties of radial symmetry and decreases with respect to the spherical radius. Then, $Ah = (G * h) \in \dot{L}_0 \cap K$, and we can apply the result of Corollary 2.1. Therefore, assertions (2.24) and (3.10) hold for this operator.

Example 4. Generalized Hardy operator

For $v \in L_1^{loc}(R)$, we use the following notation (see (2.19))

$$\dot{K}_{v,c} = \left\{\psi : \mathbb{R} \to \mathbb{R},\ \psi(-x) = -\psi(x),\ x \in \mathbb{R};\ \psi(x)\int_0^x v dy \in \tilde{K}_c\right). \tag{3.19}$$

Let us consider a Hardy-type operator

$$A : E_1(\mathbb{R}) \to \dot{L}_0(\mathbb{R}).$$

Here,

$$Af(x) = \psi(x)\int_0^x f(y)v(y)dy,\ f \in E_1(\mathbb{R}). \tag{3.20}$$

Theorem 3.1. *Let*

$$0 \leq v \in L_1^{loc}(\mathbb{R});\ v(-x) = v(x),\ \psi(-x) = -\psi(x), x \in \mathbb{R};\ \psi(x) \in [0,\infty), x \in \mathbb{R}_+.$$

Then, the following assertions take place:

$$1.\ v \in \tilde{K}_C(\mathbb{R}),\ C \in [1,\infty) \Rightarrow |(Af)| \leq 2C(Af^{\#}). \tag{3.21}$$

$$2.\ \textit{Let } \psi \in \dot{K}_{v,D},\ D \in [1,\infty).\ \textit{Then, } f \in \tilde{K}_1 \Rightarrow Af \in \tilde{K}_D. \tag{3.22}$$

Proof. 1. For $v \in \tilde{K}_C(\mathbb{R})$ we introduce $\tilde{v}(t) = \sup_{\tau \geq t} v(\tau)$, $t \geq 0$; $\tilde{v}(y) = \tilde{v}(|y|), y \in \mathbb{R}$.

Then,

$$\tilde{v} \in \tilde{K}_1(\mathbb{R}),\ v \leq \tilde{v} \leq Cv;\ v^* \leq \tilde{v}^* \leq Cv^*. \tag{3.23}$$

First, we prove that

$$|(Af)(x)| \leq 2\psi(|x|) \int_0^{|x|/2} f^{\#}(\xi)\tilde{v}(2\xi)d\xi, |x| > 0. \tag{3.24}$$

For $x > 0$ we have (see explanation below)

$$|(Af)(x)| \leq \psi(x) \int_0^x |f|\tilde{v}dy \leq \psi(x) \int_0^x f_+^*(y)\tilde{v}_+^*(y)dy.$$

The first step is evident; at the second one we define

$$f_+(y) = f(y)\chi_{(0,\infty)}(y);\ \tilde{v}_+(y) = \tilde{v}(y)\chi_{(0,\infty)}(y),$$

and apply the well-known property of rearrangements (see [2, Theorem 2.2]). Furthermore,

$$|f_+| \leq |f| \Rightarrow f_+^*(y) \leq f^*(y) = f^{\#}(\frac{y}{2}), y > 0;$$

$$\tilde{v} \in \tilde{K}_1 \Rightarrow \tilde{v}_+^*(y) = \tilde{v}^*(2y) = \tilde{v}^{\#}(y) = \tilde{v}(y), y > 0.$$

Therefore,

$$|(Af)(x)| \leq \psi(x) \int_0^x f^{\#}(\frac{y}{2})\tilde{v}(y)dy = 2\psi(x) \int_0^{x/2} f^{\#}(\xi)\tilde{v}(2\xi)d\xi,$$

and (3.24) holds.

For $x < 0$ we have (see explanation below)

$$|(Af)(x)| \leq \psi(|x|) \int_0^{|x|} |f(-y)|\tilde{v}(-y)dy \leq \psi(|x|) \int_0^{|x|} f_-^*(y)\tilde{v}_-^*(y)dy.$$

At the first step we change variables in (3.20), and take into account that $v \leq \tilde{v}$. At the second one we define

$$f_-(y) = f(y)\chi_{(-\infty,0)}(y);\ \tilde{v}_-(y) = \tilde{v}(y)\chi_{(-\infty,0)}(y).$$

Then we apply the well-known property of rearrangements. Further steps are the same as before. Namely,

$$|f_-| \leq |f| \Rightarrow f_-^*(y) \leq f^*(y) = f^{\#}(\frac{y}{2}), y > 0;$$

$$\tilde{v} \in \tilde{K}_1 \Rightarrow \tilde{v}_-^*(y) = \tilde{v}^*(2y) = \tilde{v}^{\#}(y) = \tilde{v}(y), y > 0.$$

Thus, we obtain estimate

$$|(Af)(x)| \leq \psi(|x|) \int_0^{|x|} f^{\#}(\frac{y}{2})\tilde{v}(y)dy = 2\psi(|x|) \int_0^{|x|/2} f^{\#}(\xi)\tilde{v}(2\xi)d\xi,$$

which implies (3.24) for $x < 0$. Since $\tilde{v}(2\xi) \leq \tilde{v}(\xi) \leq Cv(\xi)$, we obtain

$$|(Af)(x)| \leq 2C\psi(|x|) \int_0^{|x|} f^{\#}(\xi)v(\xi)d\xi = 2C(Af^{\#})(|x|), |x| > 0. \tag{3.25}$$

Now, we take into account that $(Af^{\#})(|x|) = (Af^{\#})(x)$, and (3.21) follows.

2. Let us prove (3.22). The functions $v, f \in \tilde{K}_1$ are even, and $\psi \in \dot{K}_{v,D}$ is odd, thus Af is even, and $Af \geq 0$. Moreover, the function

$$F(x) = \left(\int_0^x v dy\right)^{-1} \int_0^x f v dy, x \in \mathbb{R},$$

decreases on $\mathbb{R}_+$ (as a mean value of the decreasing function f). For function $\psi(x) \int_0^x v dy \in \tilde{K}_D$ we have:

$$0 < x_1 < x_2 \Rightarrow \psi(x_2) \int_0^{x_2} v dy \leq D\psi(x_1) \int_0^{x_1} v dy.$$

Since, for $0 < x_1 < x_2$

$$Af(x_2) = \left[\psi(x_2) \int_0^{x_2} v dy\right] \left[\left(\int_0^{x_2} v dy\right)^{-1} \int_0^{x_2} f v dy\right]$$

$$\leq D \left[\psi(x_1) \int_0^{x_1} v dy\right] \left[\left(\int_0^{x_1} v dy\right)^{-1} \int_0^{x_1} f v dy\right] = D\psi(x_1) \int_0^{x_1} f v dy = DAf(x_1).$$

This proves (3.22).

□

Remark 3. As a result, for the operator A we have inequalities (2.13), (2.14) with $c_0 = 2C$, and (2.16), (2.17) with $c_1 = D$.

Corollary 3.1. *Let*

$$v \in \tilde{K}_1;\ \psi(x) = \left(\int_0^x v dy\right)^{-1}. \tag{3.26}$$

Then, $\psi \in \dot{K}_{v,1}$*, and assertions* (3.21) *and* (3.22) *hold with* $C = D = 1$*. In particular, this is true for the classical Hardy operator*

$$Hf(x) = \frac{1}{x} \int_0^x f(y) dy. \tag{3.27}$$

Here we have $v = 1$, $\psi(x) = 1/x \in \dot{K}_{v,1}$.

Acknowledgments

This work is supported by the Ministry of Science and Higher Education of the Russian Federation: agreement no. 075-03-2020-223/3 (FSSF-2020-0018).

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Mikhail Lvovich Goldman, Elza Gizarovna Bakhtigareeva
S.M. Nikol'skii Mathematical Institute
Peoples' Friendship University of Russia (RUDN University)
6 Miklukho-Maklaya St,
117198, Moscow, Russian Federation
E-mails: seulydia@yandex.ru, bakhtigareeva-eg@rudn.ru

Received: 10.05.2020