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CORRECT SINGULAR PERTURBATIONS OF THE LAPLACE OPERATOR

B.N. Biyarov, D.L. Svistunov, G.K. Abdrasheva

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Key words: maximal (minimal) operator, singular perturbation of an operator, correct restriction, correct extension, system of eigenvectors.

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Abstract. The work is devoted to the study of the Laplace operator when the potential is a singular generalized function and plays the role of a singular perturbation of the Laplace operator. Abstract theorem obtained earlier by B.N. Biyarov and G.K. Abdrasheva can be applied in this case. The main purpose of the paper is studying the related spectral problems. Singular perturbations for differential operators have been studied by many authors for the mathematical substantiation of solvable models of quantum mechanics, atomic physics, and solid state physics. In all those cases, the problems were self-adjoint. In this paper, we consider non-self-adjoint singular perturbation problems. A new method has been developed that allows investigating the considered problems.

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1 Introduction

Let us present some definitions, notation, and terminology.

In a Hilbert space H, we consider a linear operator L with domain $D(L)$ and range $R(L)$. By the *kernel* of L we mean the set

$$\operatorname{Ker} L = \big\{f \in D(L) : \ Lf = 0\big\}.$$

Definition 1. An operator L is called a *restriction* of an operator L_1, and L_1 is called an *extension* of L, briefly $L \subset L_1$, if:
1) $D(L) \subset D(L_1)$,
2) $Lf = L_1 f$ for all f from $D(L)$.

Definition 2. A linear closed operator L_0 in a Hilbert space H is called *minimal* if there exists the bounded inverse operator L_0^{-1} on $R(L_0)$ and $R(L_0) \neq H$.

Definition 3. A linear closed operator $\widehat{L}$ in a Hilbert space H is called *maximal* if $R(\widehat{L}) = H$ and $\operatorname{Ker} \widehat{L} \neq \{0\}$.

Definition 4. A linear closed operator L in a Hilbert space H is called *correct* if there exists the bounded inverse operator L^{-1} defined on the whole space H.

Definition 5. We say that a correct operator L in a Hilbert space H is a *correct extension* of a minimal operator L_0 (*a correct restriction* of a maximal operator $\widehat{L}$) if $L_0 \subset L$ ($L \subset \widehat{L}$).

Definition 6. We say that a correct operator L in a Hilbert space H is a *boundary correct* extension of a minimal operator L_0 with respect to a maximal operator $\widehat{L}$ if L is simultaneously a correct restriction of the maximal operator $\widehat{L}$ and a correct extension of the minimal operator L_0, that is, $L_0 \subset L \subset \widehat{L}$.

Let $\widehat{L}$ be a maximal linear operator in a Hilbert space H, let L be any known correct restriction of $\widehat{L}$, and let K be an arbitrary linear bounded (in H) operator satisfying the following condition:

$$R(K) \subset \operatorname{Ker} \widehat{L}.$$

Then the operator L_K^{-1} defined by the formula (see [10])

$$L_K^{-1} f = L^{-1} f + K f \tag{1.1}$$

describes the inverse operators to all possible correct restrictions L_K of $\widehat{L}$, i.e., $L_K \subset \widehat{L}$.

Let L_0 be a minimal operator in a Hilbert space H, let L be any known correct extension of L_0, and let K be a linear bounded operator in H satisfying the conditions

a) $R(L_0) \subset \operatorname{Ker} K$,

b) $\operatorname{Ker}(L^{-1} + K) = \{0\}$,

then the operator L_K^{-1} defined by formula (1.1) describes the inverse operators to all possible correct extensions L_K of L_0 (see [10]).

Let L be any known boundary correct extension of L_0, i.e., $L_0 \subset L \subset \widehat{L}$. The existence of at least one boundary correct extension L was proved by Vishik in [12]. Let K be a linear bounded (in H) operator satisfying the conditions

a) $R(L_0) \subset \operatorname{Ker} K$,

b) $R(K) \subset \operatorname{Ker} \widehat{L}$,

then the operator L_K^{-1} defined by formula (1.1) describes the inverse operators to all possible boundary correct extensions L_K of L_0 (see [10]).

Definition 7. A bounded operator A in a Hilbert space H is called *quasinilpotent* if its spectral radius is zero, that is, the spectrum consists of the single point zero.

Definition 8. An operator A in a Hilbert space H is called a *Volterra operator* if A is compact and quasinilpotent.

Definition 9. A correct restriction L of a maximal operator $\widehat{L}$ ($L \subset \widehat{L}$), a correct extension L of a minimal operator L_0 ($L_0 \subset L$) or a boundary correct extension L of a minimal operator L_0 with respect to a maximal operator $\widehat{L}$ ($L_0 \subset L \subset \widehat{L}$), will be called *Volterra* if the inverse operator L^{-1} is a Volterra operator.

Definition 10. A densely defined closed linear operator A in a Hilbert space H is called *formally normal* if

$$D(A) \subset D(A^*), \quad \|Af\| = \|A^* f\| \quad \text{for all } f \in D(A).$$

Definition 11. A formally normal operator A is called *normal* if

$$D(A) = D(A^*).$$

2 Preliminaries

In this section, we present some results for correct restrictions and extensions [4] which are used in Section 3.

Let L_0 be some minimal operator, and let M_0 be another minimal operator related to L_0 by the equation $(L_0u, v) = (u, M_0v)$ for all $u \in D(L_0)$ and $v \in D(M_0)$. Then $\widehat{L} = M_0^*$ and $\widehat{M} = L_0^*$ are maximal operators such that $L_0 \subset \widehat{L}$ and $M_0 \subset \widehat{M}$. The existence of at least one boundary correct extension L was proved by Vishik in [12], that is, $L_0 \subset L \subset \widehat{L}$. In this case, L^* is a boundary correct extension of the minimal operator M_0, that is, $M_0 \subset L^* \subset \widehat{M}$. The inverse operators to all possible correct restrictions L_K of the maximal operator $\widehat{L}$ have the form (1.1), then $D(L_K)$ is dense in H if and only if $\operatorname{Ker}(I + K^*L^*) = \{0\}$. Thus, it is obvious that any correct extension M_K of M_0 is the adjoint of some correct restriction L_K with a dense domain, and vice versa [2]. Finally, all possible correct extensions M_K of M_0 have inverses of the form

$$M_K^{-1}f = (L_K^*)^{-1}f = (L^*)^{-1}f + K^*f, \tag{2.1}$$

where K is an arbitrary bounded linear operator in H with $R(K) \subset \operatorname{Ker}\widehat{L}$ such that

$$\operatorname{Ker}(I + K^*L^*) = \{0\}.$$

It is also clear that $R(M_0) \subset \operatorname{Ker} K^*$. In particular, M_K is a boundary correct extension of M_0 if and only if $R(M_0) \subset \operatorname{Ker} K^*$ and $R(K^*) \subset \operatorname{Ker}\widehat{M}$.

Lemma 2.1. *Let L be a densely defined correct restriction of the maximal operator $\widehat{L}$ in a Hilbert space H. Then the operator KL is bounded on $D(L)$ (that is, $\overline{KL}$ is bounded in H) if and only if*

$$R(K^*) \subset D(L^*).$$

Proof. Let $R(K^*) \subset D(L^*)$. Then, by virtue of $(KL)^* = L^*K^*$, we have that $\overline{KL}$ is bounded in H, where $\overline{KL}$ is the closure of the operator KL in H. Here we have used the boundedness of the operator L^*K^*. Then the operator KL is bounded on $D(L)$. Conversely, let KL be bounded on $D(L)$. Then $\overline{KL}$ is bounded on H, by virtue of $(KL)^* = (\overline{KL})^*$ and that $(KL)^*$ is defined on the whole space H. Then the operator K^* transfers any element f in H to $D(L^*)$. Indeed, for any element g of $D(L)$ we have

$$(Lg, K^*f) = (KLg, f) = (g, (KL)^*f).$$

Therefore, K^*f belongs to the domain $D(L^*)$. □

Lemma 2.2. *Let L_K be a densely defined correct restriction of the maximal operator $\widehat{L}$ in a Hilbert space H. Then $D(L^*) = D(L_K^*)$ if and only if $R(K^*) \subset D(L^*) \cap D(L_K^*)$, where L and K are the operators from representation* (1.1).

Proof. If $D(L^*) = D(L_K^*)$, then from representation (1.1) we easily get

$$R(K^*) \subset D(L^*) \cap D(L_K^*) = D(L^*) = D(L_K^*)$$

Let us prove the converse. If

$$R(K^*) \subset D(L^*) \cap D(L_K^*),$$

then we obtain

$$(L_K^*)^{-1}f = (L^*)^{-1}f + K^*f = (L^*)^{-1}(I + L^*K^*)f, \tag{2.2}$$

$$(L^*)^{-1}f = (L_K^*)^{-1}f - K^*f = (L_K^*)^{-1}(I - L_K^*K^*)f, \tag{2.3}$$

for all f in H. It follows from (2.2) that $D(L_K^*) \subset D(L^*)$, and taking into account (2.3) this implies that $D(L^*) \subset D(L_K^*)$. Thus $D(L^*) = D(L_K^*)$. □

Corollary 2.1. *Let L_K be any densely defined correct restriction of the maximal operator $\widehat{L}$ in a Hilbert space H. If $R(K^*) \subset D(L^*)$ and $\overline{KL}$ is a compact operator in H, then*

$$D(L^*) = D(L_K^*).$$

Proof. Compactness of $\overline{KL}$ implies compactness of L^*K^*. Then $R(I + L^*K^*)$ is a closed subspace in H. It follows from the dense definiteness of L_K that $R(I + L^*K^*)$ is a dense set in H. Hence $R(I + L^*K^*) = H$. Then from equality (2.2) we get $D(L^*) = D(L_K^*)$. □

Lemma 2.3. *If $R(K^*) \subset D(L^*) \cap D(L_K^*)$, then bounded operators $I + L^*K^*$ and $I - L_K^*K^*$ from* (2.2) *and* (2.3)*, respectively, have a bounded inverse defined on H.*

Proof. By virtue of the density of the domains of the operators L_K^* and L^* it follows that the operators $I + L^*K^*$ and $I - L_K^*K^*$ are invertible. Since from (2.2) and (2.3) we have $\operatorname{Ker}(I + L^*K^*) = \{0\}$ and $\operatorname{Ker}(I - L_K^*K^*) = \{0\}$, respectively. From representations (2.2) and (2.3) we also note that $R(I+L^*K^*) = H$ and $R(I-L_K^*K^*) = H$, since $D(L^*) = D(L_K^*)$. The inverse operators $(I+L^*K^*)^{-1}$ and $(I - L_K^*K^*)^{-1}$ of the closed operators $I - L_K^*K^*$ and $I + L^*K^*$, respectively, are closed. Then the closed operators $(I + L^*K^*)^{-1}$ and $(I - L_K^*K^*)^{-1}$, defined on the whole of H, are bounded. □

Under the assumptions of Lemma 2.3 the operators KL and KL_K will be (see [3]) restrictions of the bounded operators $\overline{KL}$ and $\overline{KL_K}$, respectively, where the bar denotes the closure of operators in H. Thus $(I - L_K^*K^*)^{-1} = I + L^*K^*$ and $(I - \overline{KL_K})^{-1} = I + \overline{KL}$.

Next we consider the following statement.

Theorem 2.1. *Let L_K be a densely defined correct restriction of the maximal operator $\widehat{L}$ in a Hilbert space H. If $R(K^*) \subset D(L^*) \cap D(L_K^*)$, where L and K are the operators in representation* (1.1)*, then*

1) *the operator $B_K = (I + \overline{KL})L_K$ is relatively bounded correct perturbations of correct restriction L_K and the spectra of the operators B_K and L coincide, that is, $\sigma(B_K) = \sigma(L)$;*

2) *the operator L is a quasinilpotent (the Volterra) boundary correct extension of L_0, and B_K is a quasinilpotent (the Volterra) correct operator, simultaneously;*

3) *if L is an operator with discrete spectrum, then the system of root vectors of L is complete (the basis) in H if and only if the system of root vectors of B_K is complete (the basis) in H;*

4) *in particular, when L is a normal operator with discrete spectrum, then the system of root vectors of the operator B_K forms a Riesz basis in H.*

Proof. 1. Note that $B_K^{-1} = L_K^{-1}(I - \overline{KL_K})$, and $(I - \overline{KL_K})L_K^{-1} = L_K^{-1} - K = L^{-1}$. The correctness of the operator B_K is obvious. For bounded operators R and S it is known (see [1]) that $\sigma(RS) \setminus \{0\} = \sigma(SR) \setminus \{0\}$. Thus, Statement 1) is proved.

2. Note that $B_K^{-1} = (I - \overline{KL_K})^{-1}L^{-1}(I - \overline{KL_K})$. It follows easily by Lemma 2.2 and Lemma 2.3 that the operators $I - \overline{KL_K}$ and $(I - \overline{KL_K})^{-1}$ are bounded and defined on the whole of H. It is then obvious that the operators L^{-1} and B_K^{-1} are quasinilpotent (the Volterra), simultaneously. Statement 2) is proved.

3. From the known facts of functional analysis (see [11]) it follows that the systems root vectors of the operators L and B_K are complete (the basis), simultaneously.

4. The system of root vectors of the normal discrete correct operator L forms an orthonormal basis in H. Hence, the system of root vectors of the correct operator B_K forms a Riesz basis in H.

□

Example 1. In the Hilbert space $L_2(0,1)$, let us consider the minimal operator L_0 generated by the differentiation operator

$$\widehat{L}y = y', \quad D(\widehat{L}) = W_2^1(0,1).$$

Then

$$D(L_0) = \{y \in W_2^1(0,1) : \ y(0) = y(1) = 0\}.$$

The action of the maximal operator $\widehat{M} = L_0^*$ has the form

$$\widehat{M}v = -v', \quad D(\widehat{M}) = W_2^1(0,1).$$

Then

$$D(M_0) = \{v \in W_2^1(0,1) : \ v(0) = v(1) = 0\}.$$

As a fixed boundary correct extension L of L_0 we take the operator acting as the maximal operator $\widehat{L}$ on the domain

$$D(L) = \{y \in D(\widehat{L}) : \ y(0) = 0\}.$$

Then all possible correct restrictions L_K of $\widehat{L}$ have the following inverses

$$L_K^{-1}f(x) = L^{-1}f(x) + Kf(x) = \int_0^x f(t)dt + \int_0^1 f(t)\overline{\sigma(t)}dt,$$

where $\sigma \in L_2(0,1)$ defines the operator K. The domain $D(L_K)$ of L_K is defined as

$$D(L_K) = \{y \in W_2^1(0,1) : \ y(0) = \int_0^1 y'(t)\overline{\sigma(t)}dt\}.$$

Then $D(L_K)$ is not dense in $L_2(0,1)$ if and only if $\sigma \in W_2^1(0,1)$, $\sigma(1) = 0$, and $\sigma(0) = -1$. If we exclude such σ from $L_2(0,1)$, then there exists L_K^* which has the inverse of the form

$$(L_K^*)^{-1}g = (L_K^{-1})^*g = (L^*)^{-1}g + K^*g \quad \text{for all } \ g \in L_2(0,1).$$

This is a description of inverse operators of all possible correct extensions L_K^* of M_0. Let the condition of Theorem 2.1 hold. Then $\sigma \in W_2^1(0,1)$, $\sigma(1) = 0$, and $\sigma(0) \neq -1$. Let us construct the following operators

$$\overline{KL}f = -\int_0^1 f(t)\sigma'(t)dt,$$
$$\overline{KL_K}f = -\frac{1}{1+\overline{\sigma(0)}}\int_0^1 f(t)\sigma'(t)dt.$$

Note that

$$L_K^*v(x) = -v'(x) + \frac{\sigma'(x)}{1+\sigma(0)}v(0),$$
$$D(L_K^*) = D(L^*) = \{v \in W_2^1(0,1) : \ v(1) = 0\}.$$

Then the operator B_K has the following form

$$B_K u(x) = u'(x) - \int_0^1 u'(t)\overline{\sigma'(t)}dt,$$
$$D(B_K) = D(L_K) = \{u \in W_2^1(0,1) : \ u(0) = \int_0^1 u'(t)\overline{\sigma(t)}dt\},$$

where $\sigma \in W_2^1(0,1)$, $\sigma(1) = 0$, and $\sigma(0) \neq -1$. By virtue of Theorem 2.1 B_K is a Volterra correct operator. We know that for a first order differentiation operator there are no Volterra correct restrictions or correct extensions, except for the Cauchy problem at some point $x = d$, $0 \leq d \leq 1$. However, the operator B_K is neither a correct restriction of $\widehat{L}$ nor a correct extension of L_0. This Volterra problem is obtained by perturbation of the differentiation operator itself and the Cauchy boundary conditions, simultaneously.

Example 2. If in Example 1 as a fixed boundary correct operator L we take the operator $\widehat{L}$ with the domain

$$D(L) = \{y \in W_2^1(0,1) : \ y(0) + y(1) = 0\},$$

then L is a normal operator. In this case, the operator B_K has the form

$$B_K y(x) = y'(x) - \int_0^1 y'(t)\overline{\sigma'(t)}dt,$$
$$D(B_K) = \{y \in W_2^1(0,1) : \ y(0) + y(1) = 2\int_0^1 y'(t)\overline{\sigma(t)}dt\},$$

where $\sigma(x) \in W_2^1(0,1)$, $\sigma(0) + \sigma(1) = 0$, and $\sigma(0) \neq -\frac{1}{2}$. The operator B_K is correct and its system of root vectors forms a Riesz basis in $L_2(0,1)$. The eigenvalues of the normal operator L and the correct operator B_K coincide.

Corollary 2.2. *The results of Theorem 2.1 are also valid for the operator*

$$B_K^* = L_K^*(I + L^*K^*).$$

All four statements are valid for the pair of operators B_K^* and L^*.

Remark 1. The results of Examples 1–2 are also valid for the operator B_K^*, which has the form

$$B_K^* v(x) = -\frac{d}{dx}[v(x) - \sigma'(x)\int_0^1 v(t)dt],$$
$$D(B_K^*) = \{v \in L_2(0,1) : \ v(x) - \sigma'(x)\int_0^1 v(t)dt \in D(L^*)\},$$

where $\sigma \in W_2^1(0,1)$, $\sigma(1) = 0$, and $\sigma(0) \neq -1$, in the case of Example 1, and

$$\sigma \in W_2^1(0,1), \quad \sigma(0) + \sigma(1) = 0,$$

and $\sigma(0) \neq -\frac{1}{2}$, in the case of Example 2.

We recall that the conditions $\sigma(0) \neq -1$ and $\sigma(0) \neq -\frac{1}{2}$ provide the density of the domain $D(L_K)$ in H.

The latest results that are closely related to our area of study one may find in [6]-[9]. In these papers singular perturbations of the Laplace operator were investigated. There the method of Green's function was used. They studied various spectral properties of the singular perturbations of Laplace operator.

In this paper, we study correct singular perturbations of the Laplace operator in explicit form by using Theorem 2.1.

3 Main results

In the Hilbert space $L_2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^m$ with an infinitely smooth boundary $\partial\Omega$, let us consider the minimal L_0 and maximal $\widehat{L}$ operators generated by the Laplace operator

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_m^2}\right). \tag{3.1}$$

The closure L_0, in the space $L_2(\Omega)$ of Laplace operator (3.1) with the domain $C_0^\infty(\Omega)$, is *the minimal operator corresponding to the Laplace operator*. The operator $\widehat{L}$, adjoint to the minimal operator L_0 corresponding to Laplace operator, is *the maximal operator corresponding to the Laplace operator* (see [5]). Note that

$$D(\widehat{L}) = \{u \in L_2(\Omega) : \ \widehat{L}u = -\Delta u \in L_2(\Omega)\}.$$

Denote by L_D the operator, corresponding to the Dirichlet problem with the domain

$$D(L_D) = \{u \in W_2^2(\Omega) : \ u|_{\partial\Omega} = 0\}.$$

Then, by virtue of (1.1), the inverse operators L^{-1} to all possible correct restrictions of the maximal operator $\widehat{L}$ corresponding to the Laplace operator (3.1) have the following form:

$$L^{-1}f = L_D^{-1}f + Kf,$$

where, by virtue of (1.1), K is an arbitrary linear operator bounded in $L_2(\Omega)$ with

$$R(K) \subset \operatorname{Ker}\widehat{L} = \{u \in L_2(\Omega) : \ -\Delta u = 0\}.$$

Then the operator L is determined by

$$\widehat{L}u = -\Delta u,$$
$$D(L) = \{u \in D(\widehat{L}) : \ [(I - K\widehat{L})u]|_{\partial\Omega} = 0\},$$

where I is the identity operator in $L_2(\Omega)$. There are no other linear correct restrictions of the operator $\widehat{L}$ (see [2]). The operators $(L^*)^{-1}$, corresponding to the adjoint operators L^*

$$(L^*)^{-1}g = L_D^{-1}g + K^*g,$$

describe the inverse operators to all possible correct extensions of L_0 if and only if K satisfies the condition (see [2])

$$\operatorname{Ker}(I + K^*L^*) = \{0\}.$$

Note that the last condition is equivalent to $\overline{D(L)} = L_2(\Omega)$.

By applying Theorem 2.1 to this particular case we have

Theorem 3.1. *Let the operator K have the form*

$$Kf(x) = \omega(x)\iint\limits_{\Omega} f(\xi)\overline{g(\xi)}d\xi, \quad x,\ \xi \in \Omega \subset \mathbb{R}^m,$$

where ω is a harmonic function in $L_2(\Omega)$, $g \in L_2(\Omega)$, and

$$K^*f(x) = g(x)\iint\limits_{\Omega} f(\xi)\overline{\omega(\xi)}d\xi.$$

If K satisfies the assumptions of Theorem 2.1, then $g \in W_2^2(\Omega)$, $g(x)\mid_{\partial\Omega}=0$,

$$\iint\limits_{\Omega} (\Delta g)(\xi)\overline{\omega(\xi)}d\xi \neq 1,$$

and the correct operator

$$B_K u(x) = -\Delta u(x) - \omega(x) \iint\limits_{\Omega} (\Delta u)(\xi)(\Delta \overline{g})(\xi)d\xi,$$

$$D(B_K) = \Big\{u \in W_2^2(\Omega):\ \big(u(x) + \omega(x) \iint\limits_{\Omega} (\Delta u)(\xi)\overline{g(\xi)}d\xi\big)\mid_{\partial\Omega}=0\Big\}$$

describes a relatively bounded perturbation of L which has the same eigenvalues as the Dirichlet operator L_D.

The system of root vectors of B_K forms a Riesz basis in $L_2(\Omega)$. Morever, if $\{v_k\}$ is an orthonormal system of eigenfunctions of L_D (Dirichlet problem), then the system of eigenvectors $\{u_k\}$ of B_K has the form

$$u_k(x) = ((I + \overline{KL})v_k)(x) = v_k(x) + \omega(x) \iint\limits_{\Omega} v_k(\xi)(\Delta \overline{g})(\xi)d\xi, \quad k = 1, 2, \ldots$$

Consider a more visual case when $m = 2$, that is, $\Omega \subset \mathbb{R}^2$. To do this, we define the operator K by using the function g constructed in the following way. Let $x^{(k)} = (x_1^{(k)}, x_2^{(k)}) \in \Omega,\ k = 1, 2, ..., n$ be points lying strictly inside the closed domain $\overline{\Omega}$. We take a holomorphic function $F(z) \in L_2(\Omega)$ in the domain Ω such that $F(z_k) = 0$, where $z_k = x_1^{(k)} + ix_2^{(k)}$, $k = 1, 2, \ldots, n$, with multiplicities m_k. As functions $g(x_1, x_2)$ we take the solution of the following Dirichlet problem

$$-(\Delta g)(x) = \ln|F(z)|, \quad g|_{\partial\Omega} = 0. \tag{3.2}$$

Then, in a neighborhood of the point z where $F(z) \neq 0$ there is an analytic branch $\Phi(z)$ of the function $\ln F$, hence $\ln|F| = \operatorname{Re}\Phi$ is a harmonic function. In a neighborhood of z_k we can write

$$\begin{aligned} &F(z) = (z - z_k)^{m_k}\Phi(z), \\ &\ln|F(z)| = m_k \ln|z - z_k| + \ln|\Phi(z)|, \end{aligned}$$

where $\Phi(z_k) \neq 0$, $k = \overline{1, n}$. Then by Theorem 3.3.2 of [5] and the harmonicity of the function $\ln|\Phi(z)|$ we get

$$(\Delta \ln|F|)(x) = 2\pi m_k \delta(x - x^{(k)})$$

in this neighborhood. We verify the assumptions of Theorem 2.1, taking into account that Ω and

$$Kf(x) = w(x) \iint\limits_{\Omega} f(\xi)\overline{g(\xi)}d\xi, \quad x \in \Omega,$$

where w is a harmonic function from $L_2(\Omega)$ and g is a solution of the Dirichlet problem (3.2). Then $g \in W_2^2(\Omega)$, $g(x)|_{\partial\Omega} = 0$, and

$$\iint\limits_{\Omega} \ln|F(\zeta)|\overline{w(\xi)}d\xi \neq 1,$$

where $\zeta = \xi_1 + i\xi_2$ and $\xi = (\xi_1, \xi_2)$. If we denote by T the following bounded operator in $L_2(\Omega)$

$$Tu(x) = w(x) \int\limits_{\partial\Omega} \left[\frac{\partial u(\xi)}{\partial n} \ln |F(\zeta)| - u(\xi) \frac{\partial}{\partial n} \ln |F(\zeta)| \right] ds,$$

we get the following

$$B_K u(x) = -(\Delta u)(x) + 2\pi w(x) \sum_{k=1}^{n} m_k u(x^{(k)}) - (Tu)(x),$$

where $x^{(k)} = (x_1^{(k)}, x_2^{(k)}) \in \Omega$. The domain of the operator B_K has the form

$$D(B_K) = \Big\{ u \in W_2^2(\Omega) : \\ \left[u(x) + w(x) \int\limits_{\partial\Omega} u(\xi) \frac{\partial \ln |F(\zeta)|}{\partial n} ds - w(x) \iint\limits_{\Omega} u(\xi) \ln |F(\zeta)| d\xi \right] \Big|_{\partial\Omega} = 0 \Big\}.$$

We have obtained a relatively bounded perturbation B_K of L_D which has the same eigenvalues as the operator L_D. The system of root vectors of B_K forms a Riesz basis in $L_2(\Omega)$. If $\{v_k\}$ is an orthonormal system of eigenfunctions of L_D, then the system of eigenfunctions $\{u_k\}$ of B_K has the form

$$u_k(x) = ((I + \overline{KL}) v_k)(x) = v_k(x) + w(x) \iint\limits_{\Omega} v_k(\xi) \ln \overline{|F(\zeta)|} d\xi, \quad k = 1, 2, \ldots .$$

Thus, we have constructed a singular perturbation of the Dirichlet problem for the Laplace operator with a basic system of root vectors. This perturbation is a correct non-self-adjoint operator with real spectrum, which is not a restriction of the maximal operator $\widehat{L}$ and is not an extension of the minimal operator L_0.

Using the properties of subharmonic functions, one can obtain a similar result in the case $m > 2$.

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