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ALMOST PERIODIC AT INFINITY FUNCTIONS FROM HOMOGENEOUS SPACES AS SOLUTIONS TO DIFFERENTIAL EQUATIONS WITH UNBOUNDED OPERATOR COEFFICIENTS

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Key words: almost periodic at infinity function, vanishing at infinity function, homogeneous space, Banach module, differential equations, asymptotically finite-dimensional operator semigroup.

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Abstract. By using the subspace of functions from homogeneous spaces with integrals decreasing at infinity we define new classes of functions almost periodic at infinity. We obtain spectral criteria for a function to be almost periodic at infinity and asymptotically almost periodic (with respect to the chosen subspace). These results are used for deriving criteria for almost periodicity at infinity of bounded solutions to differential equations with unbounded operator coefficients. In addition, for the new class of asymptotically finite-dimensional operator semigroups we prove the almost periodicity at infinity of their orbits.

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1 Introduction

Many authors studied the problem of almost periodicity of bounded functions on the real axis $\mathbb{R} = (-\infty, +\infty)$ and semi-axis $\mathbb{R}_+ = [0, \infty)$ presenting solutions to differential and functional equations (see [1–10]). One of the first articles in this direction was written by Bochner and von Neumann [17]. They established the almost periodicity of bounded solutions to difference equations. In [24] Sobolev obtained a criterion for almost periodicity of solutions to parabolic equations with a selfadjoint operator on a Hilbert space.

In this article, a family is introduced of subspaces of vanishing at infinity (in certain senses) functions belonging to homogeneous spaces. We give four equivalent definitions (see Theorem 5.1) of functions almost periodic at infinity with respect to a chosen subspace of functions vanishing at infinity. These functions are used in studying bounded solutions to differential equations with unbounded operator coefficients.

While studying differential equations we use the method developed in [4, 5]. It is based on showing that the set of nonbelonging (see Definition 19) of a bounded solution to the differential equation

$$\dot{x}(t) - Ax(t) = \psi(t),\ t \in \mathbb{R}, \tag{1.1}$$

is included in $\sigma(A) \cap (i\mathbb{R})$, with $\sigma(A)$ standing for the spectrum of an operator A, where $A : D(A) \subset X \to X$ is a generator of a strongly continuous operator semigroup, X is a complex Banach space and $\psi : \mathbb{R} \to X$ is a continuous almost periodic function. This method has become the most common for studying the almost periodicity of bounded solutions to differential equations (see also [1, 3, 22]).

As in [5, 6], the derivation of the criterion of almost periodicity at infinity of bounded solutions to (1.1) with the function ψ almost periodic at infinity (with respect to a subspace) is based on the description of the Beurling spectrum of a bounded solution to this equation (Theorem 4.1) and a criterion for almost periodicity of a vector (Theorem 4.1). The main result of this article appears in Theorems 6.3 and 6.4: we obtain a sufficient condition for almost periodicity at infinity of a mild solution to (1.1) with the function ψ almost periodic at infinity with respect to some subspace of functions vanishing at infinity.

Section 7 introduces the new class of asymptotically finite-dimensional operator semigroups. We establish a theorem on the almost periodicity at infinity of the orbits of the operator semigroup under consideration.

2 Homogeneous function spaces

Let X be a complex Banach space, $End\, X$ be a Banach algebra of bounded linear operators on X. Let $\mathbb{J}$ be either $\mathbb{R}_+ = [0,\infty)$ or $\mathbb{R} = (-\infty,\infty)$. By $L^1_{loc}(\mathbb{J},X)$ we denote the space of Bochner measurable locally integrable on $\mathbb{J}$ (classes of) functions with values in X. By $S^p(\mathbb{J},X)$, $p\in[1,\infty)$, we denote the Stepanov space of functions [21] $x\in L^1_{loc}(\mathbb{J},X)$ with the norm

$$\|x\|_{S^p} = \sup_{s\in\mathbb{J}}\left(\int\limits_0^1 \|x(s+t)\|^p_X dt\right)^{1/p}, \ p\in[1,\infty).$$

The Stepanov function spaces play an important role when studying differential equations in Banach spaces (see [9]). By $C_b(\mathbb{J},X)$ we denote the space of bounded continuous functions with the norm $\|x\|_\infty = \sup\limits_{t\in\mathbb{J}}\|x(t)\|_X$, $x\in C_b(\mathbb{J},X)$.

In the case $X=\mathbb{C}$ we omit 'X' in designations of the spaces mentioned above. For example, $C_b(\mathbb{R}) = C_b(\mathbb{R},\mathbb{C})$.

Definition 1. A Banach space $\mathscr{F}(\mathbb{R},X)$ of functions defined on $\mathbb{R}$, with values in X is called *homogeneous,* if the following conditions are satisfied:
(a) the space $\mathscr{F}(\mathbb{R},X)$ is injectively (that means injectivity of the inclusion operator) and continuously embedded in $S^1(\mathbb{R},X)$;
(b) the group of shift operators $S(t)$ for $t\in\mathbb{R}$, where

$$(S(t)x)(s) = x(s+t),\ s,t\in\mathbb{R},\ x\in\mathscr{F}(\mathbb{R},X), \tag{2.1}$$

is defined and bounded on $\mathscr{F}(\mathbb{R},X)$;
(c) for any $f\in L^1(\mathbb{R})$ and $x\in\mathscr{F}(\mathbb{R},X)$ their convolution defined by

$$(f*x)(t) = \int\limits_{\mathbb{R}} f(\tau)x(t-\tau)d\tau = \int\limits_{\mathbb{R}} f(\tau)(S(-\tau)x)(t)d\tau,\ t\in\mathbb{R}, \tag{2.2}$$

belongs to $\mathscr{F}(\mathbb{R},X)$ and satisfies the inequality $\|f*x\|\le C\|f\|\|x\|$ for some $C\ge 1$ independent of x and f (usually, $C=1$);
(d) the inclusion $\varphi x\in\mathscr{F}(\mathbb{R},X)$ holds for any $x\in\mathscr{F}(\mathbb{R},X)$ and any infinitely differentiable function $\varphi\in C_b(\mathbb{R})$ which has a compact support $\operatorname{supp}\varphi$; in addition, the inequality $\|\varphi x\|\le\|\varphi\|\|x\|$ holds and the mapping $t\mapsto\varphi S(t)x : \mathbb{R}\to\mathscr{F}(\mathbb{R},X)$ is continuous.

By $\mathscr{F}_0(\mathbb{R},X)$ we denote the least closed subspace of $\mathscr{F}(\mathbb{R},X)$ containing all functions φx, $x\in\mathscr{F}(\mathbb{R},X)$, where $\varphi\in C_b(\mathbb{R},X)$ is infinitely differentiable and $\operatorname{supp}\varphi$ is a compact set.

Definition 2. A Banach space $\mathscr{F}(\mathbb{R}_+, X)$ of functions from $S^1(\mathbb{R}_+, X)$ is called *homogeneous,* if there exists a homogeneous space $\mathscr{F}(\mathbb{R}, X)$ associated with the space $\mathscr{F}(\mathbb{R}_+, X)$ such that for any function $x \in \mathscr{F}(\mathbb{R}_+, X)$ there is an extension $y \in \mathscr{F}(\mathbb{R}, X)$ with the following properties:

1) $y(t) = x(t)$ for all $t \in \mathbb{R}_+$;
2) $\|y\| \leq C\|x\|$, where $C > 0$ is independent of x and y;
3) $y \in \mathscr{F}_0(\mathbb{R}_-, X)$;
4) $S(t)x \in \mathscr{F}(\mathbb{R}_+, X)$ for all $t \in \mathbb{R}_+$ and $x \in \mathscr{F}(\mathbb{R}_+, X)$;
5) for any other extension $z \in \mathscr{F}(\mathbb{R}, X)$ possessing properties 1)–4) the condition $y - z \in \mathscr{F}_0(\mathbb{R}, X)$ holds.

Further in this article we denote a homogeneous space by $\mathscr{F}(\mathbb{J}, X)$, while in case $X = \mathbb{C}$ we use the notation $\mathscr{F}(\mathbb{J})$ (see [26]).

Definition 3. A closed subspace of $\mathscr{F}(\mathbb{J}, X)$ defined as $\{x \in \mathscr{F}(\mathbb{J}, X) :$ the function $t \mapsto S(t)x : \mathbb{J} \to \mathscr{F}(\mathbb{J}, X)$ is continuous$\}$ is denoted by $\mathscr{F}_c(\mathbb{J}, X)$.

For example, $(L^p(\mathbb{J}, X))_c = \{x \in L^p(\mathbb{J}, X) :$ function $t \mapsto S(t)x : \mathbb{J} \to L^p(\mathbb{J}, X)$ is continuous$\}$.

Example 1. The following Banach spaces of functions defined on $\mathbb{J}$ and with values in a Banach space X are homogeneous. All of them are linear subspaces of $L^1_{loc}(\mathbb{J}, X)$.

1. The spaces $L^p = L^p(\mathbb{J}, X)$, $p \in [1, \infty)$, of Lebesgue measurable and integrable with power $p \in [1, \infty)$ (classes of) functions with the norm $\|x\|_p = \left(\int\limits_{\mathbb{J}} \|x(t)\|^p_X dt \right)^{1/p}$, $p \in [1, \infty)$. It should be noted that $(L^p(\mathbb{J}, X))_c = L^p(\mathbb{J}, X)$, $(L^p(\mathbb{J}, X))_0 = L^p(\mathbb{J}, X)$.

2. The space $L^\infty = L^\infty(\mathbb{J}, X)$ of essentially bounded (classes of) functions with the norm $\|x\|_\infty = \operatorname*{supvrai}_{t \in \mathbb{J}} \|x(t)\|_X$. Note that $(L^\infty(\mathbb{J}, X))_c = C_{b,u}(\mathbb{J}, X)$.

3. The Stepanov spaces $S^p = S^p(\mathbb{J}, X)$, $p \in [1, \infty)$.

4. The Wiener amalgam spaces $(L^p, l^q) = (L^p(\mathbb{J}, X), l^q(\mathbb{J}, X))$, $p \in [1, \infty)$, $q \in [1, \infty]$, (see [9]) of functions in $L^1_{loc}(\mathbb{J}, X)$ with the norm $\|x\|_{p,q} = \left(\sum\limits_{k \in \mathbb{Z}} \left(\int_0^1 \|x(s + k)\|^p ds \right)^{q/p} \right)^{1/q} < \infty$, $p, q \in [1, \infty)$. These spaces are homogeneous with respect to the following equivalent norm:
$$\sup_{t \in [0,1]} \left(\sum_{k \in \mathbb{Z}} \left(\int_0^1 \|x(s + t + k)\|^p ds \right)^{q/p} \right)^{1/q} < \infty, \ p, q \in [1, \infty).$$

5. The space $C_b = C_b(\mathbb{J}, X)$ of bounded continuous functions. Note that $C_b(\mathbb{J}, X)$ is a closed subspace of $L^\infty(\mathbb{J}, X)$ and $(C_b(\mathbb{J}, X))_c = C_{b,u}(\mathbb{J}, X)$, $(C_b(\mathbb{J}, X))_0 = C_0(\mathbb{J}, X)$.

6. The subspace $C_{b,u} = C_{b,u}(\mathbb{J}, X)$ of $C_b(\mathbb{J}, X)$ of uniformly continuous functions belonging to C_b. It should be noted that $(C_{b,u}(\mathbb{J}, X))_c = C_{b,u}(\mathbb{J}, X)$, $(C_{b,u}(\mathbb{J}, X))_0 = C_0(\mathbb{J}, X)$.

7. The subspace $C_0 = C_0(\mathbb{J}, X)$ of $C_{b,u}(\mathbb{J}, X)$ of continuous vanishing at infinity functions. These functions satisfy the condition $\lim\limits_{|t| \to \infty} \|x(t)\| = 0$, $t \in \mathbb{J}$.

8. The subspace $C_{sl,\infty} = C_{sl,\infty}(\mathbb{J}, X)$ of $C_{b,u}(\mathbb{J}, X)$ of continuous slowly varying at infinity functions. These functions satisfy the condition $\lim\limits_{|t| \to \infty} \|x(t + \tau) - x(t)\| = 0$, $t, \tau \in \mathbb{J}$ (see [15, 27]).

9. The subspace $C_{\omega,\infty} = C_{\omega,\infty}(\mathbb{J}, X)$ of $C_{b,u}(\mathbb{J}, X)$ of continuous ω-periodic at infinity functions, $\omega \in \mathbb{R}_+$. These functions satisfy the condition $\lim\limits_{|t|\to\infty} \|x(t+\omega) - x(t)\| = 0$, $t \in \mathbb{J}$ (see [15, 27]).

10. The subspace $AP_\infty = AP_\infty(\mathbb{J}, X)$ of $C_{b,u}(\mathbb{J}, X)$ of continuous almost periodic at infinity functions (see [9, 10]).

11. The subspaces $C^k = C^k(\mathbb{J}, X)$, $k \in \mathbb{N}$, of k times continuously differentiable functions with bounded k-th derivative and the norm $\|x\|_{(k)} = \|x\|_\infty + \|x^{(k)}\|_\infty$.

12. The Hölder spaces $C^{k,\alpha} = C^{k,\alpha}(\mathbb{J}, X)$, $k \in \mathbb{N} \cup \{0\}$, $\alpha \in (0, 1]$,

$$C^{k,\alpha} = \left\{ x \in C^k : \|x^{(k)}\|_{C^{0,\alpha}} = \sup_{t \neq s \in \mathbb{J}} \frac{|x^{(k)}(t) - x^{(k)}(s)|}{|t-s|^\alpha} < \infty \right\},$$

$\|x\|_{C^{k,\alpha}} = \|x\|_{C^k} + \|x^{(k)}\|_{C^{0,\alpha}}$.

13. The subspace $\mathbb{V} = \mathbb{V}(\mathbb{J}, X)$ of functions from $L^\infty(\mathbb{J}, X)$ with bounded variation $\|x\|_{\mathbb{V}} = \sup\limits_{t\in\mathbb{J}} V_t^{t+1}(x) + \sup\limits_{t\in\mathbb{J}} \|x\|_X$ which is used as a norm.

Definition 1 implies that each of the homogeneous spaces $\mathcal{F}(\mathbb{R}, X)$ mentioned above is endowed with the structure of a Banach $L^1(\mathbb{R})$-module by using convolution (2.2), where S is a group of shifts defined by (2.1). Thus, there is an opportunity to use classic notions and results of the theory of Banach $L^1(\mathbb{R})$-modules given below. In particular, in this terminology the spaces $\mathcal{F}_c(\mathbb{R}, X)$ coincide with the spaces of S-continuous vectors (see Definition 16).

3 Almost periodic at infinity functions

Let $\mathscr{F}(\mathbb{J}, X)$ be a homogeneous space of functions that satisfies Definition 1 for $\mathbb{J} = \mathbb{R}$ and Definition 2 for $\mathbb{J} = \mathbb{R}_+$. In the Banach space $\mathscr{F}(\mathbb{J}, X)$ we consider the (semi-) group $S : \mathbb{J} \to End\,\mathscr{F}(\mathbb{J}, X)$ of shift operators $(S(t)x)(\tau) = x(t+\tau)$ for $t, \tau \in \mathbb{J}$.

Definition 4. A function x from $\mathscr{F}(\mathbb{J}, X)$ is called *integrally vanishing at infinity*, if the following condition holds:

$$\lim_{\alpha\to\infty} \frac{1}{\alpha} \sup_{t\in\mathbb{J}} \int_0^\alpha \|x(t+s)\|\, ds = 0.$$

The set of functions from $\mathscr{F}(\mathbb{J}, X)$ integrally vanishing at infinity is denoted by $\mathscr{F}_{0,int} = \mathscr{F}_{0,int}(\mathbb{J}, X)$.

Definition 4 directly implies that $\mathscr{F}_{0,int}(\mathbb{J}, X)$ is a closed subspace of $\mathscr{F}(\mathbb{J}, X)$.

Definition 5. By $\mathscr{F}_{mid} = \mathscr{F}_{mid}(\mathbb{J}, X)$ we denote a closed subspace of $\mathscr{F}(\mathbb{J}, X)$ with the following properties:

1) $S(t)x \in \mathscr{F}_{mid}$ for every $t \in \mathbb{J}$ and $x \in \mathscr{F}_{mid}$;
2) $\mathscr{F}_0 \subset \mathscr{F}_{mid} \subset \mathscr{F}_{0,int}$;
3) $e_\lambda x \in \mathscr{F}_{mid}$ for every $\lambda \in \mathbb{R}$, where $e_\lambda(t) = e^{i\lambda t}$ for $t \in \mathbb{R}$.

The spaces of this kind are called *the subspaces of $\mathscr{F}(\mathbb{J}, X)$ of functions vanishing at infinity.*

In particular, such subspaces include the set of closed subspaces $\mathscr{F}_{0,p}$ for $p \in [1,\infty)$ of a homogeneous space $\mathscr{F}(\mathbb{J},X)$ defined by

$$\mathscr{F}_{0,p} = \mathscr{F}_{0,p}(\mathbb{J},X) = \{x \in \mathscr{F}(\mathbb{J},X) : \lim_{t\to\infty} \frac{1}{\alpha} \sup_{t\in\mathbb{J}} \int\limits_0^\alpha \|x(s+t)\|^p \, ds = 0\}.$$

Note that $\mathscr{F}_{0,1}(\mathbb{J},X) = \mathscr{F}_{0,int}(\mathbb{J},X)$.

Definition 6. A function $x \in \mathscr{F}_c(\mathbb{J},X)$ is called *slowly varying at infinity with respect to the subspace* $\mathscr{F}_{mid}(\mathbb{J},X)$ if and only if the condition $(S(\alpha)x - x) \in \mathscr{F}_{mid}(\mathbb{J},X)$ is fulfilled for every $\alpha \in \mathbb{J}$.

Note that in articles [9, 10, 11, 12] slowly varying at infinity with respect to the subspace $C_0(\mathbb{R},X)$ functions are defined and their properties are studied. Slowly varying and periodic at infinity (with respect to $\mathscr{F}_0(\mathbb{R},X)$) functions in $\mathscr{F}(\mathbb{R},X)$ were studied in [26]). Particularly, in [11] it was established that the solutions of the heat equation are slowly varying at infinity (with respect to time and the subspace $C_0(\mathbb{R},X)$).

By $(\mathscr{F}_{mid})_{sl,\infty}$ we denote the set of functions in $\mathscr{F}(\mathbb{J},X)$ slowly varying at infinity with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J},X)$.

Definition 6 directly implies that the space $(\mathscr{F}_{mid})_{sl,\infty} \subset \mathscr{F}(\mathbb{J},X)$ of functions slowly varying at infinity with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J},X)$ is a closed subspace of $\mathscr{F}(\mathbb{J},X)$.

Here are some examples of functions from $(C_0)_{sl,\infty}(\mathbb{J}) = (C_0)_{sl,\infty}(\mathbb{J},\mathbb{C})$ slowly varying at infinity:

1) $x_1(t) = \sin\ln(1+|t|)$, $t \in \mathbb{J}$;
2) $x_2(t) = c + x_0(t)$ for $t \in \mathbb{J}$, where $c \in \mathbb{C}$ and $x_0 \in C_0(\mathbb{J})$;
3) $x_3(t) = \operatorname{arctg} t$, $t \in \mathbb{J}$;
4) any continuously differentiable function $x \in C_b(\mathbb{J})$ with the property $\lim\limits_{|t|\to\infty} \dot{x}(t) = 0$.

Let us state four definitions of functions from $\mathscr{F}(\mathbb{J},X)$ almost periodic at infinity with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J},X)$ of vanishing at infinity functions from $\mathscr{F}(\mathbb{J},X)$. After that we are going to prove them to be equivalent.

To start with, let us introduce a definition of continuous almost periodic at infinity function (see [10, 16]), which is based on the notion of ε-periodicity at infinity.

Definition 7. Assume that $\varepsilon > 0$. A number $\omega > 0$ is called an *ε-period at infinity* of $x \in C_b(\mathbb{J},X)$ if there exists a number $a(\varepsilon) > 0$ such that $\sup\limits_{|t|\geq a(\varepsilon)} \|x(t+\omega) - x(t)\| < \varepsilon$.

The set of ε-periods at infinity of $x \in C_b(\mathbb{J},X)$ is denoted by $\Omega_\infty(\varepsilon,x)$.

Definition 8. A subset Ω of $\mathbb{J}$ is called *relatively dense* in $\mathbb{J}$ whenever there exists $l > 0$ with $[t,t+l] \cap \Omega \neq \varnothing$ for every $t \in \mathbb{J}$.

Definition 9. A function $x \in C_b(\mathbb{J},X)$ is called *almost periodic at infinity* whenever for every $\varepsilon > 0$ the set $\Omega_\infty(\varepsilon,x)$ of its ε-periods at infinity is relatively dense in $\mathbb{J}$.

Definitions 7 and 9 imply that every function $x \in C_b(\mathbb{J},X)$ almost periodic in Bohr's sense (see [7, 21]) is almost periodic at infinity.

The set of classical almost periodic functions is denoted by $AP(\mathbb{J},X)$. The set of continuous almost periodic at infinity functions is denoted by $AP_\infty(\mathbb{J},X)$. These functions were considered in (see [9, 10, 16]).

Definition 10. Assume that $\varepsilon > 0$. A number $\omega > 0$ is called an *ε-period at infinity of $x \in \mathscr{F}(\mathbb{J}, X)$ with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$* whenever there exists a function $x_0 \in \mathscr{F}_{mid}(\mathbb{J}, X)$ such that $\|S(\omega)x - x - x_0\| < \varepsilon$.

The set of ε-periods at infinity of $x \in \mathscr{F}(\mathbb{J}, X)$ with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$ is denoted by $\Omega_\infty(x; \mathscr{F}_{mid}; \varepsilon)$.

Definition 11. A function x from $\mathscr{F}_c(\mathbb{J}, X)$ is called *almost periodic at infinity with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$* whenever for any $\varepsilon > 0$ the set $\Omega_\infty(x; \mathscr{F}_{mid}; \varepsilon)$ of its ε-periods at infinity (with respect to $\mathscr{F}_{mid}(\mathbb{J}, X)$) is relatively dense in $\mathbb{J}$.

Definitions 10 and 11 imply that every function $x \in C_b(\mathbb{J}, X)$ almost periodic in Bohr's sense (see [7, 21]) is almost periodic at infinity with respect to every subspace $(C_b)_{mid}$ of vanishing at infinity functions from $C_b(\mathbb{J}, X)$.

The set of functions from $\mathscr{F}(\mathbb{J}, X)$ almost periodic at infinity with respect to $\mathscr{F}_{mid}(\mathbb{J}, X)$ is denoted by $AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$. Definition 6 directly implies that $\Omega_\infty(x; \mathscr{F}_{mid}; \varepsilon) = \mathbb{J}$ for all $x \in (\mathscr{F}_{mid})_{sl,\infty}$ and $\varepsilon > 0$, hence, $x \in AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$. Consequently, $(\mathscr{F}_{mid})_{sl,\infty} \subset AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$.

Definition 12. A set $\mathcal{M} \subset \mathscr{F}(\mathbb{J}, X)$ is called *precompact at infinity with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$* whenever for any $\varepsilon > 0$ there exist finitely many functions $b_1, \dots, b_N \in \mathcal{M}$ called an ε-net at infinity such that for every $x \in \mathcal{M}$ there exist a function $b_k, k \in \{1, \dots, N\}$, and a function $\alpha_\varepsilon \in \mathscr{F}_{mid}(\mathbb{J}, X)$ such that $\|x - b_k - \alpha_\varepsilon\| < \varepsilon$.

Definition 13. A function $x \in \mathscr{F}_c(\mathbb{J}, X)$ is called *almost periodic at infinity with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$* whenever the set of shifts $S(t)x$ of x for $t \in \mathbb{J}$ is precompact at infinity with respect to $\mathscr{F}_{mid}(\mathbb{J}, X)$.

For $\mathscr{F}(\mathbb{J}, X) = C_{b,u}(\mathbb{J}, X)$ Definition 13 corresponds to Bochner's criterion (see [21]) of almost periodicity.

Observe that the functions of the form

$$x(t) = \sum_{k=1}^{N} x_k(t)e^{i\lambda_k t},\ x_1, \dots, x_N \in (\mathscr{F}_{mid})_{sl,\infty},\ \lambda_1, \dots, \lambda_N \in \mathbb{R},$$

called generalized trigonometric polynomials, are almost periodic at infinity with respect to $\mathscr{F}_{mid}(\mathbb{J}, X)$ in the sense of Definition 13.

Definition 14. A function $x \in \mathscr{F}_c(\mathbb{J}, X)$ is called *almost periodic at infinity with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$* whenever given $\varepsilon > 0$ we can indicate a finite collection $\lambda_1, \dots, \lambda_N$ of real numbers and functions $x_1, \dots, x_N$ from $(\mathscr{F}_{mid})_{sl,\infty}$ such that $\sup\limits_{t \in \mathbb{J}} \|x(t) - \sum\limits_{k=1}^{N} x_k(t)e^{i\lambda_k t}\| < \varepsilon$.

Definition 15. A function $x \in \mathscr{F}_c(\mathbb{J}, X)$ is called *asymptotically almost periodic with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$* whenever $x = x_1 + x_0$ with $x_1 \in AP(\mathbb{J}, X)$, $x_0 \in \mathscr{F}_{mid}(\mathbb{J}, X)$.

The set $AAP\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$ of asymptotically almost periodic functions with respect to $\mathscr{F}_{mid}(\mathbb{J}, X)$ is a closed subspace of $\mathscr{F}(\mathbb{J}, X)$. Moreover,

$$AAP\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid}) = AP(\mathbb{J}, X) \oplus \mathscr{F}_{mid}(\mathbb{J}, X).$$

Thus, the strict inclusions

$$AP(\mathbb{J}, X) \subset AAP\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid}) \subset AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$$

hold.

It should be noted that for $\mathscr{F} = C_b(\mathbb{R}_+, X)$ the space $AAP\mathscr{F}(\mathbb{R}_+, X; C_0)$ was considered in [22]. It was denoted by $AAP(\mathbb{R}_+, X)$ there.

The fourth definition of a function almost periodic at infinity with respect to $\mathscr{F}_{mid}(\mathbb{J}, X)$ is given after introducing some definitions and results of the spectral theory of Banach $L^1(\mathbb{R})$-modules.

4 Almost periodic vectors from Banach $L^1(\mathbb{R})$-modules

Let $\mathscr{X}$ be a complex Banach space and $End\ \mathscr{X}$ be a Banach algebra of linear operators on $\mathscr{X}$. By $L^1(\mathbb{R})$ we denote the algebra of complex Lebegue's measurable (classes of) functions summable on $\mathbb{R}$ with the convolution as multiplication:

$$(f * g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds,\ t \in \mathbb{R},\ f, g \in L^1(\mathbb{R}).$$

We endow $\mathscr{X}$ with the structure of a nondegenerate Banach $L^1(\mathbb{R})$-module (see [5, 7, 8, 13, 20]) associated with some bounded isometric representation $T : \mathbb{R} \to End\ \mathscr{X}$. This means that the following properties hold.

Assumption 1. *A Banach $L^1(\mathbb{R})$-module $\mathscr{X}$ satisfies the conditions below:*

1) if $fx = 0$ for every function $f \in L^1(\mathbb{R})$ then the vector $x \in \mathscr{X}$ is null (a nondegeneracy property of $\mathscr{X}$);

2) for every $f \in L^1(\mathbb{R})$ and $x \in \mathscr{X}$ the following equations hold true:

$$T(t)(fx) = (T(t)f)x = f(T(t)x) \text{ for t} \in \mathbb{R},$$

which means that the module structure on $\mathscr{X}$ is associated with the representation $T : \mathbb{R} \to End\ \mathscr{X}$.

If $T : \mathbb{R} \to End\ \mathscr{X}$ is a strongly continuous isometric representation then the formula

$$T(f)x = fx = \int_{\mathbb{R}} f(t)T(-t)xdt,\ f \in L^1(\mathbb{R}),\ x \in \mathscr{X},$$

endows $\mathscr{X}$ with the structure of a Banach $L^1(\mathbb{R})$-module satisfying the conditions of Assumption 1. This structure is associated with the representation T.

Remark 1. *With every nondegenerate Banach $L^1(\mathbb{R})$-module $\mathscr{X}$ we associate a unique representation $T : \mathbb{R} \to End\ \mathscr{X}$ (see [13]). In order to emphasise that sometimes we use the notation $(\mathscr{X}, T)$.*

Definition 16. A vector from a Banach $L^1(\mathbb{R})$-module $(\mathscr{X}, T)$ is called *continuous* (with respect to the representation T) or *T-continuous* if a function $\varphi_x : \mathbb{R} \to \mathscr{X}$ defined by $\varphi_x(t) = T(t)x$ for $t \in \mathbb{R}$ is continuous at $t = 0$ (hence, it is continuous on $\mathbb{R}$).

A set of all continuous vectors from a Banach $L^1(\mathbb{R})$-module $\mathscr{X}$ denoted by $\mathscr{X}_c$ or $(\mathscr{X}, T)_c$ is a closed submodule in $\mathscr{X}$, i.e., $\mathscr{X}_c$ is a closed linear subspace in $\mathscr{X}$ invariant under shift operators $T(f)$, $T(t)$ for $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$.

Every homogeneous space $\mathscr{F}(\mathbb{R}, X)$ is endowed with the structure of a Banach $L^1(\mathbb{R})$-module using the convolution (2.2), where $S : \mathbb{R} \to End\ \mathscr{F}(\mathbb{R}, X)$ is the group of shifts defined by (2.1). However, formula (2.2) does not define the structure of $L^1(\mathbb{R})$-module for $\mathscr{F}(\mathbb{R}_+, X)$. Nevertheless, the quotient space $\mathscr{F}(\mathbb{R}_+, X)/\mathscr{F}_{mid}(\mathbb{R}_+, X)$ is endowed with the structure of a Banach $L^1(\mathbb{R})$-module.

In what follows we use some definitions and results of the spectral theory of Banach modules (see [2, 4, 5, 6, 7, 8, 14]) which stem from Domar [18] and Reiter [23].

Given a function f from $L^1(\mathbb{R})$, the Fourier transform $\widehat{f} : \mathbb{R} \to \mathbb{C}$ is defined as

$$\widehat{f}(\lambda) = \int_{\mathbb{R}} f(t)e^{-i\lambda t}\, dt,\ \lambda \in \mathbb{R}.$$

Definition 17. By the *Beurling spectrum* of a vector $x \in \mathscr{X}$ we call the set of numbers $\Lambda(x)$ in $\mathbb{R}$ defined by

$$\Lambda(x) = \{\lambda_0 \in \mathbb{R} : fx \neq 0 \text{ for every function } f \in L^1(\mathbb{R}) \text{ with } \widehat{f}(\lambda_0) \neq 0\}.$$

The definition implies that $\Lambda(x) = \mathbb{R}\setminus\{\mu_0 \in \mathbb{R}:$ there exists a function $f \in L^1(\mathbb{R})$ such that $\widehat{f}(\mu_0) \neq 0$ and $fx = 0\}$.

The Beurling spectrum of vectors in a Banach $L^1(\mathbb{R})$-module $\mathscr{X}$ has the properties listed below.

Lemma 4.1. *For every $x \in \mathscr{X}$ and $f \in L^1(\mathbb{R})$ the following properties hold true:*

1) *the set $\Lambda(x)$ is closed and $\Lambda(x) = \emptyset$ if and only if $x = 0$;*

2) *$\Lambda(fx) \subseteq (\operatorname{supp} \widehat{f}) \bigcup \Lambda(x)$;*

3) *$fx = 0$ when $(\operatorname{supp} \widehat{f}) \bigcap \Lambda(x) = \emptyset$ and $fx = x$ if the set $\Lambda(x)$ is compact and $\widehat{f} = 1$ in its neighbourhood;*

4) *the set $\Lambda(x)$ is a singleton $((\Lambda(x) = \{\lambda_0\}))$ if and only if $x \neq 0$ and $T(t)x = e^{i\lambda_0 t}x$ for $t \in \mathbb{R}$.*

Remark 2. *As we indicated above, a homogeneous space $\mathscr{F}(\mathbb{R}, X)$ is a Banach $L^1(\mathbb{R})$-module. If a function $x \in \mathscr{F}(\mathbb{R}, X)$ has the property $\Lambda(x) = \{\lambda_0\}$ then it can be represented as $x(t) = x_0 e^{i\lambda_0 t}$ for $t \in \mathbb{R}$, where $x_0 \in X$.*

Further in this paper we use the following concept of an almost periodic vector in a Banach space $\mathscr{X}$ (see [5, 6, 7, 8]) with a strongly continuous isometric representation $T : \mathbb{R} \to End\,\mathscr{X}$.

Definition 18. A vector $x_0 \in \mathscr{X}$ is called *almost periodic* (with respect to a representation T) if one of the following conditions is met:

1) for every $\varepsilon > 0$ the set $\Omega(x_0, \varepsilon) = \{\omega \in \mathbb{R} : \|T(\omega)x_0 - x_0\| < \varepsilon\}$ of ε-periods of the vector x_0 is relatively dense in $\mathbb{R}$;

2) the orbit $\{T(t)x_0,\ t \in \mathbb{R}\}$ of x_0 is precompact in $\mathscr{X}$;

3) $t \mapsto \varphi(t) = T(t)x_0$, $t \in \mathbb{R}$, is a continuous almost periodic function, i.e., $\varphi \in AP(\mathbb{R}, X)$ (see [21]);

4) for any $\varepsilon > 0$ there are eigenvalues $\lambda_1, \dots, \lambda_N$ and associated eigenvectors $x_1, \dots, x_N$ of the representation T, i.e., $T(t)x_k = e^{i\lambda_k t}x_k$ for $t \in \mathbb{R}$ and $k = 1, \dots, N$ such that $\|x_0 - \sum\limits_{k=1}^{N} x_k\| < \varepsilon$.

The set $AP(\mathscr{X}) = AP(\mathscr{X}, T)$ of almost periodic vectors (with respect to the representation T) is a closed submodule of $\mathscr{X}$. Observe that $AP(C_{b,u}(\mathbb{R}, X), S) = AP(\mathbb{R}, X)$ and $AP(\mathscr{X}) \subset \mathscr{X}_c$.

If $\mathscr{X}_0$ is a closed submodule of $\mathscr{X}$ invariant under operators $T(t)$, $t \in \mathbb{R}$, then the quotient space $\mathscr{X}/\mathscr{X}_0$ is also a Banach $L^1(\mathbb{R})$-module whose structure for all $f \in L^1(\mathbb{R})$ and equivalence classes $\widetilde{x} = x + \mathscr{X}_0$, $x \in \mathscr{X}$, is defined as $f\widetilde{x} = fx + \mathscr{X}_0 = \widetilde{fx}$. This structure is associated with the representation

$$\widetilde{T} : \mathbb{R} \to End\,\mathscr{X}/\mathscr{X}_0,\ \widetilde{T}(t)\widetilde{x} = \widetilde{T(t)x} = T(t)x + \mathscr{X}_0,\ x \in \mathscr{X}.$$

Definition 19. For any vector $x \in \mathscr{X}$ the Beurling spectrum $\Lambda(\widetilde{x})$ of the equivalence class $\widetilde{x} = x + \mathscr{X}_0$ is called *a set of nonbelonging* of x in the submodule $\mathscr{X}_0$ and denoted by $\Lambda(x, \mathscr{X}_0)$. the set $\Lambda(x, AP(\mathscr{X}))$ is called *a set of nonalmost periodicity* of x.

The following statement established in [5] is important for further.

Theorem 4.1. *If the set $\Lambda(x)$ for $x \in \mathscr{X}$ lacks limit points on $\mathbb{R}$ then $x \in AP(\mathscr{X})$.*

Theorem 4.2. *Assume that the set $\Lambda(x, AP(\mathscr{X}))$ for $x \in \mathscr{X}$ is no-more-than countable. In this case x is almost periodic whenever every limit point of $\Lambda(x, AP(\mathscr{X}))$ is ergodic for x. In particular, if the set $\Lambda(x, AP(\mathscr{X}))$ lacks limit points then $x \in AP(\mathscr{X})$.*

Theorem 4.2 was established in [4]. It can also be found in [2, 21].

Let us recall that the notation $\mathscr{F}_{mid} = \mathscr{F}_{mid}(\mathbb{J}, X)$ stands for the subspace of $\mathscr{F}(\mathbb{J}, X)$ of functions vanishing at infinity, where $\mathscr{F}_0 \subset \mathscr{F}_{mid} \subset \mathscr{F}_{0,int}$. The definitions of the subspaces $\mathscr{F}_{mid}(\mathbb{R}, X)$ and $(\mathscr{F}_{mid})_{sl,\infty}$ imply that they are closed subspaces of $\mathscr{F}(\mathbb{J}, X)$ invariant under shift operators. Thus, they are closed submodules of $\mathscr{F}(\mathbb{R}, X)$.

By $\mathscr{X}(\mathbb{J})$, $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$, we denote the quotient space $\mathscr{F}(\mathbb{J}, X)/\mathscr{F}_{mid}(\mathbb{J}, X)$, which is a Banach space under the norm $\|\widetilde{x}\| = \inf\limits_{y \in x + \mathscr{F}_{mid}} \|y\|$, where $\widetilde{x} = x + \mathscr{F}_{mid}(\mathbb{J}, X)$ is the equivalence class containing the function $x \in \mathscr{F}(\mathbb{J}, X)$. The Banach space $\mathscr{X}(\mathbb{J})$ is a Banach algebra with multiplication rule defined by $\widetilde{x}\widetilde{y} = \widetilde{xy}$, $\widetilde{x}, \widetilde{y} \in \mathscr{X}(\mathbb{J})$.

On $\mathscr{X}(\mathbb{R})$ we define the strongly continuous group of isometries $\widetilde{S} : \mathbb{R} \to End\, \mathscr{X}(\mathbb{R})$ by the formula

$$\widetilde{S}(t)\widetilde{x} = \widetilde{S(t)x},\ x \in \mathscr{F}(\mathbb{R}, X),\ t \in \mathbb{R}.$$

The quotient space $\mathscr{X}(\mathbb{R})$ is endowed with the structure of a Banach $L^1(\mathbb{R})$-module using the formula $f\widetilde{x} = \widetilde{f * x}$, $f \in L^1(\mathbb{R})$, $x \in \mathscr{F}(\mathbb{R}, X)$.

Remark 3. *Assume that $\mathbb{J} = \mathbb{R}_+$. Each function $x \in \mathscr{F}(\mathbb{R}_+, X)$ can be extended to the function y on $\mathbb{R}$ so that y satisfied all 5 conditions of Definition 2. Note that the equivalence class $\widetilde{x} \in \mathscr{X}(\mathbb{R})$ does not depend on the certain extension and, consequently, the Banach space $\mathscr{X}(\mathbb{R}_+)$ is isometrically embedded in $\mathscr{X}(\mathbb{R})$ as a closed submodule. In this case the group $\widetilde{S}$ is correctly defined on $\mathscr{X}(\mathbb{R}_+)$ and the space $\mathscr{X}(\mathbb{R}_+)$ is endowed with the structure of a Banach $L^1(\mathbb{R})$-module as well.*

5 Criteria of almost periodicity at infinity

Now let us give the fourth definition of function from $\mathscr{F}(\mathbb{J}, X)$ almost periodic at infinity with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$.

Definition 20. A function x from $\mathscr{F}_c(\mathbb{J}, X)$ is called *almost periodic at infinity with respect to the subspace* $\mathscr{F}_{mid}(\mathbb{J}, X)$ of functions from $\mathscr{F}(\mathbb{J}, X)$ vanishing at infinity whenever the equivalence class $\widetilde{x} = x + \mathscr{F}_{mid}(\mathbb{J}, X)$ is an almost periodic vector from $\mathscr{X}(\mathbb{J})$ with respect to the isometric representation $\widetilde{S} : \mathbb{R} \to End\, \mathscr{X}(\mathbb{J})$.

The functions in $C_{b,u}(\mathbb{J}, X)$ almost periodic at infinity (with respect to the subspace $C_0(\mathbb{J}, X)$) appeared for the first time in [9, 10]. In those papers the definition similar to Definition 20 was used. The main results of those papers deal with the asymptotic behaviour of bounded operator semigroups.

On the Banach space $AP(\mathscr{X}) = AP(\mathscr{X}, T)$ of almost periodic vectors there exists a unique linear operator $\mathscr{J} \in End\, AP(\mathscr{X})$ with the properties:

1) $\|\mathscr{J}\| = 1$;

2) $\mathscr{J}(T(t)x) = \mathscr{J}x$ for $t \in \mathbb{J}$ and $x \in \mathscr{X}$.

For every vector x from $AP(\mathscr{X})$ let us consider a function $\widehat{x}_B : \mathbb{R} \to AP(\mathscr{X})$, $\widehat{x}_B(\lambda) = \mathscr{J}(T_\lambda x)$, where $T_\lambda(t) = T(t)e^{-i\lambda t}$ for $t \in \mathbb{R}$. This function is called the *Bohr transform* of the vector x. Its support $\operatorname{supp} \widehat{x}_B$ is at most countable set, i.e., $\operatorname{supp} \widehat{x}_B = \{\lambda_1, \lambda_2, \dots\}$, and

$$T(t)x_k = e^{i\lambda_k t}x_k,\ t \in \mathbb{R},\ k \geq 1,$$

where x_k, $k \geq 1$, are eigenvectors of the representation T (they are also eigenvectors of the generator iA of the operator group T, i.e., $iAx_k = i\lambda_k x_k$, $k \geq 1$). Moreover, $\Lambda(x_k) = \{\lambda_k\}$ for $k \geq 1$. The set $\Lambda_B(x) = \{\lambda_1, \lambda_2, \dots\}$ is called the *Bohr spectrum* of the vector $x \in AP(\mathscr{X})$.

The series

$$x \sim \sum_{k \geq 1} x_k \tag{5.1}$$

is called the *Fourier series* of the vector $x \in AP(\mathscr{X})$. Note that if the series converges then $x = \sum\limits_{k \geq 1} x_k$.

Lemma 5.1. *Given a function f from $L^1(\mathbb{R})$ and an almost periodic vector $x \in AP(\mathscr{X}, T)$ with the Fourier series of form (5.1), the vector fx is almost periodic with the Fourier series of the form*

$$fx \sim \sum_{k \geq 1} \widehat{f}(\lambda_k) x_k.$$

The following lemma uses the bounded approximative identity (b.a.i.) (f_n), $n \geq 1$, in $L^1(\mathbb{R})$ constructed below (see [5, 6, 7, 8]).

Let us consider a function $\widehat{f_0}$ in the space $\widehat{L}^1(\mathbb{R})$ of Fourier transforms of functions in $L^1(\mathbb{R})$ (with pointwise multiplication) with compact support $\operatorname{supp} \widehat{f_0}$ on the interval $[-1, 1]$ such that $\widehat{f_0}(0) = 1$. Then the sequence (f_n), $n \geq 1$, of functions of the form

$$f_n(t) = n f_0(nt), \ t \in \mathbb{R}, \tag{5.2}$$

is a bounded approximation of the identity in $L^1(\mathbb{R})$. Let us note that $\|f_n\| = \|f_0\|$ for $n \geq 1$.

Lemma 5.1 implies the following

Lemma 5.2. *Let x be an almost periodic vector from $AP(\mathscr{X}, T)$ with the Bohr spectrum $\Lambda_B(x) = \{\lambda_1, \lambda_2, ...\}$, where (λ_n), $n \geq 1$, satisfies the condition $\lim\limits_{n \to \infty} |\lambda_n| = \infty$. In this case*

$$\lim_{n \to \infty} \|x - f_n x\| = \lim_{n \to \infty} \|x - \sum_{|\lambda_k| < n} \widehat{f_0}(\frac{\lambda_k}{n}) x_k\|,$$

where x_k, $k \geq 1$, are taken from expansion into its Fourier series (5.1).

Theorem 5.1. *All definitions of almost periodic at infinity functions from $\mathscr{F}(\mathbb{J}, X)$ with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$ (Definitions* 11, 13, 14, 20*) are equivalent.*

Proof. We assume $\mathbb{J} = \mathbb{R}$ (the result for $\mathbb{J} = \mathbb{R}_+$ follows from Remark 3). Let us consider the quotient space $\mathscr{X}(\mathbb{R}) = \mathscr{F}(\mathbb{R}, X)/\mathscr{F}_{mid}(\mathbb{R}, X)$ and the group of isometries $T = \widetilde{S} : \mathbb{R} \to End\, \mathscr{X}(\mathbb{R})$ defined above. For this representation Definition 14 corresponds to property 4) of Definition 18. Since all properties of Definition 18 are equivalent, it suffices to show that the first three properties are equivalent to Definitions 11,13 and 20 respectively.

Given $x \in \mathscr{F}(\mathbb{R}, X)$, take the equivalence class $\widetilde{x}$ in $\mathscr{X}(\mathbb{R})$, constructed from the function x. Then for each $\varepsilon > 0$ the set $\Omega_\infty(x; \mathscr{F}_{mid}; \varepsilon) \cup (-\Omega_\infty(x; \mathscr{F}_{mid}; \varepsilon))$ coincides with the set $\Omega(\widetilde{x}, \varepsilon)$ of ε-periods of $\widetilde{x}$. Hence, the corresponding definitions are equivalent.

The equivalence of Definition 13 and property 2) of Definition 18 follows directly from the definition of the quotient module $\mathscr{X}(\mathbb{R})$.

In order to verify the equivalence of the approximative Definition 14 and property 4) of Definition 18, it suffices to establish that the Beurling spectrum $\Lambda(\widetilde{y})$ of the equivalence class $\widetilde{y} \in \mathscr{X}(\mathbb{R})$

with $\widetilde{y} = y + \mathscr{F}_{mid}$ is a singleton ($\Lambda(\widetilde{y}) = \{\lambda_0\}$) if and only if we can represent $y \in \mathscr{F}(\mathbb{R}, X)$ in the form $y(t) = y_0(t)e^{i\lambda_0 t}$ for $t \in \mathbb{R}$, where $y_0 \in (\mathscr{F}_{mid})_{sl,\infty}$.

If $\Lambda(\widetilde{y}) = \{\lambda_0\}$, then $\widetilde{S}(t)\widetilde{y} = e^{i\lambda_0 t}\widetilde{y}$ for every $t \in \mathbb{R}$ (see property 4) of Lemma 4.1). Hence, $\Lambda(\widetilde{y}_0) = \{0\}$, where $y_0(s) = y(s)e^{-i\lambda_0 s}$ for $s \in \mathbb{R}$. Therefore, $\widetilde{S}(t)\widetilde{y}_0 = \widetilde{y}_0$ for every $t \in \mathbb{R}$. Thus, $S(t)y_0 - y_0 \in \mathscr{F}_{mid}(\mathbb{R}, X)$ for $t \in \mathbb{R}$, i.e., $y_0 \in (\mathscr{F}_{mid})_{sl,\infty}$.

Conversely, if $y(t) = y_0(t)e^{i\lambda_0 t}$ for $t \in \mathbb{R}$, where $y_0 \in \mathscr{F}_{mid}(\mathbb{R}, X)$, then $\widetilde{S}(t)\widetilde{y} = e^{i\lambda_0 t}\widetilde{y}$ for $t \in \mathbb{R}$ and so property 4) of Lemma 4.1 implies that $\Lambda(\widetilde{y}) = \{\lambda_0\}$ (see Remark 2). □

Observe that the closedness of the space $AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$ follows, for instance, from the approximative Definition 14.

Note the following inclusions:

$$\mathscr{F}_{mid}(\mathbb{J}, X) \subset (\mathscr{F}_{mid})_{sl,\infty} \subset AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid}),$$

$$AP(\mathbb{J}, X) \subset AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid}).$$

In the results mentioned below by $\mathscr{X}$ we denote the quotient space $\mathscr{F}(\mathbb{J}, X)/\mathscr{F}_{mid}(\mathbb{J}, X)$ and by $\mathscr{Y}$ — the quotient space $\mathscr{F}(\mathbb{J}, X)/AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$.

Theorem 5.2. *If the set $\Lambda_\infty(x, \mathscr{F}_{mid})$ lacks limit points then $x \in AP(\mathbb{J}, X; \mathscr{F}_{mid})$.*

Proof. As we mentioned before, the quotient space $\mathscr{X}$ is a Banach $L^1(\mathbb{R})$-module whose structure is endowed by the representation $\widetilde{S} : \mathbb{R} \to End\,\mathscr{X}$ mentioned above. Therefore, the set $\Lambda(\widetilde{x}) = \Lambda_\infty(x)$ lacks limit points in $\mathbb{R}$ and, consequently, Theorem 4.1 implies that the equivalence class $\widetilde{x} = x + \mathscr{F}_{mid}(\mathbb{J}, X)$ is an almost periodic vector in $AP(\mathscr{X})$, which means that $x \in AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$ (see Definition 20). □

Definition 21. The set $\Lambda(\widetilde{x})$, where $\widetilde{x} = x + \mathscr{F}_{mid}(\mathbb{J}, X)$ with $x \in \mathscr{F}(\mathbb{J}, X)$ is called *a spectrum of the function x at infinity with respect to the subspace $\mathscr{F}_{mid} = \mathscr{F}_{mid}(\mathbb{J}, X)$* and denoted by $\Lambda_\infty(x; \mathscr{F}_{mid})$. The set $\Lambda(x, AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})) = \Lambda(\widetilde{x})$ for $\widetilde{x} = x + AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid}) \in \mathscr{Y}$ is called *a set of nonalmost periodicity* of x (see Definition 19).

Theorem 5.3. *Assume that the set $\Lambda(x, AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid}))$ for $x \in \mathscr{F}_c(\mathbb{J}, X)$ is no more than countable. In this case x is almost periodic at infinity with respect to the subspace $\mathscr{F}_{mid} = \mathscr{F}_{mid}(\mathbb{J}, X)$ whenever every limit point of $\Lambda(x, AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid}))$ is ergodic for the equivalence class $\widetilde{x} = x + AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid}))$ from the factor-module $\mathscr{Y}$. In particular, if the set $\Lambda(x, AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid}))$ lacks limit points then $x \in AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$.*

The proof of Theorem 5.3 follows directly from Theorem 4.2 by using the fact that the quotient space $\mathscr{Y}$ is endowed by the structure of a Banach $L^1(\mathbb{R})$-module.

These results are of great importance in the derivation of spectral criteria for almost periodicity at infinity of solutions to differential equations.

We also derived the following spectral criterion of asymptotic almost periodicity with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$.

Theorem 5.4. *A function $x \in AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$ is asymptotically almost periodic with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$ (i.e., $x \in AAP\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$) whenever every point of the spectrum $\Lambda_\infty(x; \mathscr{F}_{mid})$ of the function x at infinity with respect to the subspace $\mathscr{F}_{mid}(\mathbb{J}, X)$ is ergodic for x.*

Proof. The necessity of the statement of the theorem follows from the fact that every point of the set $\Lambda_\infty(x)$ is egrodic for $x_1 \in AP(\mathbb{J}, X; \mathscr{F}_{mid})$ and $x_0 \in \mathscr{F}_{mid}(\mathbb{J}, X)$, where $x = x_1 + x_0$ (see Definition 15). The sufficiency follows from Definition 20 and Theorem 4.2 by using Lemma 2 from [10]. □

An analogous result for uniformly continuous functions was obtained in [10].

6 Almost periodic at infinity solutions to differential equations

In this section, as above, the notation $\mathscr{F}_{mid} = \mathscr{F}_{mid}(\mathbb{J}, X)$ stands for the subspace of $\mathscr{F}(\mathbb{J}, X)$ of functions vanishing at infinity. Observe the inclusions $\mathscr{F}_0(\mathbb{J}, X) \subset \mathscr{F}_{mid}(\mathbb{J}, X) \subset \mathscr{F}_{0,int}(\mathbb{J}, X)$ and nonseparability of the Banach space $(\mathscr{F}_{mid})_{sl,\infty}$ of functions slowly varying at infinity with respect to $\mathscr{F}_{mid}(\mathbb{J}, X)$ of functions vanishing at infinity.

Take the linear operator $A : D(A) \subset X \to X$ with domain $D(A)$ which is a generator of a strongly continuous operator semigroup $U : \mathbb{R}_+ \to End\, X$.

Consider the differential equation

$$\dot{x}(t) - Ax(t) = \psi(t),\ t \in \mathbb{J},\ \psi \in \mathscr{F}(\mathbb{J}, X). \tag{6.1}$$

Refer as a *classical solution* to differential equation (6.1) to a differentiable function $x : \mathbb{J} \to X$ such that $x(t) \in D(A)$ for every $t \in \mathbb{J}$ satisfying (6.1) for all $t \in \mathbb{J}$.

Let us state two definitions of a mild solution to (6.1) with $\psi \in \mathscr{F}(\mathbb{J}, X)$.

Definition 22. A continuous function $x : \mathbb{J} \to X$ is called a *mild solution* to the differential equation (6.1) whenever a function $z : \mathbb{J} \to X$ of the form $z(t) = \int\limits_0^t x(s)\, ds$ for $t \in \mathbb{J}$ has the following properties:

1) $z(t) \in D(A)$ for every $t \in \mathbb{J}$;

2) $x(t) - x(0) = Az(t) + \int\limits_0^t \psi(s)\, ds$ for $t \in \mathbb{J}$.

Definition 23. A function $x : \mathbb{J} \to X$ is called a *mild solution* to (6.1) whenever

$$x(t) = U(t-s)x(s) + \int_s^t U(t-\tau)\psi(\tau)\, d\tau \tag{6.2}$$

for all $s, t \in \mathbb{J}$ such that $s \le t$.

For continuous functions the equivalence of these definitions of a mild solution to (6.1) is proved in [2, Chapter 3].

In the Banach space $\mathscr{F} = \mathscr{F}(\mathbb{R}, X)$ consider the linear operator

$$\mathcal{L} = \frac{d}{dt} - A : D(\mathcal{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X).$$

Definition 24. A function $x \in \mathscr{F}(\mathbb{R}, X)$ lies in the *domain* $D(\mathcal{L})$ of the operator $\mathcal{L}$ whenever there exists a function $\psi \in \mathscr{F}(\mathbb{R}, X)$ such that equation (6.2) holds for all $s \le t$ in $\mathbb{R}$.

For $x \in D(\mathcal{L})$ put $\mathcal{L}x = \psi$ provided that x and ψ satisfy (6.2). For $\mathscr{F}(\mathbb{R}, X) = C_{b,u}(\mathbb{R}, X)$ this definition of the operator $\mathcal{L}$ appeared for the first time in [4, 5] (see also [21, 9, 14]).

By $S(f) : \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ we denote the operator $S(f)x = f * x$ of convolution of $x \in \mathscr{F}(\mathbb{R}, X)$ with $f \in L^1(\mathbb{R})$. The following commutation theorem was established in [14].

Theorem 6.1. *For every f from $L^1(\mathbb{R})$ and every x from $D(\mathcal{L})$ the function $S(f)x$ belongs to $D(\mathcal{L})$ and satisfies the condition $\mathcal{L}S(f)x = S(f)\mathcal{L}x$.*

Remark 4. *Given $\varphi \in AP_\infty\mathscr{F}(\mathbb{R}_+, X; \mathscr{F}_{mid})$ every its extension φ_1 on $\mathbb{R}$ satisfying the conditions 1)-5) of Definition 2 with the property $\lim\limits_{t\to-\infty} \varphi_1(t) = 0$ lies in the space $AP_\infty\mathscr{F}(\mathbb{R}, X; \mathscr{F}_{mid})$. It follows from the inclusions $\mathscr{F}_0(\mathbb{J}, X) \subset \mathscr{F}_{mid}(\mathbb{J}, X)$, $\pm\Omega_\infty(\varphi, \varepsilon) \subset \Omega_\infty(\varphi_1, \varepsilon)$, where $\varepsilon > 0$.*

The following result is of great importance for the derivation of a spectral criterion for almost periodicity at infinity of mild solutions to differential equation (6.1).

Theorem 6.2. *For every mild solution $x \in \mathscr{F}(\mathbb{J}, X)$ to differential equation (6.1) with $\psi \in \mathscr{F}(\mathbb{J}, X)$ we have*

$$\Lambda_\infty(x, \mathscr{F}_{mid}) \subset (i^{-1}\sigma(A) \cap \mathbb{R}) \cup \Lambda_\infty(\psi, \mathscr{F}_{mid}). \tag{6.3}$$

Proof. To start with assume that $\mathbb{J} = \mathbb{R}$. Choose a number λ_0 so that $i\lambda_0$ lies outside $\Delta = (\sigma(A) \cap (i\mathbb{R})) \cup \Lambda_\infty(\psi, \mathscr{F}_{mid})$. Consider f in $L^1(\mathbb{R})$ such that $\widehat{f}(\lambda_0) \neq 0$ and $\operatorname{supp} \widehat{f} \cap \Delta = \emptyset$. Theorem 6.1 implies that $f * x$ lies in the domain $D(\mathcal{L})$ of the operator $\mathcal{L} = -\frac{d}{dt} + A$ and

$$\mathcal{L}(f * x) = f * \mathcal{L}x = f * \psi. \tag{6.4}$$

Inclusion (6.3) implies that the resolvent of A is defined in some neighbourhood $iV_0 \subset i\mathbb{R}$ of $i\lambda_0$, furthermore, $V_0 \cap \Lambda_\infty(\psi, \mathscr{F}_{mid}) = \emptyset$. Then from the definition of $\Lambda_\infty(\psi, \mathscr{F}_{mid})$ it follows that the function $\psi_0 = f * \psi$ belongs to the subspace $\mathscr{F}_{mid}$. Choose a number $\delta_0 > 0$ so that $[\lambda_0 - \delta_0; \lambda_0 + \delta_0] \subset V_0$.

Consider an infinitely differentiable function $\widehat{\varphi_0} : \mathbb{R} \to \mathbb{C}$ such that $\widehat{\varphi_0}(\lambda_0) \neq 0$ and $\operatorname{supp} \widehat{\varphi_0} \subset [\lambda_0 - \delta_0; \lambda_0 + \delta_0]$. It is the Fourier transform of some function $\varphi_0 \in L^1(\mathbb{R})$, while

$$\widehat{F}(\lambda) = \begin{cases} \widehat{\varphi}_0 R(i\lambda, A), & \lambda \in [\lambda_0 - \delta_0; \lambda_0 + \delta_0], \\ 0, & \lambda \notin [\lambda_0 - \delta_0; \lambda_0 + \delta_0], \end{cases}$$

is the Fourier transform of a summable function $F : \mathbb{R} \to End\, \mathscr{X}$.

Equation (6.4) implies that

$$F * \mathcal{L}(y * x) = \varphi_0 * x = F * \varphi * \psi \in \mathscr{F}_{mid}(\mathbb{R}, X).$$

Since $\widehat{\varphi_0}(\lambda_0) \neq 0$ it follows that λ_0 lies outside the set $\Lambda_\infty(x, \mathscr{F}_{mid})$. Thus, inclusion (6.3) is proved for functions defined on $\mathbb{R}$.

Now we assume that $\mathbb{J} = \mathbb{R}_+$. Consider a continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ with the properties $\operatorname{supp} \varphi \subset [1, \infty)$ and $\varphi \equiv 1$ on $[2, \infty)$. By φx we denote the function equal to zero on $(-\infty, 0]$ and the product of φ and x on $\mathbb{R}_+$. Then $\varphi x \in D(\mathcal{L})$ and $\mathcal{L}(\varphi x) = \dot{\varphi} x + \varphi\psi$, where $\varphi\psi$ is regarded as the identical zero on $\mathbb{R}_-$. Since $\dot{\varphi} x \in \mathscr{F}_{mid}(\mathbb{R}, X)$ and $\varphi = 0$ on $(-\infty, 0]$, we have

$$\Lambda_\infty(\psi, \mathscr{F}_{mid}(\mathbb{R}_+, X)) = \Lambda_\infty(\varphi\psi, \mathscr{F}_{mid}(\mathbb{R}, X)),$$

$$\Lambda_\infty(x, \mathscr{F}_{mid}(\mathbb{R}_+, X)) = \Lambda_\infty(\varphi x, \mathscr{F}_{mid}(\mathbb{R}, X)),$$

where $\mathscr{F}_{mid}(\mathbb{R}, X)$ is the subspace of functions in $\mathscr{F}(\mathbb{R}, X)$, which are extensions of functions in $\mathscr{F}_{mid}(\mathbb{R}_+, X)$ to $(-\infty, 0]$ with the property $\lim\limits_{t \to -\infty} y_1(t) = 0$ for every extension y_1 of y. □

Theorem 6.3. *Assume that a function ψ in (6.1) belongs to $AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$ and the set*

$$((i^{-1}\sigma(A)) \cap \mathbb{R}) \cup \Lambda_\infty(\psi, \mathscr{F}_{mid})$$

lacks limit points on $\mathbb{R}$. Then each bounded mild solution x to (6.1) is almost periodic at infinity with respect to $\mathscr{F}_{mid}(\mathbb{J}, X)$ ($x \in AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$).

The statement of Theorem 6.3 immediately follows from Theorem 6.2 and the spectral criterion of almost periodicity at infinity (Theorem 5.2).

Theorem 6.4. *Assume that a function ψ in (6.1) belongs to $\mathscr{F}_{mid} = \mathscr{F}_{mid}(\mathbb{J}, X)$ and the set $\sigma(A) \cap (i\mathbb{R})$ lacks limit points on $i\mathbb{R}$. Then each bounded mild solution x to (6.1) belongs to $AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$ and for each $\varepsilon > 0$ we can find numbers $i\lambda_1, \dots, i\lambda_m \in \sigma(A) \cap (i\mathbb{R})$ and functions $x_1, \dots, x_m$ in $(\mathscr{F}_{mid})_{sl,\infty}$ such that*

$$\sup_{t\in\mathbb{J}} \|x(t) - \sum_{k=1}^{m} x_k(t) e^{i\lambda_k t}\| < \varepsilon,$$

furthermore, $x(t) = \sum\limits_{k=1}^{m} x_k(t)e^{i\lambda_k t}$, $t \in \mathbb{R}$, whenever $\sigma(A) \cap (i\mathbb{R}) = \{i\lambda_1, \dots, i\lambda_m\}$ is a finite set.

Proof. Since x belongs to $\mathscr{F}_{mid}(\mathbb{J}, X)$ we have $\Lambda_\infty(\psi, \mathscr{F}_{mid}) = \emptyset$. Therefore,

$$\Lambda_\infty(x, \mathscr{F}_{mid}) \subset (i^{-1}\sigma(A)) \cap \mathbb{R}.$$

Suppose that $\Lambda_\infty(x, \mathscr{F}_{mid}) = \{\lambda_1, \lambda_2, \dots\}$ is a countable set. Then $\lim\limits_{n\to\infty} |\lambda_n| = \infty$. Regarding the quotient space $\mathscr{X} = \mathscr{F}(\mathbb{J}, X)/\mathscr{F}_{mid}(\mathbb{J}, X)$ as a Banach $L^1(\mathbb{R})$-module, the definition of $\Lambda_\infty(x, \mathscr{F}_{mid})$ directly implies that $\Lambda(\widetilde{x}, \widetilde{S}) = \{\lambda_1, \lambda_2, \dots\}$. Theorem 5.2 yields that the equivalence class $\widetilde{x} \in \mathscr{X}$ is an almost periodic vector in the $L^1(\mathbb{R})$-module $(\mathscr{X}, \widetilde{S})$.

Lemma 5.2 implies that the bounded approximation of the identity (f_n), $n \geq 1$, specified in (5.2) satisfies the condition

$$\lim_{n\to\infty} \|\widetilde{x} - \sum_{|\lambda_k|<n} \widehat{f}_0(\frac{\lambda_k}{n})\widetilde{x}_k\| = 0. \tag{6.5}$$

Since $\Lambda(\widetilde{x}_k) = \{\lambda_k\}$, it follows (see the proof of Theorem 5.1) that we can represent each function x_k as $x_k(t) = x_k^0(t)e^{i\lambda_k t}$ for $t \in \mathbb{J}$, where $x_k^0 \in (\mathscr{F}_{mid})_{sl,\infty}$ and $|\lambda_k| < n$. Therefore, (6.5) implies the existence of some sequence (y_n) in $\mathscr{F}_{mid}$ such that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{J}} \|x(t) - \sum_{k=1}^{n} \widehat{f}_0(\frac{\lambda_k}{n}) x_k^0(t) e^{i\lambda_k t} - y_n(t)\| = 0.$$

It remains to note that by property 3) of the subspace $\mathscr{F}_{mid}$ (see Definition 5) we can represent each function y_n, $n \geq 1$, in the form $y_n(t) = x_{n,k}^0(t)e^{i\lambda_k t}$ for $t \in \mathbb{J}$, where $|\lambda_k| < n$ and $x_{n,k}^0 \in \mathscr{F}_{mid}$. □

Corollary 6.1. *Consider a finite-dimensional Banach space X, an operator A in $End\, X$ and a function $\psi \in \mathscr{F}_{mid}(\mathbb{J}, X)$. Then every bounded solution $x : \mathbb{J} \to X$ to differential equation (6.1) belongs to $AP_\infty\mathscr{F}(\mathbb{J}, X; \mathscr{F}_{mid})$ and can be represented in the form*

$$x(t) = \sum_{k=1}^{m} x_k(t) e^{i\lambda_k t},\ x_k \in (\mathscr{F}_{mid})_{sl,\infty},\ t \in \mathbb{J},$$

provided that $\sigma(A) \cap (i\mathbb{R}) = \{i\lambda_1, \dots, i\lambda_m\}$.

Thus, our results are of some interest even for systems of ordinary differential equations.

7 Asymptotically finite-dimensional operator semigroups

Let us consider a bounded strongly continuous operator semigroup $T : \mathbb{R}_+ \to End\, X$ and a subspace of vanishing at infinity functions $\mathscr{F}_{mid} = \mathscr{F}_{mid}(\mathbb{R}_+, X)$ (see Definition 5.)

By $X_0 = X_0(T; \mathscr{F}_{mid})$ we denote a closed subspace of X defined by

$$X_0(T; \mathscr{F}_{mid}) = \{x_0 \in X : \text{a function } t \mapsto T(t)x : \mathbb{R}_+ \to X \text{ belongs to } \mathscr{F}_{mid}(\mathbb{R}_+, X)\}.$$

Definition 25. The semigroup T is called *asymptotically finite-dimensional* with respect to the subspace $\mathscr{F}_{mid}(\mathbb{R}_+, X)$, if the quotient space $X/X_0(T;\mathscr{F}_{mid})$ is finite-dimensional.

Asymptotically finite-dimensional semigroups were defined and studied in [19, 25].

Theorem 7.1. *Let $T : \mathbb{R}_+ \to End\, X$ be an asymptotic finite-dimensional semigroup of operators with respect to the subspace $\mathscr{F}_{mid}(\mathbb{R}_+, X)$ of vanishing at infinity functions. Then each function $t \mapsto T(t)x : \mathbb{R}_+ \to X$, $x \in X$, can be represented as*

$$T(t)x = \sum_{k=1}^{m} x_k(t)e^{i\lambda_k t},\ t \in \mathbb{R}_+,$$

where $x_k \in (\mathscr{F}_{mid})_{sl,\infty}$, $k = 1, \dots, m$, and $\{i\lambda_1, \dots, i\lambda_m\} \subset \sigma(A) \cap (i\mathbb{R})$.

Proof. The subspace $X_0 = X_0(T;\mathscr{F}_{mid})$ is closed and invariant under $T(t)$ for $t \geq 0$. Therefore, the quotient group of operators

$$\widetilde{T} : \mathbb{R}_+ \to End\, \mathscr{X},\ \widetilde{T}(t)\widetilde{x} = \widetilde{T(t)x},\ \widetilde{x} = x + X_0,$$

is well defined, where $\mathscr{X} = X/X_0$. The quotient space $\mathscr{X}$ is finite-dimensional, and therefore the operators $\widetilde{T}(t)$ for $t \geq 0$ are invertible. The boundedness of the semigroup T implies the uniform boundedness of the semigroup of inverse operators $\widetilde{T}(t) = \widetilde{T}(t)^{-1}$ for $t < 0$. Its generator $\widetilde{A}$ belongs to $End\, \mathscr{X}$, while its spectrum $\sigma(\widetilde{A}) = \{i\lambda_1, \dots, i\lambda_m\} \subset (i\mathbb{R}) \cap \sigma(A)$ is a finite set of cardinality m not exceeding $\dim \mathscr{X}$.

Since $\widetilde{T}$ is a bounded operator group acting in the finite-dimensional space $\mathscr{X}$, for any vector $x \in X$ and the equivalence class $\widetilde{x} \in \mathscr{X}$ containing x there exist vectors $x_1, \dots, x_n \in X$ such that

$$\widetilde{T}(t)\widetilde{x} = \sum_{k=1}^{m} \widetilde{x}_k e^{i\lambda_k t},\ t \geq 0,$$

$$\widetilde{T}(t)\widetilde{x}_k = e^{i\lambda_k t}\widetilde{x}_k,\ t \geq 0,\ 1 \leq k \leq m.$$

Consequently, the inclusion $T(t)x_k - e^{i\lambda_k t}x_k \in X_0$ holds for any $t \geq 0$. Hence, the function $s \mapsto T(t+s)x_k - e^{i\lambda_k(t+s)}x_k : \mathbb{R}_+ \to X$ belongs to $\mathscr{F}_{mid}(\mathbb{R}_+, X)$ for all $t \geq 0$.

Put $x_k(s) = T(s)x_k$ for $s \geq 0$ and $k = 1, \dots, m$. with $x_0(s) = T(s)x$ for $s \geq 0$ we infer that the function $x_0(s) = x(s) - \sum\limits_{k=1}^{m} x_k(s)e^{i\lambda_k s}$ for $s \geq 0$ belongs to $\mathscr{F}_{mid}(\mathbb{R}_+, X)$. □

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