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# ON $\mathbb{R}$-LINEAR CONJUGATION PROBLEM ON THE UNIT CIRCLE

**L. Primachuk, S. Rogosin, M. Dubatovskaya**

Communicated by M. Lanza de Cristoforis

**Key words:** $\mathbb{R}$-linear conjugation problem, vector-matrix $\mathbb{C}$-linear conjugation problem, continuous fractions, factorization, partial indices.

**AMS Mathematics Subject Classification:** 15A23, 30E25, 15A54.

**Abstract.** A new method of finding a solution to the $\mathbb{R}$-linear conjugation problem on the unit circle is proposed. The problem is studied under the assumption that its main coefficient is a segment of the Fourier series. The applied method is based on reducing the considered problem to the vector-matrix boundary value problem and applying the recently suggested generalization of G.N. Chebotarev's approach to the factorization of triangular matrix functions to its matrix coefficient.

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## 1 Introduction

We consider the solvability of the so called $\mathbb{R}$-linear boundary value problem (or Markushevich problem)

$$\varphi^+(t) = a(t)\varphi^-(t) + b(t)\overline{\varphi^-(t)} + f(t), \quad t \in \mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}, \tag{1.1}$$

where $\varphi^+(t), \varphi^-(t)$ are the boundary values of the unknown functions, analytic respectively inside and outside of the unit disc $\mathbb{D}$. The second name of this problem is related to the paper by A.I. Markushevich [8] in which the particular case of (1.1)

$$\varphi^+(t) = \overline{\varphi^-(t)}, \quad t \in \mathbb{T},$$

was studied. Problem (1.1) was considered later by several authors (see a brief description of the results in [6, § 20], [10]). In particular, in [5] it was shown that (1.1) on the unit circle is equivalent to the vector-matrix $\mathbb{C}$-linear conjugation problem

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \mathbb{T}, \tag{1.2}$$

where

$$G(t) = \begin{pmatrix} \frac{1}{\overline{a(t)}} & 0 \\ 0 & \frac{1}{\overline{a(t)}} \end{pmatrix} \begin{pmatrix} |a(t)|^2 - |b(t)|^2 & b(t) \\ -\overline{b(t)} & 1 \end{pmatrix},$$

$$g(t) = \frac{1}{\overline{a(t)}} \begin{pmatrix} \overline{a(t)}f(t) - b(t)\overline{f(t)} \\ -\overline{f(t)} \end{pmatrix},$$

and

$$\Phi^+(z) = \begin{pmatrix} \Phi_1^+(z) \\ \Phi_2^+(z) \end{pmatrix} = \begin{pmatrix} \varphi^+(z) \\ \overline{\varphi^-(\frac{1}{\overline{z}})} \end{pmatrix}, \Phi^-(z) = \begin{pmatrix} \Phi_1^-(z) \\ \Phi_2^-(z) \end{pmatrix} = \begin{pmatrix} \varphi^-(z) \\ \overline{\varphi^+(\frac{1}{\overline{z}})} \end{pmatrix}.$$

It follows (see, e.g. [7, 12]) that the solvability of $\mathbb{R}$-linear problem (1.1) is reduced to the factorization of the matrix coefficient $G(t)$ of vector-matrix problem (1.2).

## 2 Problem formulation. Notations

Let us introduce some preliminary facts and assumptions under which the problem will be studied below. In order to avoid additional technical difficulties, we assume that coefficients of problem (1.1) are Hölder-continuous functions on the unit circle.

Let $æ = \mathrm{ind}_{\mathbb{T}}\, a(t)$ be the Cauchy index (winding number) of the coefficient $a(t)$ in (1.1). Factorization of the scalar function $a(t)$ (see [4]) yields

$$a(t) = \chi^+(t) t^{æ} \chi^-(t), t \in \mathbb{T},$$

where $\chi^+(t), \chi^-(t)$ are boundary values of functions, analytic and nonvanishing in $D^+ = \mathbb{D}$ and $D^- = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, respectively. Denoting

$$\phi^+(z) = \frac{\varphi^+(z)}{\chi^+(z)}, \ \phi^-(z) = \varphi^-(z)\chi^-(z),$$

we rewrite boundary condition (1.1) in the following equivalent form:

$$\phi^+(t) = t^{æ}\phi^-(t) + q(t)\overline{\phi^-(t)} + h(t), \ t \in \mathbb{T}, \tag{2.1}$$

with

$$q(t) = \frac{b(t)}{\chi^+(t)\overline{\chi^-(t)}}, \ h(t) = \frac{f(t)}{\chi^+(t)}.$$

Any Hölder-continuous function can be represented (see, e.g., [4]) by the Sokhotsky-Plemelj formulas as

$$q(t) = q^+(t) + q^-(t)$$

with functions $q^{\pm}(t)$ analytically extended to $D^{\pm}$, respectively. Since the function $q^+(t)\overline{\phi^-(t)}$ admits an analytic extension to $D^+$, then finally we have the following equivalent form of (1.1):

$$\psi^+(t) = t^{æ}\psi^-(t) + p(t)\overline{\psi^-(t)} + h(t), \quad t \in \mathbb{T}, \tag{2.2}$$

where $\psi^+(t) := \phi^+(t) - q^+(t)\overline{\phi^-(t)}$, $\psi^-(t) := \phi^-(t)$, $p(t) := q^-(t)$.

In the above notation problem (2.2) is equivalent to the vector-matrix $\mathbb{C}$-linear conjugation problem

$$\Psi^+(t) = \begin{pmatrix} t^{æ} & 0 \\ 0 & t^{æ} \end{pmatrix} \begin{pmatrix} 1 - p(t)\overline{p(t)} & p(t) \\ -\overline{p(t)} & 1 \end{pmatrix} \Psi^-(t) + H(t), \ t \in \mathbb{T}. \tag{2.3}$$

Each Hölder-continuous function on the unit circle analytically extendible to $D^-$ can be expanded to absolutely converging Fourier series

$$p(t) = q^-(t) = \sum_{n=k}^{\infty} \frac{c_n}{t^n}, \quad k \geq 1.$$

In what follows we assume that the function $p(t)$ is rational, namely, we consider problem (2.2) under the following

**Assumption**. Let the function $p(t)$ be a finite segment of the Fourier series

$$p(t) = q^-(t) = \sum_{n=k}^{m} \frac{c_n}{t^n} =: \frac{C_{m-k}(t)}{t^m}, \tag{2.4}$$

with $C_{m-k}(t) = c_m + c_{m-1}t + \ldots + c_k t^{m-k}$ being a polynomial of order $m - k$.

**Remark 1.** The above assumption means the following condition on the coefficients $a(t), b(t)$:

$$\frac{b(t)}{|a(t)|}e^{\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\ln|a(\tau)|\frac{\tau+t}{\tau-t}d\tau} = \sum_{n=0}^{\infty} d_n t^n + \sum_{n=k}^{m} \frac{c_n}{t^n}, \tag{2.5}$$

with converging series $\sum\limits_{n=0}^{\infty} d_n t^n$ and a finite sum corresponding to negative powers of $t$.

This follows from the direct calculation of the expression $q(t) = \frac{b(t)}{\chi^+(t)\overline{\chi^-(t)}}$ and relation between the Cauchy kernel and the Schwarz kernel on the unit circle $\mathbb{T}$.

We also use the notation

$$\widetilde{p}(t) = \overline{p(t)} = \sum_{n=k}^{m} \overline{c_n} t^n.$$

Our aim is to establish a constructive algorithm for factorization of the coefficient of the homogeneous vector-matrix boundary value problem

$$\Psi^+(t) = \begin{pmatrix} t^{\text{æ}} & 0 \\ 0 & t^{\text{æ}} \end{pmatrix} \begin{pmatrix} 1 - p(t)\widetilde{p}(t) & p(t) \\ -\widetilde{p}(t) & 1 \end{pmatrix} \Psi^-(t), \quad t \in \mathbb{T}, \tag{2.6}$$

or what it the same, to factorize the matrix

$$A(t) = \begin{pmatrix} 1 - p(t)\widetilde{p}(t) & p(t) \\ -\widetilde{p}(t) & 1 \end{pmatrix}.$$

This algorithm is based on the recently proposed method of factorization of triangular matrix functions of an arbitrary order [11] which generalizes G.N. Chebotarev's approach [3].

Note that the matrix $A(t)$ has a special property det $A(t) \equiv 1$ and by the assumption the entries of $A(t)$ are rational functions. An algorithm proposed here is simpler than known algorithm for factorization of rational matrix [1] and uses the above property. At every step of transformation we keep the value of determinant of the transformed matrix.

For convenience we use the following explicit matrix solutions $X^{\pm}(z)$ to the homogeneous vector-matrix problem with the coefficient $A(t)$

$$X^+(t) = A(t)X^-(t), \quad t \in \mathbb{T}, \tag{2.7}$$

namely, $X^+(z) = E_2$ is the unit $2 \times 2$ matrix, and

$$X^-(z) = \begin{matrix} 0 \\ -m \end{matrix} \begin{pmatrix} 1 & -p(z) \\ \widetilde{p}(z) & 1 - p(z)\widetilde{p}(z) \end{pmatrix} \begin{matrix} k \\ -m+k \end{matrix}. \tag{2.8}$$

We write in (2.8) outside of the matrix $X^-(z)$ the orders at infinity of the corresponding elements of $X^-(z)$. It follows that the order of the columns of this matrix are equal to $-m$ and $-m+k$, respectively. Hence, the partial indices of the matrix $A(t)$ are integer numbers $\text{æ}_1, \text{æ}_2 = -\text{æ}_1$ from the interval $[k-m, m-k]$. Therefore, the first result on partial indices of vector-matrix boundary value problem (2.6) follows.

**Theorem 2.1.** *The partial indices of vector-matrix boundary value problem* (2.6) *are equal to* $\text{æ} + \text{æ}_1, \text{æ} + \text{æ}_2$, *where* $\text{æ}_1, \text{æ}_2 = -\text{æ}_1$ *are integer numbers from the interval* $[k-m, m-k]$ *and* $\text{æ} = \operatorname{ind}_{\mathbb{T}} a(t)$.

Note that the order of a column at infinity is equal to minimal order of its elements.

To illustrate the results of this theorem let us consider the following special case of the problem (2.2).

**Example 1.** Let $p(t) = \frac{c_1}{t} + \frac{c_2}{t^2}$. Then the partial indices of the matrix $A(t)$ are either $æ_1 = æ_2 = 0$ or $æ_1 = -1, æ_2 = 1$.

In this case $p(t) = \frac{c_1 t + c_2}{t^2}$, $\widetilde{p}(t) = \overline{c_1} t + \overline{c_2} t^2$. Thus

$$\frac{1}{p(t)} = \frac{t^2}{c_1 t + c_2} = \frac{t}{c_1} - \frac{c_2}{c_1^2} + \frac{\frac{c_1^2}{c_2^2}}{c_1 t + c_2}.$$

Right multiplying both sides of equality (2.7) by the matrix

$$\begin{pmatrix} 1 & 0 \\ \frac{t}{c_1} - \frac{c_2}{c_1^2} & 1 \end{pmatrix}$$

we get

$$X_1^-(t) = \begin{pmatrix} \frac{c_2^2}{c_1^2}\frac{1}{t^2} & -p(t) \\ \frac{t}{c_1} - \frac{c_2}{c_1^2} + \frac{c_2^2}{c_1^2}\frac{\widetilde{p}(t)}{t^2} & 1 - p(t)\widetilde{p}(t) \end{pmatrix}.$$

Then, multiplying both sides of the equality $X_1^+(t) = A(t)X_1^-(t)$ by the matrix

$$\begin{pmatrix} 1 & \frac{c_1^2}{c_2^2}(c_1 t + c_2) \\ 0 & 1 \end{pmatrix}$$

we get

$$X_2^-(t) = \begin{pmatrix} \frac{c_2^2}{c_1^2}\frac{1}{t^2} & 0 \\ \frac{t}{c_1} + \frac{c_2}{c_1^2}(|c_2|^2 - 1) + \frac{c_2^2}{c_1^2}\frac{\overline{c_1}}{t} & \frac{c_1^2}{c_2^2}t^2 \end{pmatrix}.$$

Let us consider two different cases: $a)$ $|c_2| = 1$; $b)$ $|c_2| \neq 1$.

In the case $a)$ we multiply the equality $X_2^+(t) = A(t)X_2^-(t)$ by the matrix

$$\begin{pmatrix} 1 & -\frac{c_1^3}{c_2^2}t \\ 0 & 1 \end{pmatrix}$$

and get

$$X_3^-(t) = \begin{pmatrix} \frac{c_2^2}{c_1^2}\frac{1}{t^2} & -\frac{c_1}{t} \\ \frac{t}{c_1} + \frac{c_2^2}{c_1^2}\frac{\overline{c_1}}{t} & -|c_1|^2 \end{pmatrix}.$$

Finally, multiplying by

$$\begin{pmatrix} 1 & 0 \\ \frac{t}{c_1|c_1|^2} & 1 \end{pmatrix}$$

we obtain in the case $a)$ the function $X_4^-$ having the normal form at infinity

$$X_4^-(t) = \begin{matrix} 0 \\ 1 \end{matrix} \begin{pmatrix} \frac{c_2^2}{c_1^2}\frac{1}{t^2} - \frac{1}{|c_1|^2} & -\frac{c_1}{t} \\ \frac{c_2^2}{c_1^2}\frac{\overline{c_1}}{t} & -|c_1|^2 \end{pmatrix} \begin{matrix} 1 \\ 0 \end{matrix}.$$

It follows that with $|c_2| = 1$ the partial indices of the matrix $A(t)$ are $æ_1 = æ_2 = 0$.

In the case $b$) we multiply $X_2^+(t) = A(t)X_2^-(t)$ by the matrix

$$\begin{pmatrix} 1 & \frac{c_1^2}{c_2}(|c_2|^2-1) \\ 0 & 1 \end{pmatrix}$$

and get

$$X_3^-(t) = \begin{pmatrix} \frac{c_2^2}{c_1^2}\frac{1}{t^2} & \frac{c_2}{t^2}(|c_2|^2-1) - \frac{c_1}{t} \\ \frac{t}{c_1} + \frac{c_2}{c_1^2}(|c_2|^2-1) + \frac{c_2^2}{c_1^2}\frac{\overline{c_1}}{t} & (|c_2|^2-1)^2 - |c_1|^2 + \frac{c_2\overline{c_1}}{t}(|c_2|^2-1) \end{pmatrix}.$$

Two different situation occur:

$b_1$) if $(|c_2|^2-1)^2 \neq |c_1|^2$, then $æ_1 = æ_2 = 0$;

$b_2$) if $(|c_2|^2-1)^2 = |c_1|^2$, then $æ_1 = -1, æ_2 = 1$.

Note that $m > k$ matrix (2.8) does not have in general the normal form at infinity and thus $X^{\pm}(z)$ is not a canonical matrix of problem (2.7) (see [9]). Therefore, in order to obtain a more exact description of partial indices of the matrix $A(t)$ we perform in the next section certain transformations reducing $X^-(z)$ to the normal form at infinity.

## 3 Solution algorithm

Here we describe an algorithm of construction of the canonical matrix for boundary value (2.7) from the matrix $X^{\pm}(z)$. The applied technique is similar to that in [11], i.e. we transform the matrix $X^{\pm}(z)$ by multiplying by the rational matrix with the unit determinant whose elements are obtained by expanding certain elements of the initial matrix into continuous fraction.

Let us start with the one-term representation of $\frac{1}{p(t)}$ as the continuous fraction:

$$\frac{1}{p(t)} = \frac{t^m}{C_{m-k}(t)} = Q_k(t) + \frac{C_{r_1}(t)}{C_{m-k}(t)}, \quad 0 \le r_1 < m-k. \tag{3.1}$$

Here $Q_k(t)$ is a polynomial of (exact) order $k$ and $C_{r_1}(t)$ is a polynomial of order $r_1, 0 \le r_1 < m-k$.

It follows from (3.1) that

$$1 - p(t)Q_k(t) = \frac{C_{r_1}(t)}{t^m}. \tag{3.2}$$

Now we right multiply both sides of (2.7) (or what is the same matrices $X^+(t) = E_2$ and $X^-(t)$) by the polynomial matrix with the unit determinant

$$P_1(t) = \begin{pmatrix} 1 & 0 \\ Q_k(t) & 1 \end{pmatrix}.$$

After transformation the matrix $X^-(t)$ becomes

$$X_1^-(t) = \begin{pmatrix} \frac{C_{r_1}(t)}{t^m} & -p(t) \\ F_1(t) & 1 - p(t)\widetilde{p}(t) \end{pmatrix},$$

where

$$F_1(t) = Q_k(t) + \frac{C_{r_1}(t)}{t^m}\widetilde{p}(t).$$

The order at infinity of this rational function is equal to

$$d_1 = -\max\{-(-k), -(-r_1+m-m)\} = -\max\{-(-k), -(-r_1)\} \le 0.$$

Note [9] that the matrix (2.7) is called the canonical matrix of $X^{\pm}(z)$ if it satisfies the boundary condition, nowhere vanishes in $\mathbb{C}$ and has the normal form at infinity, i.e. the sum of orders of its columns is equal to the index of the determinant of the matrix coefficient $A(t)$.

**Remark 2.** If $d_1 = 0$, then it means that the main (polynomial) part of $\frac{C_{r_1}(t)}{t^m}\widetilde{p}(t)$ is equal to $-Q_k(t) + Q_k(0)$, i.e. $F_1(t)$ contains only non-positive powers of $t$ and $F_1(\infty) \neq 0$ ($F_1(t) = a_0 + a_1 t^{-1} + \ldots + a_{m-k}t^{k-m}, a_0 \neq 0$); necessarily $r_1 = k$. In what follows we discuss this situation separately.

Note that such situation can occur only if $k < \frac{m}{2}$. Vice versa, if $k \geq \frac{m}{2}$, then necessarily $d_1 < 0$.

Let $d_1 < 0$. Then the orders at infinity of the matrix $X_1^-$ are described in the following diagram:

$$X_1^-(t) = \begin{matrix} m - r_1 \\ d_1 \end{matrix} \begin{pmatrix} \frac{C_{r_1}(t)}{t^m} & -p(t) \\ F_1(t) & 1 - p(t)\widetilde{p}(t) \end{pmatrix} \begin{matrix} k \\ -m + k \end{matrix}. \tag{3.3}$$

Thus, the orders at infinity of the column of $X_1^-(z)$ are equal to $d_1 < 0$ and $-m+k < 0$, respectively. It follows that this matrix also does not have normal form at infinity and we have to continue transformations. To do this we continue expanding $\frac{1}{p(t)}$ as a continuous fraction:

$$\frac{1}{p(t)} = \frac{t^m}{C_{m-k}(t)} = Q_k(t) + \frac{1}{\frac{C_{m-k}(t)}{C_{r_1}(t)}} = Q_k(t) + \frac{1}{Q_{m-k-r_1}(t) + \frac{C_{r_2}(t)}{C_{r_1}(t)}}, \quad 0 \leq r_2 < r_1, \tag{3.4}$$

where $C_{r_2}(t)$ is a polynomial of order $r_2$. It gives, in particular, the identity

$$C_{m-k}(t) = Q_{m-k-r_1}(t)C_{r_1}(t) + C_{r_2}(t). \tag{3.5}$$

Right multiplying both sides of the equality $P_1(t) = A(t)X_1^-(t)$ by the matrix

$$P_2(t) = \begin{pmatrix} 1 & Q_{m-k-r_1}(t) \\ 0 & 1 \end{pmatrix}$$

we obtain by using (3.5) the minus-matrix in the form

$$X_2^-(t) = \begin{pmatrix} \frac{C_{r_1}(t)}{t^m} & -\frac{C_{r_2}(t)}{t^m} \\ F_1(t) & F_2(t) \end{pmatrix},$$

where

$$\begin{aligned} F_2(t) &= F_1(t)Q_{m-k-r_1}(t) + 1 - p(t)\widetilde{p}(t) \\ &= Q_k(t)Q_{m-k-r_1}(t) + 1 + \widetilde{p}(t)\left[\frac{C_{r_1}(t)Q_{m-k-r_1}(t)}{t^m} - \frac{C_{m-k}(t)}{t^m}\right] \\ &= Q_k(t)Q_{m-k-r_1}(t) + 1 - \widetilde{p}(t)\frac{C_{r_2}(t)}{t^m}. \end{aligned}$$

Denote by $d_2$ the order at infinity of the function $F_2(t)$,

$$d_2 = -\max\{-(r_1 - m), -(-r_2)\} \leq 0.$$

Thus, the orders at infinity of the matrix $X_2^-$ are described in the following diagram:

$$X_2^-(t) = \begin{matrix} m - r_1 \\ d_1 \end{matrix} \begin{pmatrix} \frac{C_{r_1}(t)}{t^m} & -\frac{C_{r_2}(t)}{t^m} \\ F_1(t) & F_2(t) \end{pmatrix} \begin{matrix} m - r_2 \\ d_2 \end{matrix}, \tag{3.6}$$

and the order at infinity of the columns of $X_2^-(z)$ are equal to $d_1$ and $d_2$, respectively.

**Remark 3.** The main (polynomial) part of $p(t)\widetilde{p}(t)$ is equal to $p_0(t) = b_1 t + b_2 t^2 + \ldots + b_{m-k} t^{m-k}$, $b_{m-k} = c_k \overline{c_m} \neq 0$.

Let $d_1 = 0$. Then right multiplying both sides of the equality $X_1^+(t) = A(t)X_1^-(t)$ by the matrix

$$P_2^{(0)}(t) = \begin{pmatrix} 1 & -\frac{b_{m-k}t^{m-k}}{a_0} \\ 0 & 1 \end{pmatrix},$$

we get that the order at infinity of the $(1,2)$-element of the matrix $X_1^-(t)P_2^{(0)}(t)$ is equal to 0, but another element of the second column has the non- negative order at infinity. Hence the matrix $X_1^-(t)P_2^{(0)}(t)$ has the normal form at infinity and the partial indices are equal to $æ_1 = æ_2 = 0$.

If both $d_1, d_2$ are negative we can continue expansion of the function $\frac{1}{p(t)}$ in a continuous fraction (next step is a division of $\frac{C_{r_1}(t)}{C_{r_2}(t)}$ etc.) Since the sequence $r_1, r_2, \ldots, r_s, \ldots$ is strictly decreasing, we note that this sequence is finite: $r_1, r_2, \ldots, r_{\nu-1}, r_\nu, 0$.

Therefore, two situations are possible with $C$ being a complex constant:

- i) $\frac{C_{r_{\nu-1}}(t)}{C_{r_\nu}(t)} = Q_{r_{\nu-1}-r_\nu}(t) + \frac{C}{C_{r_\nu}(t)} \Leftrightarrow C_{r_{\nu-1}}(t) = Q_{r_{\nu-1}-r_\nu}(t)C_{r_\nu}(t) + C$;
- ii) $\frac{C_{r_{\nu-1}}(t)}{C_{r_\nu}(t)} = Q_{r_{\nu-1}-r_\nu}(t) + 0 \Leftrightarrow C_{r_{\nu-1}}(t) = Q_{r_{\nu-1}-r_\nu}(t)C_{r_\nu}(t)$.

Further transformation and formulation of the final result depends on the division of the polynomials on the former step (i.e. i) or ii) occurres) and on the parity of the number $\nu$.

If $\nu$ is an odd number, then in the case ii) the matrix $X_\nu^-(z)$ has the form

$$X_\nu^-(t) = \begin{pmatrix} \frac{C_{r_\nu}(t)}{t^m} & -\frac{C_{r_{\nu-1}}(t)}{t^m} \\ F_\nu(t) & F_{\nu-1}(t) \end{pmatrix}.$$

Right multiplying both sides of the equality $P_1(t)P_2(t)\ldots P_\nu(t) = A(t)X_\nu^-(t)$ by the matrix

$$P_{\nu+1}(t) = \begin{pmatrix} 1 & Q_{r_{\nu-1}-r_\nu}(t) \\ 0 & 1 \end{pmatrix}$$

we obtain $X_{\nu+1}^-(t)$ in the form

$$X_{\nu+1}^-(t) = \begin{pmatrix} \frac{C_{r_{\nu-1}}(t)}{t^m} & 0 \\ F_\nu(t) & F_{\nu+1}(t) \end{pmatrix},$$

where $F_{\nu+1}(t) = F_{\nu-1}(t) + F_\nu(t)Q_{r_{\nu-1}-r_\nu}(t) = \frac{t^m}{C_{r_{\nu-1}}(t)}$.

If $\nu$ is an even number, then in the case ii) the matrix $X_\nu^-(z)$ has the form

$$X_\nu^-(t) = \begin{pmatrix} \frac{C_{r_{\nu-1}}(t)}{t^m} & -\frac{C_{r_\nu}(t)}{t^m} \\ F_{\nu-1}(t) & F_\nu(t) \end{pmatrix}.$$

Right multiplying both sides of the equality $P_1(t)P_2(t)\ldots P_\nu(t) = A(t)X_\nu^-(t)$ by the matrix

$$P_{\nu+1}(t) = \begin{pmatrix} 1 & 0 \\ Q_{r_{\nu-1}-r_\nu}(t) & 1 \end{pmatrix}$$

we obtain $X_{\nu+1}^-(t)$ in the form

$$X_{\nu+1}^-(t) = \begin{pmatrix} 0 & -\frac{C_{r_\nu}(t)}{t^m} \\ F_{\nu+1}(t) & F_\nu(t) \end{pmatrix},$$

where $F_{\nu+1}(t) = F_{\nu-1}(t) + F_\nu(t)Q_{r_{\nu-1}-r_\nu}(t) = \frac{t^m}{C_{r_\nu}(t)}$.

In both situations $X^-_{\nu+1}(t)$ is a triangular matrix. Hence, further transformations (if necessary) can be performed by G.N. Chebotarev's method (see [3], cf. [11]).

Therefore, during our transformations we obtain a collection of rational functions $F_1, F_2, \ldots, F_\nu, F_{\nu+1}$ having the order at infinity $d_1, d_2, \ldots, d_\nu, d_{\nu+1}$, respectively. It leads to the following result for partial indices.

**Theorem 3.1.** *Let for certain $k, 1 \le k \le \nu$, the numbers $d$ satisfy inequalities*

$$d_1 < 0, \ldots, d_{k-1} < 0, d_k \ge 0. \tag{3.7}$$

*(i) If $d_k = 0$, then $æ_1 = æ_2 = 0$.*
*(ii) If $d_k > 0$, then $æ_1 = -\min\{m - r_k, d_k\}$, $æ_2 = -\min\{m - r_k, d_k\}$.*

**Theorem 3.2.** *Let all numbers $d_j$ be negative*

$$d_1 < 0, \ldots, d_\nu < 0, d_{\nu+1} < 0, \tag{3.8}$$

*then partial indices $æ_1, æ_2$ belong to the segment $[k - m + 1, m - k - 1]$.*

The interval of possible values of the partial indices becomes smaller (cf. Theorem 2.1) since $d_1 \le -1$.

As for the case i) after the same transformation we obtain either

$$X^-_{\nu+1}(t) = \begin{pmatrix} \frac{C_{r_{\nu-1}}(t)}{t^m} & -\frac{C}{t^m} \\ F_\nu(t) & F_{\nu+1}(t) \end{pmatrix}, \text{ for } \nu \text{ odd},$$

or

$$X^-_{\nu+1}(t) = \begin{pmatrix} \frac{C}{t^m} & -\frac{C_{r_\nu}(t)}{t^m} \\ F_{\nu+1}(t) & F_\nu(t) \end{pmatrix}, \text{ for } \nu \text{ even}.$$

Right multiplying the transformed boundary condition by

$$P_{\nu+2}(t) = \begin{pmatrix} 1 & 0 \\ \frac{C_{r_\nu}(t)}{C} & 1 \end{pmatrix} \quad \text{or} \quad P_{\nu+2}(t) = \begin{pmatrix} 1 & \frac{C_{r_\nu}(t)}{C} \\ 0 & 1 \end{pmatrix}$$

we again obtain triangular minus-factors

$$X^-_{\nu+2}(t) = \begin{pmatrix} 0 & \frac{C}{t^m} \\ F_{\nu+2}(t) & F_{\nu+1}(t) \end{pmatrix} \quad \text{or} \quad X^-_{\nu+2}(t) = \begin{pmatrix} \frac{C}{t^m} & 0 \\ F_{\nu+1}(t) & F_{\nu+2}(t) \end{pmatrix},$$

where $F_{\nu+2}(t) = F_\nu(t) + \frac{C_{r_\nu}(t)}{C} F_{\nu+1}(t) = \mp\frac{t^m}{C}$, which we can also transform by G.N. Chebotarev's method.

Thus, the corresponding results for partial indices can be formulated in the same manner as in case ii).

## 4 The case of the coefficient being an infinite series

Let us consider the homogeneous problem corresponding to (2.2)

$$\psi^+(t) = t^æ\psi^-(t) + p(t)\overline{\psi^-(t)}, \quad t \in \mathbb{T}, \tag{4.1}$$

in the case when main assumption (2.4) is not satisfied, i.e. $p(t) = q^-(t)$ is an infinite (absolutely converging) Fourier series

$$p(t) = \sum_{n=k}^{\infty} \frac{c_n}{t^n}. \tag{4.2}$$

As before, boundary value problem (4.1) is equivalent to the vector-matrix $\mathbb{R}$-linear conjugation problem

$$\Psi^+(t) = \begin{pmatrix} t^{æ} & 0 \\ 0 & t^{æ} \end{pmatrix} \begin{pmatrix} 1 - p(t)\overline{p(t)} & p(t) \\ -\overline{p(t)} & 1 \end{pmatrix} \Psi^-(t),\ t \in \mathbb{T}. \tag{4.3}$$

The solvability of the latter (as well as the solvability of (4.1)) depends on the factorization of the matrix

$$A(t) = \begin{pmatrix} 1 - p(t)\overline{p(t)} & p(t) \\ -\overline{p(t)} & 1 \end{pmatrix}.$$

Following the idea of [13] (see also [2]) we find a minimal positive integer number $N$ such that

$$\left| p(t) - \sum_{n=k}^{N} \frac{c_n}{t^n} \right| < |t^{æ}| = 1. \tag{4.4}$$

The same argument as in [13] and the above results for the case of the coefficient $p(t)$ being a finite sum leads to the following theorem.

**Theorem 4.1.** *Let the coefficient $p(t)$ of the problem (4.1) be of the form (4.2) and a positive integer number $N, N \geq k$, be defined by inequality (4.4).*

*Then partial indices $æ_1$, $æ_1$ of the matrix $A(t)$ belong to the segment $[k - N, N - k]$.*

*The number $l$ of solutions to (4.1), linear independent over the field $\mathbb{R}$, satisfies the following relations:*

*(a) if $æ \geq N - k$, then $l = 2æ$;*
*(b) if $0 \leq æ < N - k$ and $N - k - æ$ is an even number, then $2æ \leq l < æ + N - k$;*
*(c) if $0 \leq æ < N - k$ and $N - k - æ$ is an odd number, then $2æ \leq l < æ + N - k - 1$;*
*(d) if $k - N < æ < 0$ and $N - k - æ$ is an even number, then $0 \leq l \leq æ + N - k$;*
*(e) if $k - N < æ < 0$ and $N - k - æ$ is an odd number, then $0 \leq l \leq æ + N - k - 1$;*
*(f) if $æ \leq k - N$, then $l = 0$.*

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