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DISTRIBUTIONS OF COUNTABLE MODELS OF QUITE O-MINIMAL EHRENFEUCHT THEORIES

B.Sh. Kulpeshov, S.V. Sudoplatov

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Key words: quite o-minimal theory, Ehrenfeucht theory, distribution of countable models, decomposition formula.

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Abstract. We describe Rudin–Keisler preorders and distribution functions of numbers of limit models for quite o-minimal Ehrenfeucht theories. Decomposition formulas for these distributions are found.

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Introduction

The notion of a quite o-minimal theory was introduced and studied in [5]. This notion is a variation of weakly o-minimality [9]. This notion occurred fruitful enough producing both the structural description of models of these theories and the generalization of Mayer theorem [10]: it was shown that any countable quite o-minimal theory has either finitely many countable models, namely $3^k \cdot 6^s$ for any integers $k, s \geq 0$, or maximum, i.e. 2^ω countable models [7].

In the present paper, using a general theory of classification of countable models of complete theories [18, 19] as well as the description [7] of specificity for quite o-minimal theories, we describe distributions of countable models of quite o-minimal Ehrenfeucht theories in terms of Rudin–Keisler preorders and distribution functions of numbers of limit models. Moreover, we derive decomposition formulas for these distributions.

1 Preliminaries

In this section we give the necessary information from [18, 19].

Throughout the paper we consider countable complete first-order theories with infinite models.

Recall that the number of pairwise non-isomorphic models of a theory T that have cardinality λ is denoted by $I(T, \lambda)$.

Definition 1. [11] A theory T is called *Ehrenfeucht* if $1 < I(T, \omega) < \omega$.

Definition 2. [2] A type $p(\bar{x}) \in S(T)$ is said to be *powerful* in a theory T if every model $\mathcal{M}$ of T realizing p also realizes every type $q \in S(T)$, i.e., $\mathcal{M} \models S(T)$.

Since for any type $p \in S(T)$ there exists a countable model $\mathcal{M}$ of T, realizing p, and the model $\mathcal{M}$ realizes exactly countably many types, the availability of a powerful type implies that T is *small*, that is, the set $S(T)$ is countable. Hence for any type $q \in S(T)$ and its realization $\bar{a}$, there exists a *prime model* $\mathcal{M}(\bar{a})$ *over* $\bar{a}$, i. e., a model of T containing $\bar{a}$ with $\mathcal{M}(\bar{a}) \models q(\bar{a})$ and such that $\mathcal{M}(\bar{a})$ is elementarily embeddable to any model realizing the type q. Since all prime models over realizations of q are isomorphic, we denote these models by $\mathcal{M}_q$. Models $\mathcal{M}_q$ are also called *finitely generated* [12, 4], *almost prime* [3], or *q-prime*.

Definition 3. [18, 8, 14] Let p and q be types in $S(T)$. We say that the type p *is dominated by the type* q, or p *does not exceed* q *under the Rudin–Keisler preorder* (denoting this by $p \leq_{\mathrm{RK}} q$), if $\mathcal{M}_q \models p$, that is, $\mathcal{M}_p$ is an elementary submodel of $\mathcal{M}_q$ (denoting this by $\mathcal{M}_p \preceq \mathcal{M}_q$). Moreover, we say that a model $\mathcal{M}_p$ *is dominated by a model* $\mathcal{M}_q$, or $\mathcal{M}_p$ *does not exceed* $\mathcal{M}_q$ *under the Rudin–Keisler preorder*, and write $\mathcal{M}_p \leq_{\mathrm{RK}} \mathcal{M}_q$.

Syntactically, the condition $p \leq_{\mathrm{RK}} q$ (and hence also $\mathcal{M}_p \leq_{\mathrm{RK}} \mathcal{M}_q$) is expressed as follows: there exists a formula $\varphi(\bar{x}, \bar{y})$ such that the set $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\}$ is consistent and $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\} \vdash p(\bar{x})$. Since we deal with a small theory (there are only countably many types over any tuple $\bar{a}$ and so any consistent formula with parameters in $\bar{a}$ is deducible from a principal formula with parameters in $\bar{a}$), $\varphi(\bar{x}, \bar{y})$ can be chosen so that for any formula $\psi(\bar{x}, \bar{y})$, the set $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$ being consistent implies that $q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\} \vdash \psi(\bar{x}, \bar{y})$. In this case the formula $\varphi(\bar{x}, \bar{y})$ is said to be (q,p)-*principal*.

Definition 4. [18, 8, 14] Types p and q are said to be *domination-equivalent*, *realization-equivalent*, *Rudin–Keisler equivalent*, or RK-*equivalent* (denoting this by $p \sim_{\mathrm{RK}} q$) if $p \leq_{\mathrm{RK}} q$ and $q \leq_{\mathrm{RK}} p$. Models $\mathcal{M}_p$ and $\mathcal{M}_q$ are said to be *domination-equivalent*, *Rudin–Keisler equivalent*, or RK-*equivalent* (denoting this by $\mathcal{M}_p \sim_{\mathrm{RK}} \mathcal{M}_q$).

As in [20], types p and q are said to be *strongly domination-equivalent*, *strongly realization-equivalent*, *strongly Rudin–Keisler equivalent*, or *strongly* RK-*equivalent* (denoting this by $p \equiv_{\mathrm{RK}} q$) if for some realizations $\bar{a}$ and $\bar{b}$ of p and q respectively, both $\mathrm{tp}(\bar{b}/\bar{a})$ and $\mathrm{tp}(\bar{a}/\bar{b})$ are principal. Models $\mathcal{M}_p$ and $\mathcal{M}_q$ are said to be *strongly domination-equivalent*, *strongly Rudin–Keisler equivalent*, or *strongly* RK-*equivalent* (denoting this by $\mathcal{M}_p \equiv_{\mathrm{RK}} \mathcal{M}_q$).

Clearly, domination relations form preorders, and (strong) domination-equivalence relations are equivalence relations. Here, $\mathcal{M}_p \equiv_{\mathrm{RK}} \mathcal{M}_q$ implies $\mathcal{M}_p \sim_{\mathrm{RK}} \mathcal{M}_q$.

If $\mathcal{M}_p$ and $\mathcal{M}_q$ are not domination-equivalent then they are non-isomorphic. Moreover, non-isomorphic models may be found among domination-equivalent ones.

In Ehrenfeucht examples, models $\mathcal{M}^n_{p_0}, \ldots, \mathcal{M}^n_{p_{n-3}}$ are domination-equivalent but pairwise non-isomorphic.

A syntactic characterization for the model isomorphism between $\mathcal{M}_p$ and $\mathcal{M}_q$ is given by the following proposition. It asserts that the existence of an isomorphism between $\mathcal{M}_p$ and $\mathcal{M}_q$ is equivalent to the strong domination-equivalence of these models.

Proposition 1.1. [18, 14] *For any types $p(\bar{x})$ and $q(\bar{y})$ of a small theory T, the following conditions are equivalent:*

(1) *the models $\mathcal{M}_p$ and $\mathcal{M}_q$ are isomorphic;*

(2) *the models $\mathcal{M}_p$ and $\mathcal{M}_q$ are strongly domination-equivalent;*

(3) *there exist (p,q)- and (q,p)-principal formulas $\varphi_{p,q}(\bar{y}, \bar{x})$ and $\varphi_{q,p}(\bar{x}, \bar{y})$ respectively, such that the set*

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y}, \bar{x}), \varphi_{q,p}(\bar{x}, \bar{y})\}$$

is consistent;

(4) *there exists a (p,q)- and (q,p)-principal formula $\varphi(\bar{x}, \bar{y})$, such that the set*

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y})\}$$

is consistent.

Definition 5. [18, 14] Denote by $\mathrm{RK}(T)$ the set $\mathbf{PM}$ of all the isomorphism types of models $\mathcal{M}_p$, $p \in S(T)$, on which the relation of domination is induced by $\leq_{\mathrm{RK}}$, a relation deciding domination among $\mathcal{M}_p$, that is, $\mathrm{RK}(T) = \langle \mathbf{PM}; \leq_{\mathrm{RK}} \rangle$. We say that isomorphism types $\mathbf{M}_1, \mathbf{M}_2 \in \mathbf{PM}$ are *domination-equivalent* (denoting this by $\mathbf{M}_1 \sim_{\mathrm{RK}} \mathbf{M}_2$) if so are their representatives.

Clearly, the preordered set $\mathrm{RK}(T)$ has a least element, which is an isomorphism type of a prime model.

Proposition 1.2. [18, 14] *If $I(T,\omega) < \omega$ then $\mathrm{RK}(T)$ is a finite preordered set whose factor set $\mathrm{RK}(T)/\!\sim_{\mathrm{RK}}$, with respect to domination-equivalence $\sim_{\mathrm{RK}}$, forms a partially ordered set with a greatest element.*

Definition 6. [18, 19, 14, 16] A model $\mathcal{M}$ of a theory T is called *limit* if $\mathcal{M}$ is not prime over tuples and $\mathcal{M} = \bigcup\limits_{n\in\omega} \mathcal{M}_n$ for some elementary chain $(\mathcal{M}_n)_{n\in\omega}$ of prime models of T over tuples. In this case the model $\mathcal{M}$ is said to be *limit over a sequence* $\mathbf{q}$ *of types* or $\mathbf{q}$-*limit*, where $\mathbf{q} = (q_n)_{n\in\omega}$, $\mathcal{M}_n = \mathcal{M}_{q_n}$, $n \in \omega$. If the sequence $\mathbf{q}$ consists of unique type q then the $\mathbf{q}$-limit model is called *limit over the type* q.

Denote by $I_p(T)$ the number of pairwise non-isomorphic countable models of the theory T, each of which is prime over a tuple, by $I_l(T)$ the number of limit models of T, and by $I_l(T,q)$ the number of limit models over a type $q \in S(T)$.

Definition 7. [19, 16] A theory T is called *p-categorical* (respectively, *l-categorical*, *p-Ehrenfeucht*, and *l-Ehrenfeucht*) if $I_p(T) = 1$ (respectively, $I_l(T) = 1$, $1 < I_p(T) < \omega$, $1 < I_l(T) < \omega$).

Clearly, a small theory T is p-categorical if and only if T countably categorical, and if and only if $I_l(T) = 0$; T is p-Ehrenfeucht if and only if the structure $\mathrm{RK}(T)$ finite and has at least two elements; and T is p-Ehrenfeucht with $I_l(T) < \omega$ if and only if T is Ehrenfeucht.

Let $\widetilde{\mathbf{M}} \in \mathrm{RK}(T)/\!\sim_{\mathrm{RK}}$ be the class consisting of isomorphism types of domination-equivalent models $\mathcal{M}_{p_1}, \ldots, \mathcal{M}_{p_n}$. Denote by $\mathrm{IL}(\widetilde{\mathbf{M}})$ the number of isomorphism types of models each of which is limit over some type p_i.

Theorem 1.1. [18, 14] *For any countable complete theory T, the following conditions are equivalent:*

(1) *$I(T,\omega) < \omega$;*

(2) *T is small, $|\mathrm{RK}(T)| < \omega$ and $\mathrm{IL}(\widetilde{\mathbf{M}}) < \omega$ for any $\widetilde{\mathbf{M}} \in \mathrm{RK}(T)/\!\sim_{\mathrm{RK}}$.*

If (1) or (2) holds then T possesses the following properties:

(a) *$\mathrm{RK}(T)$ has a least element $\mathbf{M}_0$ (an isomorphism type of a prime model) and $\mathrm{IL}(\widetilde{\mathbf{M}_0}) = 0$;*

(b) *$\mathrm{RK}(T)$ has a greatest $\sim_{\mathrm{RK}}$-class $\widetilde{\mathbf{M}_1}$ (a class of isomorphism types of all prime models over realizations of powerful types) and $|\mathrm{RK}(T)| > 1$ implies $\mathrm{IL}(\widetilde{\mathbf{M}_1}) \geq 1$;*

(c) *if $|\widetilde{\mathbf{M}}| > 1$ then $\mathrm{IL}(\widetilde{\mathbf{M}}) \geq 1$.*

Moreover, the following decomposition formula holds:

$$I(T,\omega) = |\mathrm{RK}(T)| + \sum_{i=0}^{|\mathrm{RK}(T)/\sim_{\mathrm{RK}}|-1} \mathrm{IL}(\widetilde{\mathbf{M}_i}), \tag{1.1}$$

where $\widetilde{\mathbf{M}_0}, \ldots, \widetilde{\mathbf{M}}_{|\mathrm{RK}(T)/\sim_{\mathrm{RK}}|-1}$ are all elements of the partially ordered set $\mathrm{RK}(T)/\!\sim_{\mathrm{RK}}$.

Definition 8. [22] The *disjoint union* $\bigsqcup_{n\in\omega}\mathcal{M}_n$ of pairwise disjoint structures $\mathcal{M}_n$ for pairwise disjoint predicate languages Σ_n, $n\in\omega$, is the structure of language $\bigcup_{n\in\omega}\Sigma_n\cup\{P_n^{(1)}\mid n\in\omega\}$ with the universe $\bigsqcup_{n\in\omega}M_n$, $P_n=M_n$, and interpretations of predicate symbols in Σ_n coinciding their interpretations in $\mathcal{M}_n$, $n\in\omega$. The *disjoint union of theories* T_n for pairwise disjoint languages Σ_n accordingly, $n\in\omega$, is the theory

$$\bigsqcup_{n\in\omega}T_n\rightleftharpoons\mathrm{Th}\left(\bigsqcup_{n\in\omega}\mathcal{M}_n\right),$$

where $\mathcal{M}_n\models T_n$, $n\in\omega$.

Clearly, the theory $T_1\sqcup T_2$ does not depend on the choice of disjoint union $\mathcal{M}_1\sqcup\mathcal{M}_2$ of models $\mathcal{M}_1\models T_1$ and $\mathcal{M}_2\models T_2$. Moreover, the cardinality of $\mathrm{RK}(T_1\sqcup T_2)$ is equal to the product of cardinalities for $\mathrm{RK}(T_1)$ and $\mathrm{RK}(T_2)$, and the relation $\leq_{\mathrm{RK}}$ on $\mathrm{RK}(T_1\sqcup T_2)$ equals the Pareto relation [17] defined by preorders in $\mathrm{RK}(T_1)$ and $\mathrm{RK}(T_2)$. Indeed, each type $p(\bar{x})$ of $T_1\sqcup T_2$ is isolated by set consisting of some types $p_1(\bar{x}^1)$ and $p_2(\bar{x}^2)$ of theories T_1 and T_2 respectively, as well as of formulas $P^1(x_i^1)$ and $P^2(x_j^2)$ for all coordinates in tuples $\bar{x}^1$ and $\bar{x}^2$. For types $p(\bar{x})$ and $p'(\bar{y})$ of $T_1\sqcup T_2$, we have $p(\bar{x})\leq_{\mathrm{RK}}p'(\bar{y})$ if and only if $p_1(\bar{x}^1)\leq_{\mathrm{RK}}p_1'(\bar{y}^1)$ (in T_1) and $p_2(\bar{x}^2)\leq_{\mathrm{RK}}p_2'(\bar{y}^2)$ (in T_2).

Thus, the following proposition holds.

Proposition 1.3. [19, 15] *For any small theories T_1 and T_2 of disjoint predicate languages Σ_1 and Σ_2 respectively, the theory $T_1\sqcup T_2$ is mutually* RK*-coordinated with respect to its restrictions to Σ_1 and Σ_2. The cardinality of* $\mathrm{RK}(T_1\sqcup T_2)$ *is equal to the product of cardinalities for* $\mathrm{RK}(T_1)$ *and* $\mathrm{RK}(T_2)$*, i. e.,*

$$I_p(T_1\sqcup T_2,\omega)=I_p(T_1,\omega)\cdot I_p(T_2,\omega),\tag{1.2}$$

and the relation $\leq_{\mathrm{RK}}$ on $\mathrm{RK}(T_1\sqcup T_2)$ *equals the Pareto relation defined by preorders in* $\mathrm{RK}(T_1)$ *and* $\mathrm{RK}(T_2)$*.*

Remark 1. [19, 15] An isomorphism of limit models of theory $T_1\sqcup T_2$ is defined by isomorphisms of restrictions of these models to the sets P_1 and P_2. In this case, a countable model is limit if and only if some its restriction (to P_1 or to P_2) is limit and the following equality holds:

$$I(T_1\sqcup T_2,\omega)=I(T_1,\omega)\cdot I(T_2,\omega).\tag{1.3}$$

Thus, the operation $\sqcup$ preserves both p-Ehrenfeuchtness and l-Ehrenfeuchtness (if components are p-Ehrenfeucht), and, by (1.3), we obtain the equality

$$I_l(T_1\sqcup T_2)=I_l(T_1)\cdot I_p(T_2,\omega)+I_p(T_1,\omega)\cdot I_l(T_2)+I_l(T_1)\cdot I_l(T_2).\tag{1.4}$$

2 *O*-minimal and quite *o*-minimal theories

Recall [13] that a linearly ordered structure $\mathcal{M}$ is *o-minimal* if any definable (with parameters) subset of M is a finite union of singletons and open intervals (a,b), where $a\in M\cup\{-\infty\}$, $b\in M\cup\{+\infty\}$. A theory T is *o-minimal* if each model of T is o-minimal.

As examples of Ehrenfeucht o-minimal theories, we mention the theories $T^1\rightleftharpoons\mathrm{Th}((\mathbb{Q};<,c_n)_{n\in\omega}$ and $T^2\rightleftharpoons\mathrm{Th}((\mathbb{Q};<,c_n,c_n')_{n\in\omega}$, where $<$ is an ordinary strict order on the set $\mathbb{Q}$ of rationals, constants c_n form a strictly increasing sequence, and constants c_n' form a strictly decreasing sequence, $c_n<c_n'$, $n\in\omega$.

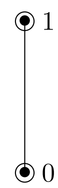

Figure 1:

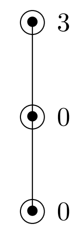

Figure 2:

The theory T^1 is an Ehrenfeucht's example [21] with $I(T^1, \omega) = 3$. It has two almost prime models and one limit model:

• a prime model with empty set of realizations of type $p(x)$ isolated by the set $\{c_n < x \mid n \in \omega\}$ of formulas;

• a prime model over a realization of the type $p(x)$, with the least realization of that type;

• one limit model over the type $p(x)$, with the set of realizations of $p(x)$ forming an open convex set.

The Hasse diagram for the Rudin–Keisler preorder $\leq_{\rm RK}$ and values of the function IL of distributions of numbers of limit models for $\sim_{\rm RK}$-classes of T^1 is represented in Fig. 1.

The theory T^2 has six pairwise non-isomorphic countable models:

• a prime model with empty set of realizations of type $p(x)$ isolated by the set $\{c_n < x \mid n \in \omega\} \cup \{x < c'_n \mid n \in \omega\}$;

• a prime model over a realization of $p(x)$, with a unique realization of this type;

• a prime model over a realization of type $q(x, y)$ isolated by the set $p(x) \cup p(y) \cup \{x < y\}$; here the set of realizations of $p(x)$ forms a closed interval $[a, b]$;

• three limit models over the type $q(x, y)$, in which the sets of realizations of $q(x, y)$ are convex sets of forms $(a, b]$, $[a, b)$, (a, b) respectively.

In Figure 2 we represent the Hasse diagram of Rudin–Keisler preorders $\leq_{\rm RK}$ and values of distribution functions IL of numbers of limit models on $\sim_{\rm RK}$-equivalence classes for the theory T^2.

The following theorem shows that the number of countable models of Ehrenfeucht *o*-minimal theories is exhausted by combinations of these numbers for the theories T_1 and T_2.

Theorem 2.1. [10] *Let T be an o-minimal theory in a countable language. Then either T has 2^ω countable models or T has exactly $3^k \cdot 6^s$ countable models, where k and s are natural numbers. Moreover, for any $k, s \in \omega$ there is an o-minimal theory T with exactly $3^k \cdot 6^s$ countable models.*

The notion of *weak o-minimality* was initially deeply studied by D. Macpherson, D. Marker, and C. Steinhorn in [9]. A subset A of a linearly ordered structure M is *convex* if for any $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. A *weakly o-minimal structure* is a linearly ordered structure $\mathcal{M} = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters) subset of the structure $\mathcal{M}$ is a finite union of convex sets in $\mathcal{M}$. Real closed fields with a proper convex valuation ring provide an important example of weakly o-minimal (not o-minimal) structures.

In the following definitions we assume that M is a weakly o-minimal structure, $A, B \subseteq M$, M is $|A|^+$-saturated, and $p, q \in S_1(A)$ are non-algebraic types.

Definition 9. (B.S. Baizhanov, [1]) We say that p is not *weakly orthogonal* to q ($p \not\perp^w q$) if there are an A-definable formula $H(x, y)$, $a \in p(M)$, and $b_1, b_2 \in q(M)$ such that $b_1 \in H(M, a)$ and $b_2 \notin H(M, a)$.

Lemma 2.1. ([1], Corollary 34 (iii)) *The relation $\not\perp^w$ of the weak non-orthogonality is an equivalence relation on $S_1(A)$.*

In [5], quite o-minimal theories were introduced forming a subclass of the class of weakly o-minimal theories and preserving a series of properties for o-minimal theories. For instance, in [6], $\aleph_0$-categorical quite o-minimal theories were completely described. This description implies their binarity (the similar result holds for $\aleph_0$-categorical o-minimal theories).

Definition 10. [5] We say that p is not *quite orthogonal* to q ($p \not\perp^q q$) if there is an A-definable bijection $f : p(M) \to q(M)$. We say that a weakly o-minimal theory is *quite o-minimal* if the relations of weak and quite orthogonality coincide for 1-types over arbitrary sets of models of the given theory.

Clearly, any o-minimal theory is quite o-minimal, since for non-weakly orthogonal 1-types over an arbitrary set A there is an A-definable strictly monotone bijection between the sets of realizations of these types.

Example 1. Let $\mathcal{M} = \langle M, <, P_1^1, P_2^1, E_1^2, E_2^2, f^1\rangle$ be a linearly ordered structure such that $\mathcal{M}$ is a disjoint union of interpretations of unary predicates P_1 and P_2, where $P_1(\mathcal{M}) < P_2(\mathcal{M})$. We identify the interpretations of P_1 and P_2 with $\mathbb{Q} \times \mathbb{Q}$ having the lexicographical order. For the interpretations of binary predicates $E_1(x, y)$ and $E_2(x, y)$ we take equivalence relations on $P_1(\mathcal{M})$ and $P_2(\mathcal{M})$, respectively, such that for every $x = (n_1, m_1), y = (n_2, m_2) \in \mathbb{Q} \times \mathbb{Q}$,

$$E_i(x, y) \Leftrightarrow n_1 = n_2, \text{ where } i = 1, 2.$$

The symbol f is interpreted by partial unary function with $\mathrm{Dom}(f) = P_1(\mathcal{M})$ and $\mathrm{Range}(f) = P_2(\mathcal{M})$ such that $f((n, m)) = (n, -m)$ for all $(n, m) \in \mathbb{Q} \times \mathbb{Q}$.

It is easy to see that $E_1(x, y)$ and $E_2(x, y)$ are $\emptyset$–definable equivalence relations dividing $P_1(\mathcal{M})$ and $P_2(\mathcal{M})$, respectively, into infinitely many infinite convex classes. We assert that f is strictly decreasing on each class $E_1(a, \mathcal{M})$, where $a \in P_1(\mathcal{M})$, and f is strictly increasing on $P_1(\mathcal{M})/E_1$. It is clear that $\mathrm{Th}(\mathcal{M})$ is a quite o-minimal theory. The theory $\mathrm{Th}(\mathcal{M})$ is not o-minimal, since $E_1(a, M)$ defines a convex set which is not a union of finitely many intervals in $\mathcal{M}$.

The following theorem, proved in [7], strengthens Theorem 2.1.

Theorem 2.2. *Let T be a quite o-minimal theory in a countable language. Then either T has 2^ω countable models or T has exactly $3^k \cdot 6^s$ countable models, where k and s are natural numbers. Moreover, for any $k, s \in \omega$ there is an o-minimal theory T with exactly $3^k \cdot 6^s$ countable models.*

It was shown in [7] that quite o-minimal Ehrenfeucht theories are binary. But this does not hold in general:

Example 2. Let $\mathcal{M} = \langle M; <, P_1^1, P_2^1, P_3^1, f^2\rangle$ be a linearly ordered structure such that M is a disjoint union of interpretations of unary predicates P_1, P_2, and P_3, where $P_1(\mathcal{M}) < P_2(\mathcal{M}) < P_3(\mathcal{M})$. We identify each interpretation of P_i ($1 \le i \le 3$) with the set $\mathbb{Q}$ of rational numbers, with ordinary orders. The symbol f is interpreted by partial binary function with $\mathrm{Dom}(f) = P_1(\mathcal{M}) \times P_2(\mathcal{M})$ and $\mathrm{Range}(f) = P_3(\mathcal{M})$ such that $f(a, b) = a + b$ for all $(a, b) \in \mathbb{Q} \times \mathbb{Q}$.

Clearly, $\mathrm{Th}(\mathcal{M})$ is a quite o-minimal theory. Take arbitrary $a \in P_1(\mathcal{M}), b \in P_2(\mathcal{M})$. Obviously, the functions $f_b(x) := f(x, b)$ and $g_a(y) := f(a, y)$ are strictly increasing on $P_1(\mathcal{M})$ and $P_2(\mathcal{M})$, respectively. Take an arbitrary $a_1 \in P_1(\mathcal{M})$ with $a < a_1$ and consider the following formulas:

$$\Phi_1(y, a, a_1, b) := (f_b(a) = f_y(a_1) \wedge P_2(y)),$$

$$\Phi_n(y, a, a_1, b) := \exists y_0[\Phi_{n-1}(y_0, a, a_1, b) \wedge f_{y_0}(a) = f_y(a_1) \wedge P_2(y)], \quad n \ge 2.$$

Clearly, $\mathcal{M} \models \exists! y \Phi_n(y, a, a_1, b)$ for each $n < \omega$, i.e., $\mathrm{dcl}(\{a, a_1, b\})$ infinite. Then considering the following set of formulas:

$$\{P_2(x)\} \cup \{x < b\} \cup \{\forall y [\Phi_n(y, a, a_1, b) \to x < y] \mid n \in \omega\}$$

and checking its local consistency, we obtain that there exists a non-principal 1-type over $\{a, a_1, b\}$ extending the given set of formulas. Whence, $\mathrm{Th}(\mathcal{M})$ has 2^ω countable models. Since for each finite set $A \subseteq M$ there are only at most countably many 1-types over A, we conclude that the theory $\mathrm{Th}\mathcal{M})$ is small.

Thus, the following proposition is proved:

Proposition 2.1. *There exists a small quite o-minimal theory which is not binary.*

Definition 11. [18, 15] We say that small theories T_1 and T_2 are *characteristically equivalent* and write $T_1 \sim_{\mathrm{ch}} T_2$ if the structure $\mathrm{RK}(T_1)$ is isomorphic to the structure $\mathrm{RK}(T_2)$ and, by the corresponding replacement of isomorphism types in $\mathrm{RK}(T_1)$ to isomorphism types in $\mathrm{RK}(T_2)$, the distribution function IL for numbers of limit models of T_1 is transformed to the distribution function for numbers of limit models of T_2.

Recall that theories T_0 and T_1 of languages Σ_0 and Σ_1 respectively are said to be *similar* if for any models $\mathcal{M}_i \models T_i$, $i = 0, 1$, there are formulas of T_i, defining in $\mathcal{M}_i$ predicates, functions and constants of language Σ_{1-i} such that the corresponding structure of Σ_{1-i} is a model of T_{1-i}.

The following theorem is a refinement of Theorem 2.2 for quite o-minimal Ehrenfeucht theories producing the direct generalization of Theorem 1.1.5.3 in [18].

Theorem 2.3. *Any model of a quite o-minimal Ehrenfeucht theory T is densely ordered except, possibly, finitely many elements with successors or predecessors laying in the definable closure of the empty set. The theory T is characteristically equivalent to some finite disjoint union of theories of form T^1, T^2 ($T \sim_{\mathrm{ch}} \bigsqcup\limits_{i=1}^{k} T_i^1 \sqcup \bigsqcup\limits_{j=1}^{l} T_j^2$, where T_i^1 are similar to T^1 and T_j^2 are similar to T^2) and has $3^k \cdot 6^l$ pairwise non-isomorphic countable models.*

3 Distributions of countable models

In this section, using Theorems 1.1 and 2.3 we give a description of Rudin–Keisler preorders and distribution functions of numbers of limit models for quite o-minimal Ehrenfeucht theories, as well as propose representations of this distributions, based on decomposition formula (1.1).

In view of Proposition 1.3 and Theorem 2.3 the Hasse diagrams for distributions of countable models for quite o-minimal Ehrenfeucht theories are constructed as figures of Pareto relations for disjoint unions of copies of theories T^1 and T^2, i.e., they are combinations of the Hasse diagrams shown in Fig. 1 and 2.

Now we describe the distributions above for the theories $\bigsqcup\limits_{i=1}^{k} T_i^1$.

In Fig. 3 and 4 the Hasse diagrams are shown for the theories $T_1^1 \sqcup T_2^1$ and $T_1^1 \sqcup T_2^1 \sqcup T_3^1$, respectively.

Adding new disjoint copies of T^1 we note that $\mathrm{RK}(T)$, where $T = \bigsqcup\limits_{i=1}^{k} T_i^1$, forms a k-dimensional cube Q_k [17], i.e., represented as a finite Boolean algebra $\mathcal{B}_k$ with k atoms $u_1, \ldots, u_k$. These atoms correspond to models realizing unique 1-types in the set $\{p_1(x), \ldots, p_k(x)\}$ of all non-principal 1-types. Thus, each element $u_{i_1} \vee \ldots \vee u_{i_t}$ of the Boolean algebra $\mathcal{B}_k$ corresponds to an almost prime model of T, realizing only non-principal 1-types $p_{i_1}(x), \ldots, p_{i_t}(x)$.

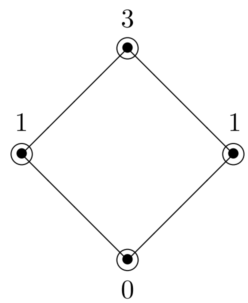

Figure 3:

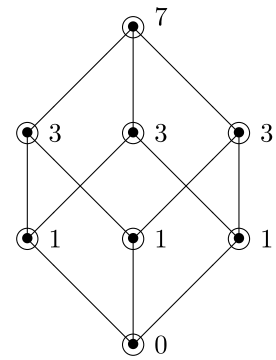

Figure 4:

The number of limit models for the element $u_{i_1} \vee \ldots \vee u_{i_t}$, i. e., of limit models over (unique) completion $q_{i_1,\ldots,i_t}(x_1,\ldots,x_t)$ of the type $p_{i_1}(x_1) \cup \ldots \cup p_{i_t}(x_t)$ equals $2^t - 1$. Indeed, choosing a prime model over the type $q_{i_1,\ldots,i_t}$ we have 2^t possibilities characterizing an independent choice either prime or limit model over each type p_{i_j}. Removing the (unique) possibility of choice of prime model for each type p_{i_j}, i. e., of prime model over the type $q_{i_1,\ldots,i_t}$, we obtain the following value of the number of limit models over the type $q_{i_1,\ldots,i_t}$:

$$I_l(T, q_{i_1,\ldots,i_t}) = 2^t - 1 \tag{3.1}$$

Since there are 3^k countable models, 2^k of them are almost prime, and the remaining are limit ones, the total number of limit models, calculated on the basis of relations (3.1) (see also (1.4)) leads to the following:

$$\sum_{q_{i_1,\ldots,i_m}} I_l(T, q_{i_1,\ldots,i_t}) = \sum_{t=1}^{k} (2^t - 1) \cdot C_k^t = 3^k - 2^k. \tag{3.2}$$

By (3.2) for the theories $\bigsqcup\limits_{i=1}^{k} T_i^1$, we have the following representation of decomposition formula (1.1):

$$3^k = 2^k + \sum_{t=1}^{k} (2^t - 1) \cdot C_k^t. \tag{3.3}$$

For $k = 1$ we have $3 = 2 + 1$, for $k = 2$: $9 = 4 + 1 \cdot 2 + 3 \cdot 1$, for $k = 3$: $27 = 8 + 1 \cdot 3 + 3 \cdot 3 + 7 \cdot 1$, as shown in Fig. 1, 3, 4, respectively.

Now we describe the distributions for the theories $\bigsqcup\limits_{j=1}^{s} T_j^2$.

In Fig. 5 and 6 the Hasse diagrams are shown for the theories $T_1^2 \sqcup T_2^2$ and $T_1^2 \sqcup T_2^2 \sqcup T_3^2$, respectively. These diagrams form distributive lattices, which are obtained as Cartesian products, respectively, from four-element and eight-element distributive lattices by extensions of each two-dimensional cube by four new elements such that each edge of given Boolean algebra contains new intermediate element. The theory $T_1^2 \sqcup T_2^2$ has $6^2 = 36$ countable models, where 9 of them are almost prime and 27 are limit. The theory $T_1^2 \sqcup T_2^2 \sqcup T_3^2$ has $6^3 = 216$ countable models, where 27 of them are almost prime and 189 are limit.

Continuing the process of adding disjoint copies of the theory T^2, we observe that $\mathrm{RK}(T)$, where $T = \bigsqcup\limits_{j=1}^{s} T_j^2$, is obtained from s-dimensional cube replacing edges by three-element lines and forming

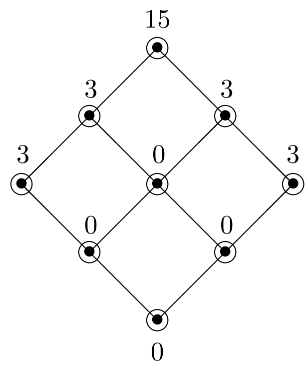

Figure 5:

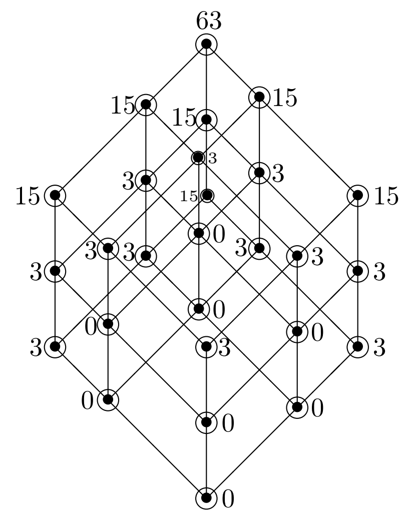

Figure 6:

s-dimensional linear space $\mathcal{L}_{s,3}$ over the field $\mathbb{Z}_3$. Therefore, $|\mathrm{RK}(T)| = 3^s$. Here, the theory T has exactly s non-principal 1-types $p_1(x), \dots, p_s(x)$, each of which, in almost prime models, either does not have realizations, or has unique realization, or has infinitely many realizations including the least and the greatest ones.

To calculate the number of limit models, we note that the structure $\mathcal{L}_{s,3}$ contains the s-dimensional cube, whose vertices, 2^s ones, symbolize prime models over completions $q_{j_1,\dots,j_m}(x_1,\dots,x_m)$ of types $p_{j_1}(x_1) \cup \ldots \cup p_{j_m}(x_m)$ such that these prime models have at most one realization for each type $p_1(x), \dots, p_s(x)$ and do not generate limit models. Furthermore, we choose among s types p_j some m types, responsible for the existence of limit models generated by realizations of these types, and obtain $4^m - 1$ possibilities for these limit models by variations of existence or absence of least and greatest realizations. Together with the choice of m types we choose among remaining $s-m$ types some r types having unique realizations. Under these conditions of choice we have $(4^m - 1) \cdot C_s^m \cdot C_{s-m}^r$ possibilities. Summarizing these values we obtain the following equations:

$$\sum_{q_{i_1,\dots,i_m}} I_l(T, q_{i_1,\dots,i_m}) = \sum_{m=1}^{s} \sum_{r=0}^{s-m} (4^m - 1) \cdot C_s^m \cdot C_{s-m}^r$$

$$= \sum_{m=1}^{s} \left(\sum_{r=0}^{s-m} C_{s-m}^r \right) (4^m - 1) \cdot C_s^m = \sum_{m=1}^{s} 2^{s-m} \cdot (4^m - 1) \cdot C_s^m = 6^s - 3^s. \tag{3.4}$$

By (3.4) for the theory $\bigsqcup\limits_{j=1}^{s} T_j^2$, we have the following representation of decomposition formula (1.1):

$$6^s = 3^s + \sum_{m=1}^{s} 2^{s-m} \cdot (4^m - 1) \cdot C_s^m. \tag{3.5}$$

For $s = 1$ we have $6 = 3 + 1 \cdot 3 \cdot 1$, for $s = 2$: $36 = 9 + 2 \cdot 3 \cdot 2 + 1 \cdot 15 \cdot 1$, for $s = 3$: $216 = 27 + 4 \cdot 3 \cdot 3 + 2 \cdot 15 \cdot 3 + 1 \cdot 63 \cdot 1$, as shown in Fig. 2, 5, 6, respectively.

Finally, we describe the indicated distributions for the theories $\bigsqcup\limits_{i=1}^{k} T_i^1 \sqcup \bigsqcup\limits_{j=1}^{s} T_j^2$.

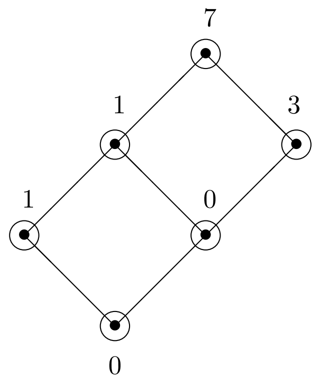

Figure 7:

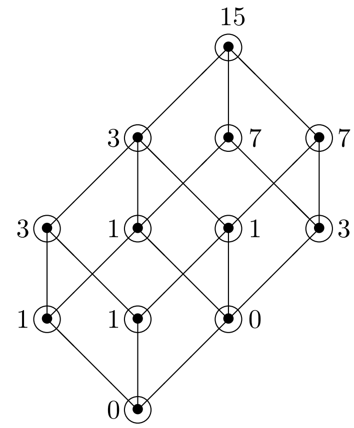

Figure 8:

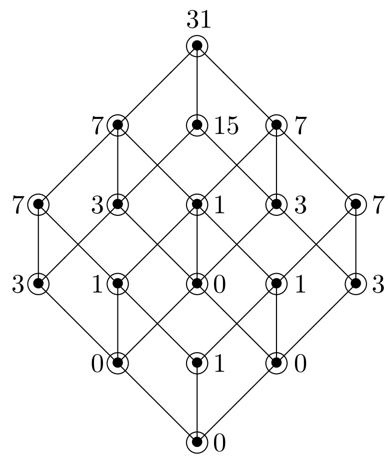

Figure 9:

In Fig. 7, 8 and 9, the Hasse diagrams are shown for the theories $T_1^1 \sqcup T_1^2$, $T_1^1 \sqcup T_2^1 \sqcup T_1^2$, and $T_1^1 \sqcup T_1^2 \sqcup T_1^2$, respectively. The theory $T_1^1 \sqcup T_1^2$ has $3 \cdot 6 = 18$ countable models, 6 of them are almost prime and 12 are limit ones. The theory $T_1^1 \sqcup T_2^1 \sqcup T_1^2$ has $3^2 \cdot 6 = 54$ countable models, 12 of them are almost prime and 42 are limit ones. The theory $T_1^1 \sqcup T_1^2 \sqcup T_1^2$ has $3 \cdot 6^2 = 108$ countable models,18 of them are almost prime and 90 are limit ones.

To calculate the number of limit models, we note that in the structure $\mathrm{RK}(T)$, where $T = \bigsqcup\limits_{i=1}^{k} T_i^1 \sqcup \bigsqcup\limits_{j=1}^{s} T_j^2$, has the k-dimensional cube Q_k and the graph structure $L_{s,3}$ defined by the space $\mathcal{L}_{s,3}$. Here, the structure $\mathrm{RK}(T)$ is represented as the lattice with the Hasse diagram defined by the product $Q_k \times L_{s,3}$ of graphs, and therefore it has $2^k \cdot 3^s$ elements. Below we will also denote the correspondent lattices by $Q_k \times L_{s,3}$.

Each vertex in $\mathrm{RK}(T)$ symbolizes a prime model over (unique) completion

$$q_{i_1,\ldots,i_t,j_1,\ldots,j_m}(x_1,\ldots,x_r,y_1,\ldots,y_m)$$

of type $p_{i_1}(x_1) \cup \ldots \cup p_{i_t}(x_m) \cup p'_{j_1}(y_1) \cup \ldots \cup p'_{j_m}(y_m)$, where the types $p_1(x),\ldots,p_k(x)$ exhaust the list of non-principal 1-types of the theories T_i^1, and the types $p'_1(x),\ldots,p'_s(x)$ for the list of non-principal 1-types of theories T_j^2. Here, almost prime models, realizing the types $p_{i_1}(x_1),\ldots,p_{i_t}(x_m)$, have their least realizations, as well as they have either not more than one realizations of each type $p'_1(x), \ldots, p'_s(x)$, or, in the latter case $p'_j(x)$, these realizations, for a fixed type, form closed intervals.

Further, we choose among k types p_i some t types, and among s types p'_j some m types, responsible for the existence of limit models generated by realizations of these types, and obtain $(2^t \cdot 4^m - 1)$ possibilities for limit models. Together with the choice of m types we choose among remaining $s-m$ types p'_j some r types having unique realizations. Under these conditions of choice we have $(2^t \cdot 4^m - 1) \cdot C_k^t \cdot C_s^m \cdot C_{s-m}^r$ possibilities. Summarizing these values we obtain the following equalities:

$$\sum_{q_{i_1,\ldots,i_t,j_1,\ldots,j_m}} I_l(T, q_{i_1,\ldots,i_m}) = \sum_{t=0}^{k}\sum_{m=0}^{s}\sum_{r=0}^{s-m}(2^t \cdot 4^m - 1) \cdot C_k^t \cdot C_s^m \cdot C_{s-m}^r$$

$$= \sum_{t=0}^{k}\sum_{m=0}^{s}\left(\sum_{r=0}^{s-m} C_{s-m}^r\right)(2^t \cdot 4^m - 1) \cdot C_k^t \cdot C_s^m$$

$$= \sum_{t=0}^{k}\sum_{m=0}^{s} 2^{s-m} \cdot (2^t \cdot 4^m - 1) \cdot C_k^t \cdot C_s^m = 3^k \cdot 6^s - 2^k \cdot 3^s. \tag{3.6}$$

By (3.6) for the theory $\bigsqcup\limits_{i=1}^{k} T_i^1 \sqcup \bigsqcup\limits_{j=1}^{s} T_j^2$, we have the following representation of the decomposition formula (1.1):

$$3^k \cdot 6^s = 2^k \cdot 3^s + \sum_{t=0}^{k}\sum_{m=0}^{s} 2^{s-m} \cdot (2^t \cdot 4^m - 1) \cdot C_k^t \cdot C_s^m. \tag{3.7}$$

For $k = 1$ and $s = 1$ we have $18 = 6 + 2 \cdot 1 \cdot 1 \cdot 1 + 1 \cdot 3 \cdot 1 \cdot 1 + 1 \cdot 7 \cdot 1 \cdot 1$; for $k = 2$ and $s = 1$: $54 = 12 + 2 \cdot 1 \cdot 2 \cdot 1 + 2 \cdot 3 \cdot 1 \cdot 1 + 1 \cdot 3 \cdot 1 \cdot 1 + 1 \cdot 7 \cdot 2 \cdot 1 + 1 \cdot 15 \cdot 1 \cdot 1$; for $k = 1$ and $s = 2$: $108 = 18 + 4 \cdot 1 \cdot 1 \cdot 1 + 2 \cdot 3 \cdot 1 \cdot 2 + 2 \cdot 7 \cdot 1 \cdot 2 + 1 \cdot 15 \cdot 1 \cdot 1 + 1 \cdot 31 \cdot 1 \cdot 1$, as shown in Fig. 7, 8, 9, respectively.

By Theorem 2.3 and obtained decomposition formulas (3.3), (3.5), (3.7) we have the following theorem.

Theorem 3.1. *Any quite o-minimal Ehrenfeucht theory T has a Rudin–Keisler preorder, represented by a lattice $Q_k \times L_{s,3}$, and a decomposition formula of the form*

$$3^k \cdot 6^s = 2^k \cdot 3^s + \sum_{t=0}^{k}\sum_{m=0}^{s} 2^{s-m} \cdot (2^t \cdot 4^m - 1) \cdot C_k^t \cdot C_s^m.$$

For $s = 0$ the decomposition formula has form (3.3), *and for $k = 0$ we obtain* (3.5).

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