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ON EXTENDED ROTHE'S METHOD FOR NONLINEAR PARABOLIC VARIATIONAL INEQUALITIES IN NONCYLINDRICAL DOMAINS

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Communicated by L.-E. Persson

Key words: parabolic variational inequalities, parabolic PDE, numerical methods, time-discretization, Rothe's method.

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Abstract: In this paper, some nonlinear parabolic variational inequalities in noncylindrical domains are considered. Using extended Rothe's method introduced and developed in [10] an approximate solution is constructed. Existence and uniqueness results are proved. Moreover, we present some further results and comments related to the main result.

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1 Introduction

Let us consider in $\mathbb{R}^{N+1}$ the domain Q defined by

$$Q = \{(x,t) : x \in \Omega_t, 0 < t < T\},$$

where $(0,T)$ is a finite interval, $\Omega_t \in C^{0,1}$ (here, $C^{0,1}$ is a set of all bounded domains in $\mathbb{R}^N$, whose boundary can be locally described by functions belonging to $C^{0,1}(\Delta)$, where $\Delta \subset \mathbb{R}^{N-1}$ is a cube; see [6]) and for every $t, s \in (0,T), t < s$,

$$\emptyset \neq \Omega_0 \subset \Omega_t \subset \Omega_s \subset \Omega_T.$$

Let $t \in [0,T]$ and $p > 1$, let

$$V_t = W_0^{k,p}(\Omega_t)$$

be the Sobolev space and let V_t^* be its dual space. We denote by $\langle \cdot, \cdot \rangle_t$ the duality between V_t^* and V_t, and $(\cdot,\cdot)_t$ denotes the inner product in $L_2(\Omega_t)$.

We will solve the parabolic variational inequality

$$u(t) \in K_t : \quad \left(\frac{du(t)}{dt}, v - u(t)\right)_t + \langle Au(t), v - u(t)\rangle_t \geq (f(t), v - u(t))_t \quad \text{for all } v \in K_t \tag{1.1}$$

for $t \in (0,T)$, where K_t is a closed convex subset of the space $V_t \cap L_2(\Omega_t)$. The norm of the space $V_t \cap L_2(\Omega_t)$ is defined by

$$\|\cdot\|_{V_t \cap L_2(\Omega_t)} = \|\cdot\|_{V_t} + \|\cdot\|_{L_2(\Omega_t)}.$$

Moreover, let A be a nonlinear differential operator of order $2k$ ($k \in \mathbb{Z}_+$) of the form:

$$(Au)(x) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \partial^\alpha (a_\alpha(x,\, \delta_k\, u))$$

for $x \in \Omega_T$, where $\delta_k u = \{\partial^\beta u\}_{|\beta|\le k}$ and

$$\partial^\beta u(x) = \frac{d^{|\beta|}u(x)}{dx_1^{\beta_1}...dx_N^{\beta_N}},$$

$\beta = (\beta_1, ..., \beta_N)$ is a multiindex, i.e., $\beta_i \in \mathbb{N}\cup\{0\}$, $i = 1, 2, ..., N$, and $|\beta| = \beta_1 + ... + \beta_N$. The coefficients $\{a_\alpha(\cdot,\ \delta_k u)\}_{|\alpha|\le k}$ of the operator A and the function f are defined in Ω_T and Q, respectively. Together with (1.1) we consider the initial condition

$$u(0) = 0. \tag{1.2}$$

In the special case, when Q is a cylinder, there are a number of works where even more generalized versions of variational inequality (1.1) – (1.2) have been solved by the so-called Rothe's method, see e.g. [1], [5], [14] and [15]. The method of Rothe (also called the method of lines) was introduced by E. Rothe in 1930. It has been developed and applied to numerical study, e.g. of parabolic equations and the corresponding variational inequalities.

However, in the case when Q is a noncylindrical domain the problem of type (1.1) – (1.2) is much less studied. If $K_t = V_t$ then the variational inequality is equivalent to the corresponding parabolic boundary-value problem in noncylindrical domains which was probably first considered in [2]. Later on it was studied in [8] by the transformation method which requires sufficient smoothness of the boundary of the domain Q in t. Problems of such type even more generalized versions were solved in [3], [4], [9] and [10] by extended Rothe's method introduced in [10]. In this paper we show the application of extended Rothe's method to variational inequality (1.1) – (1.2).

The paper is organized as follows: in Section 2 we briefly present an idea of construction of the extended Rothe method for parabolic variational inequalities. Further, in Section 3 we prove the existence and uniqueness of solution of (1.1) – (1.2) (see Theorem 3.1). Finally, in the last section we present further results (see Propositions 4.1 and 4.2) and comments related to the main result.

2 Rothe's method for noncylindrical domains

In the proof of the main result (Theorem 3.1) we will see that the following assumptions ensure the existence and uniqueness of the solution of problem (1.1) – (1.2) in the sense of Definition 1 below.

Assumptions. The coefficients of the operator A satisfy the following conditions:

(A1) The *Carathéodory condition*, i.e. $a_\alpha(x;\cdot)$ is continuous on $\mathbb{R}^m$ for a.e. $x \in \Omega_T$ and $a_\alpha(\cdot;\xi)$ is measurable on Ω_T for every $\xi \in \mathbb{R}^m$, where m is the number of all multiindices of length $|\alpha| \le k$, i.e., $m = 1 + N + N^2 + ... + N^k$.

(A2) The *growth condition*

$$|a_\alpha(x;\xi)| \le C_\alpha \left(g_\alpha(x) + \sum_{|\beta|\le k} |\xi_\beta|^{p-1} \right) \quad \text{for a.e. } x \in \Omega_T$$

for all $\xi \in \mathbb{R}^m$, where C_α is a given positive constant and g_α is a given function in $L_{p'}(\Omega_T)$, $p' = \frac{p}{p-1}$.

(A3) The *monotonicity condition*

$$\sum_{|\alpha|\le k} [a_\alpha(x;\xi) - a_\alpha(x;\eta)](\xi_\alpha - \eta_\alpha) > 0 \quad \text{for a.e. } x \in \Omega_T$$

and every $\xi, \eta \in \mathbb{R}^m$, $\xi \ne \eta$.

(A4) The *coercivity condition*

$$\sum_{|\alpha|\le k} a_\alpha(x;\xi)\xi_\alpha \ge c_0 \sum_{|\alpha|\le k} |\xi_\alpha|^p \quad \text{for a.e. } x\in\Omega_T$$

for every $\xi \in \mathbb{R}^m$ with a suitable constant $c_0 > 0$.

(A5) The *symmetry condition* $a_{\alpha\beta}(x;\xi) = a_{\beta\alpha}(x;\xi)$ for a.e. $x \in \Omega_T$ and for all $\xi \in \mathbb{R}^m$.

(A6) The function f satisfies the following condition: there exists a function $F \in C(I, L_2(\Omega_T)) \cap V^1(I,\, L_2(\Omega_T))$ such that

$$F(x,t) = f(x,t) \qquad \text{for} \quad \text{all} \quad (x,t)\in Q$$

and we extend the function f to the set $\Omega_T \times [0,T]$ as

$$f(x,t) = \begin{cases} f(x,t), & (x,t)\in Q,\\ 0, & \Omega_T\times[0,T]\setminus Q. \end{cases}$$

(A7) The sets K_t $(t \in (0,T))$ satisfy the following condition: if we denote by $\overline{K}_t$ $(t\in[0,T])$ the set of all elements of K_t extended by zero to the whole domain Ω_T, i.e.

$$\overline{K}_t = \{u\in K_T, \quad u(t)\Big|_{\Omega_t} \in K_t, \quad u(t)\Big|_{\Omega_T\setminus\Omega_t} = 0 \quad \text{a.e. in} \quad I\},$$

then $\overline{K}_0 \subset \overline{K}_t \subset \overline{K}_s \subset \overline{K}_T$.

We apply the idea of Rothe in the following way.

We divide the interval $I = [0,T]$ into n subintervals $I_1, I_2, ..., I_n$ $(I_j = [t_{j-1}, t_j],\ j = 1,2,...,n)$ of length $h = \frac{T}{n}$. According to initial condition (1.2) we put $z_0(x) = 0$, $x \in \Omega_T$, for $t_0 = 0$ and successively for $j = 1,2,...,n$ we define functions $z_j(x)$ as the solutions of the following variational inequalities:

$$\begin{aligned} z_j \in K_{t_j}: \qquad & (\frac{z_j}{h}, v - z_j)_{t_j} + \langle Az_j, v-z_j\rangle_{t_j} \\ & \qquad \ge (f_j + \frac{z_{j-1}}{h}, v-z_j)_{t_j} \quad \text{for} \quad \text{all} \quad v\in K_{t_j}. \end{aligned} \tag{2.1}$$

We obtain problems (2.1) if in (1.1) we replace the derivative $\frac{\partial u}{\partial t}$ by the differential quotient $\frac{z_j - z_{j-1}}{h}$ in the points $t = t_j$ and put $z_{j-1} = 0$ on $\Omega_{t_j} \setminus \Omega_{t_{j-1}}$, $j = 1,2,...,n$.

Inequality (2.1) can be rewritten in the form

$$z_j \in K_{t_j}: \qquad \langle A_h z_j, v - z_j\rangle_{t_j} \ge (f_j + \frac{z_{j-1}}{h}, v - z_j)_{t_j} \quad \text{for} \quad \text{all} \quad v\in K_{t_j}, \tag{2.2}$$

where $\langle A_h u, v\rangle_t = (\frac{u}{h}, v)_t + \langle Au, v\rangle_t$. The operator $A + \frac{1}{h}I : K_t \to (V_t \cap L_2(\Omega_t))^* = V_t + L_2(\Omega_t)$ is bounded, continuous, strictly monotone and coercive. Hence, due to [6, Theorem 43.2] there exists a unique solution $z_j \in K_{t_j}$ of (2.2), which implies (2.1).

We solve problem (2.2) in the following way: first we consider (2.2) for $j = 1$, which takes the form

$$z_1 \in K_{t_1} \qquad (\frac{z_1}{h}, v - z_1)_{t_1} + \langle Az_1, v - z_1\rangle_{t_1} \ge (f_1 + \frac{z_0}{h}, v - z_1)_{t_1} \quad \text{for} \quad \text{all} \quad v\in K_{t_1},$$

then we extend the obtained solution by 0:

$$\tilde{z}_1(x) = \begin{cases} z_1(x), & x \in \Omega_{t_1}, \\ 0, & x \in \Omega_T \setminus \Omega_{t_1} \end{cases}$$

and we get $\tilde{z}_1 \in K_T$.

Repeating the above procedure for $j = 2, 3, ..., n$ we get functions

$$\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_n \in K_T.$$

Next we construct the function $u_n(x,t)$, called *Rothe's function,* and defined on $\Omega_T \times I$ by putting

$$u_n(x,t) = \tilde{z}_{j-1}(x) + \frac{t - t_{j-1}}{h}(\tilde{z}_j(x) - \tilde{z}_{j-1}(x)) \tag{2.3}$$

for $t \in I_j, j = 1, 2, ..., n$, and $x \in \Omega_T$. Below we shall write z_j instead of $\tilde{z}_j$.

In this way we get the sequence $\{u_n(x,t)\}_{n=1}^{\infty}$ which is called *Rothe's sequence* of approximate solutions of problem (1.1) – (1.2).

In the next section we prove that this sequence in fact converges to the (unique) solution of our problem.

3 Existence and uniqueness results

The notion of a solution of the problem introduced above will be given now. Let us first define the following set:

$$K_Q = \{u \in L_2(I, V_T \cap L_2(\Omega_T)), \quad u(t) \in \overline{K}_t\}.$$

By the definition of $\overline{K}_t$ it follows that the set K_Q is also a convex closed set in $L_2(I, V_T \cap L_2(\Omega_T))$.

Definition 1. A function u is called a *weak solution* of problem (1.1) – (1.2) if the following conditions are fulfilled:

1) $u \in K_Q$,

2) $u \in AC(I, L_2(\Omega_T))$,

3) $u' \in L_2(I, L_2(\Omega_T))$,

4) $u(0) = 0$,

5) $\int_0^T \langle Au(t), v(t) - u(t)\rangle_T dt + \int_0^T (u'(t), v(t) - u(t))_T dt$

$$\geq \int_0^T (f, v(t) - u(t))_T dt \quad \text{for all } v \in K_Q.$$

Our main result in this section reads as follows.

Theorem 3.1. *Assume that Assumptions A1-A7 hold. Then there exists exactly one solution of problem (1.1) – (1.2) in the sense of Definition 1, i.e. exactly one function which is a weak (strong) limit of the sequence of Rothe's functions $u_n(t)$ in the space $L_2(I, V_T \cap L_2(\Omega_T))$ $(C(I, L_2(\Omega_T)))$.*

Proof. **Uniqueness.** Let u be a solution of problem (1.1) – (1.2). Let $a \in \mathbb{R}_+$ be arbitrary and let

$$v(t) = \begin{cases} w(t), & 0 < t < a, \\ u(t), & a \le t \le T, \end{cases}$$

where $w(t) \in K_t$ for $t \in (0, a)$. Putting this function into integral inequality 5) (of Definition 1) we get that

$$\int_0^a \langle Au(t), w(t) - u(t) \rangle_T dt + \int_0^a (u'(t), w(t) - u(t))_T dt \ge \int_0^a (f(t), w(t) - u(t))_T dt.$$

Assume that u_1 and u_2 are solutions of problem (1.1) – (1.2). Replacing $u,\ w$ in the last inequality for by $u_1,\ u_2$, respectively, and then $u,\ w$ by u_2, u_1, respectively, and adding the resulting inequalities we obtain that

$$-\int_0^a \langle Au_2(t) - Au_1(t), u_2(t) - u_1(t) \rangle_T dt - \int_0^a (u_2'(t) - u_1'(t), u_2(t) - u_1(t))_T dt \ge 0.$$

From this and from (A3) we get that

$$\int_0^a (u_2'(t) - u_1'(t), u_2(t) - u_1(t))_T dt \le 0.$$

Taking into account that

$$\int_0^a (u_2'(t) - u_1'(t), u_2(t) - u_1(t))_T dt = \frac{1}{2} \int_0^a \frac{d}{dt} \|u_2(t) - u_1(t))\|^2_{L_2(\Omega_T)} dt$$

$$= \frac{1}{2} \|u_2(a) - u_1(a)\|^2_{L_2(\Omega_T)} - \frac{1}{2} \|u_2(0) - u_1(0)\|^2_{L_2(\Omega_T)} = \frac{1}{2} \|u_2(a) - u_1(a)\|^2_{L_2(\Omega_T)},$$

we find that

$$\|u_2(a) - u_1(a)\|^2_{L_2(\Omega_T)} = 0.$$

Hence $u_2 = u_1$, since a was arbitrary. The proof of the uniqueness is complete.

Existence. Let us consider the inequality

$$\langle Az_j, v - z_j \rangle_{t_j} + (\frac{z_j - z_{j-1}}{h}, v - z_j)_{t_j} \ge (f_j, v - z_j)_{t_j} \quad \text{for all } v \in K_{t_j}. \tag{3.1}$$

Choose $v = z_{j-1}$ in (3.1); by the properties of z_j we can extend the integrals in (3.1) to the whole domain Ω_T and we have that

$$\langle Az_j, z_j - z_{j-1} \rangle_T + (\frac{z_j - z_{j-1}}{h}, z_j - z_{j-1})_T \le (f_j, z_j - z_{j-1})_T.$$

Adding the resulting inequalities in both sides from $j = 1$ to i we get that

$$\sum_{j=1}^i \langle Az_j, z_j - z_{j-1} \rangle_T + \frac{1}{h} \sum_{j=1}^i (z_j - z_{j-1}, z_j - z_{j-1})_T \le \sum_{j=1}^i (f_j, z_j - z_{j-1})_T.$$

If we denote

$$S_i^1 = \sum_{j=1}^i \langle Az_j, z_j - z_{j-1} \rangle_T,$$

$$S_i^2 = \frac{1}{h} \sum_{j=1}^i (z_j - z_{j-1}, z_j - z_{j-1})_T,$$

$$S_i^3 = \sum_{j=1}^i (f_j, z_j - z_{j-1})_T,$$

then we can rewrite the last inequality as

$$S_i^1 + S_i^2 \le S_i^3. \tag{3.2}$$

According to (A5) of assumptions we find that

$$S_i^1 = \frac{1}{2}\sum_{j=1}^{i}\{2\langle Az_j, z_j\rangle_T - 2\langle Az_{j-1}, z_j\rangle_T\}$$

$$= \frac{1}{2}\{\langle Az_i, z_i\rangle_T + \sum_{j=1}^{i}[\langle Az_j, z_j\rangle_T - 2\langle Az_{j-1}, z_j\rangle_T + Az_{j-1}, z_{j-1}\rangle_T]\}$$

$$= \frac{1}{2}\{\langle Az_i, z_i\rangle_T + \sum_{j=1}^{i}\langle Az_j - Az_{j-1}, z_j - z_{j-1}\rangle_T\}.$$

From this and from (A4) and (A6) of assumptions we obtain that

$$S_i^1 \ge \frac{1}{2}\langle Az_i, z_i\rangle_T \ge C\|z_i\|^p_{W^{k,p}(\Omega_T)}, \tag{3.3}$$

$$S_i^2 = \frac{1}{h}\sum_{j=1}^{i}\|z_j - z_{j-1}\|^2_{L_2(\Omega_T)}, \tag{3.4}$$

$$S_i^3 \le \sum_{j=1}^{i}\|f_j\|_{L_2(\Omega_T)}\|z_j - z_{j-1}\|_{L_2(\Omega_T)} \le \frac{h}{2}\sum_{j=1}^{i}\|f_j\|^2_{L_2(\Omega_T)} + \frac{1}{2h}\sum_{j=1}^{i}\|z_j - z_{j-1}\|^2_{L_2(\Omega_T)}$$
$$\le \frac{h}{2}iV(f)^2 + \frac{1}{2}S_i^2 \le TV(f)^2 + \frac{1}{2}S_i^2,$$

where

$$V(f) = \sup_I \|f(t)\|_{L_2(\Omega_T)} + \sup_{\{t_i\}}\sum_{i=1}^{n}\|f(t_i) - f(t_{i-1})\|_{L_2(\Omega_T)},$$

for all finite partitions $\{t_i\}$ of the interval $[0, T]$.

From this and from (3.2) – (3.4) it follows that

$$S_i^2 \le S_i^3 \le TV(f)^2 + \frac{1}{2}S_i^2,$$

and, consequently,

$$S_i^2 \le 2TV(f)^2,$$

i.e.

$$\frac{1}{h}\sum_{j=1}^{i}\|z_j - z_{j-1}\|^2_{L_2(\Omega_T)} \le C \tag{3.5}$$

and

$$S_i^3 \le 2TV(f)^2.$$

According to (3.2) and (3.3) we find that

$$\|z_i\|_{W^{k,p}(\Omega_T)} \le C. \tag{3.6}$$

The estimate

$$\|z_i\|_{L_2(\Omega_T)} \leq C \tag{3.7}$$

follows from the following calculation:

$$\|z_i\|^2_{L_2(\Omega_T)} \leq \Big(\sum_{j=1}^{i} \|z_j - z_{j-1}\|_{L_2(\Omega_T)}\Big)^2 \leq i \sum_{j=1}^{i} \|z_j - z_{j-1}\|^2_{L_2(\Omega_T)}$$

$$= ihS_i^2 \leq T^2 V(f)^2.$$

Now we consider the *Rothe sequence* $\{u_n(t)\}_{n=1}^{\infty}$ given by (2.3). From (3.6) and (3.7) it follows that

$$\|u_n(t)\|_{V_T \cap L_2(\Omega_T)} = \|z_{j-1} + \frac{t - t_{j-1}}{h}(z_j - z_{j-1})\|_{V_T \cap L_2(\Omega_T)}$$

$$\leq (1 - \frac{t - t_{j-1}}{h})\|z_{j-1}\|_{V_T \cap L_2(\Omega_T)} + \frac{t - t_{j-1}}{h}\|z_j\|_{V_T \cap L_2(\Omega_T)} \leq C$$

for every $t \in I$ and $n = 1, 2, ...$

Thus, we get that

$$\|u_n\|^2_{L_2(I, V_T \cap L_2(\Omega_T))} = \int_0^T \|u_n(t)\|^2_{V_T \cap L_2(\Omega_T)} dt \leq C^2 T$$

for $n = 1, 2, ...$ From this and from the reflexivity of the space $L_2(I, V_T \cap L_2(\Omega_T))$ it follows that the Rothe sequence $\{u_n\}_{n=1}^{\infty}$ has a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$, which converges weakly to some function $u \in L_2(I, V_T \cap L_2(\Omega_T))$, i.e.

$$u_{n_k} \rightharpoonup u \quad \text{in} \quad L_2(I, V_T \cap L_2(\Omega_T)). \tag{3.8}$$

We will show that the function u is the desired solution. Denote $Z_j = \frac{z_j - z_{j-1}}{h}$. Then we can write (2.3) in the form

$$u_n(t) = z_{j-1} + Z_j(t - t_{j-1}) \quad \text{in} \quad I_j = [t_{j-1}, t_j], \quad j = 1, 2, ..., n.$$

Now we define the functions $U_n : t \mapsto L_2(\Omega_T)$, $(n = 1, 2, ...)$ by

$$U_n(t) = \begin{cases} Z_1^n, & t = 0, \\ Z_j, & t \in (t_{j-1}, t_j], \quad j = 1, 2, ..., n. \end{cases}$$

From (3.5) it follows that the sequence $\{U_n\}_{n=1}^{\infty}$ is bounded, because

$$\|U_n\|^2_{L_2(I, L_2(\Omega_T))} = \int_0^T \|U_n(t)\|^2_{L_2(\Omega_T)} dt = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|Z_j\|^2_{L_2(\Omega_T)} dt$$

$$= \sum_{j=1}^{n} \|\frac{z_j - z_{j-1}}{h}\|^2_{L_2(\Omega_T)}(t_j - t_{j-1}) = \frac{1}{h}\sum_{j=1}^{n} \|z_j - z_{j-1}\|^2_{L_2(\Omega_T)} \leq C.$$

Hence, we can choose a subsequence $\{U_{n_k}\}_{k=1}^{\infty}$ converging weakly to some function $U \in L_2(I, L_2(\Omega_T))$, i.e.

$$U_{n_k} \rightharpoonup U \quad \text{in} \quad L_2(I, L_2(\Omega_T)). \tag{3.9}$$

Thus, there exists ω defined by

$$\omega(t) = \int_0^t U(\tau) d\tau.$$

According to (3.8), (3.9) and the relation

$$\int_0^t U_{n_k}(\tau)d\tau = u_{n_k}(t)$$

we find that

$$w = u.$$

(To obtain the last equality we apply Lebesgue's dominated convergence theorem.) Then we get that

$$u \in AC(I, L_2(\Omega_T)),$$

$$u'(t) = U(t) \quad \text{a.e. in} \quad I,$$

i.e.,

$$u(t) = \int_0^t U(\tau)d\tau$$

and

$$u(0) = 0.$$

Now we claim that the Rothe sequence converges uniformly to the solution u, i.e.

$$u_{n_k} \to u \quad \text{in} \quad C(I, L_2(\Omega_T)). \tag{3.10}$$

In view of $\frac{\partial u_{n_k}(t)}{\partial t} = U_{n_k}(t)$ a.e. in I and from Lemma A6 of [11] it follows that the Rothe sequence $\{u_n(t)\}_{n=1}^{\infty}$ is equicontinuous, i.e. the first condition of Lemma A5 of [11] is satisfied. The second condition in the lemma holds according to the fact that

$$W^{k,p}(\Omega_T) \cap L_2(\Omega_T) \hookrightarrow\hookrightarrow L_2(\Omega_T),$$

which is well-known, where $\hookrightarrow\hookrightarrow$ denotes the compact embedding of the spaces, see e.g. [7]. Hence, our claim follows from Lemma A5, see [11].

From the above considerations and from Lemma A3 of [11] it follows that

$$u \in K_Q,$$

which implies that the sequence $\{\bar{u}_n\}_{n=1}^{\infty}$, defined by

$$\bar{u}_n(t) = \begin{cases} z_0, & t \in [t_0, t_1], \\ z_{j-1}, & t \in (t_{j-1}, t_j], \quad j = 2, 3, ..., n, \end{cases}$$

is a subset of the set K_Q and this set is a convex, closed set in $L_2(I, V_T \cap L_2(\Omega_T))$. (Here, we apply Theorem 25.2 in [6], stating that every convex, closed set in a reflexive Banach space is weakly closed.)

Thus, we have proved that the function u satisfies conditions $1) - 4)$ of Definition 1. Now, we will show that this function satisfies also integral inequality 5). We consider integral inequality (3.1) written for n_k, i.e.

$$\langle Az_j, v - z_j\rangle_{t_j} + (\frac{z_j - z_{j-1}}{h}, v - z_j)_{t_j} \geq (f_j, v - z_j)_{t_j} \quad \text{for all } v \in K_{t_j},$$

$j = 1, 2, \ldots, n_k$. Let $v \in K_Q \cap L_\infty(I, V_T \cap L_2(\Omega_T))$ be arbitrary. We can rewrite the last inequality in the form

$$\langle A\tilde{u}_{n_k}(t), v(t) - \tilde{u}_{n_k}(t)\rangle_T + (U_{n_k}(t), v(t) - \tilde{u}_{n_k}(t))_T \geq (f_{n_k}(t), v(t) - \tilde{u}_{n_k}(t))_T \tag{3.11}$$

for almost all $t \in I$, where $U_{n_k}(t)$ is defined as above and

$$\tilde{u}_{n_k}(t) = \begin{cases} z_0, & t = 0, \\ z_j, \quad t \in (t_{j-1}, t_j], & j = 1, 2, ..., n_k, \end{cases}$$

and

$$f_{n_k}(t) = \begin{cases} f_1, & t = 0, \\ f_j, \quad t \in (t_{j-1}, t_j], & j = 1, 2, ..., n_k. \end{cases}$$

From the uniform convergence of the Rothe sequence and from the fact that

$$\max_{[0,T]} \|u_{n_k}(t) - \tilde{u}_{n_k}(t)\|_{L_2(\Omega_T)} \le \frac{C}{n_k}$$

it follows that the sequence $\{\tilde{u}_{n_k}\}_{k=1}^{\infty}$ also converges uniformly to the solution u. Moreover, it can be shown (by using Lemma A6) that for this sequence also the following estimate holds:

$$\|\tilde{u}_{n_k}(t) - \tilde{u}_{n_k}(t')\|^2_{L_2(\Omega_T)} \le C\,|t - t'|. \tag{3.12}$$

By the limiting process we get that

$$\|u(t) - u(t')\|^2_{L_2(\Omega_T)} \le C\,|t - t'|. \tag{3.13}$$

From (3.12) – (3.13) and from the boundedness of the sequence $\{U_{n_k}\}_{k=1}^{\infty}$ in the space $L_2(I,\, L_2(\Omega_T))$ it follows that the sequence

$$\{(U_{n_k}(t), u(t) - \tilde{u}_{n_k}(t))_T\}_{k=1}^{\infty}$$

has a subsequence which converges to zero for all $t \in I$, i.e.

$$(U_{n_k}(t), u(t) - \tilde{u}_{n_k}(t))_T \to 0 \quad \text{as} \quad k \to \infty, \tag{3.14}$$

since, by applying Hölder's inequality, we have that

$$\int_0^T |(U_{n_k}(t), u(t) - \tilde{u}_{n_k}(t))_T|\, dt$$

$$\le \int_0^T \|U_{n_k}(t)\|_{L_2(\Omega_T)} \|u(t) - \tilde{u}_{n_k}(t)\|_{L_2(\Omega_T)}\, dt \le C \max_{\Omega_T} \|u(t) - \tilde{u}_{n_k}(t)\|_{L_2(\Omega_T)}.$$

From this we find that

$$\int_0^T |(U_{n_k}(t), u(t) - \tilde{u}_{n_k}(t))_T|\, dt \to 0 \quad \text{as} \quad k \to \infty,$$

which implies the existence of a subsequence which converges to zero almost everywhere in I. Finally, we note that (3.12) and (3.13) imply (3.14).

Putting $v(t) = u(t)$ in (3.11) we obtain that

$$\langle A\tilde{u}_{n_k}(t), \tilde{u}_{n_k}(t) - u(t)\rangle_T \le (f_{n_k}(t), \tilde{u}_{n_k}(t) - u(t))_T + (U_{n_k}(t), u(t) - \tilde{u}_{n_k}(t))_T.$$

From this and according to (3.14) we have that

$$\lim_{k\to\infty} \sup \langle A\tilde{u}_{n_k}(t), \tilde{u}_{n_k}(t) - u(t)\rangle_T\, dt \le 0.$$

The operator A is pseudomonotone (see [13, Chapter 2]), which implies that

$$\langle A\tilde{u}(t), \tilde{u}(t) - v(t)\rangle_T \leq \lim_{k\to\infty} \inf\langle A\tilde{u}_{n_k}(t), \tilde{u}_{n_k}(t) - v(t)\rangle_T. \tag{3.15}$$

Using the monotonicity of A and the boundedness of $\tilde{u}_n$ in $L_\infty(I, V_T \cap L_2(\Omega_T))$ we find that

$$\langle A\tilde{u}_{n_k}(t), \tilde{u}_{n_k}(t) - v(t)\rangle_T \geq -C(\|v\|_{L_\infty(I, V_T\cap L_2(\Omega_T))}).$$

Moreover, according to Fatou's lemma we get from (3.15) that

$$\int_0^T \langle A\tilde{u}(t), \tilde{u}(t) - v(t)\rangle_T \, dt \leq \lim_{k\to\infty} \inf \int_0^T \langle A\tilde{u}_{n_k}(t), \tilde{u}_{n_k}(t) - v(t)\rangle_T \, dt. \tag{3.16}$$

After integrating (3.11) over the interval I, we obtain that

$$\begin{aligned} \int_0^T \langle A\tilde{u}_{n_k}(t), v(t) - \tilde{u}_{n_k}(t)\rangle_T dt + &\int_0^T (U_{n_k}(t), v(t) - \tilde{u}_{n_k}(t))_T dt \\ &\geq \int_0^T (f_{n_k}(t), v(t) - \tilde{u}_{n_k}(t))_T dt. \end{aligned} \tag{3.17}$$

The convergences

$$\int_0^T (U_{n_k}(t), v(t) - \tilde{u}_{n_k}(t))_T dt \to \int_0^T (u'(t), v(t) - u(t))_T dt$$

and

$$\int_0^T (f_{n_k}(t), \tilde{u}_{n_k}(t) - v(t))_T dt \to \int_0^T (f(t), u(t) - v(t))_T dt$$

as $k \to \infty$, follow from (3.8), (3.9), (A6) and Lemma A3 in [11]. By using these facts and (3.16) we obtain that

$$\begin{aligned} \int_0^T \langle Au(t), v(t) - u(t)\rangle_T dt + &\int_0^T (u'(t), v(t) - u(t))_T dt \\ &\geq \int_0^T (f(t), v(t) - u(t))_T dt. \end{aligned}$$

Moreover, since the set $K_Q \cap L_\infty(I, V_T \cap L_2(\Omega_T))$ is dense in K_Q and due to the definition of v we conclude that the function u satisfies integral inequality 5) of Definition 1.

Thus, we have proved that there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of *Rothe's sequence* $\{u_n\}_{n=1}^\infty$, which converges to the solution u of problem (1.1) – (1.2). Moreover, from the uniqueness of the *weak solution* it follows that not only the subsequence but also the sequence itself converges weakly (strongly) in $L_2(I, V_T \cap L_2(\Omega_T))$ $(C(I, L_2(\Omega_T)))$ to the solution u.

□

4 Further results and discussion

In this section we present some results which are related to the main result in the previous section.

Proposition 4.1. *Let the assumptions of Theorem 3.1 be satisfied except that instead of* (A6) *the function f satisfies the Lipschitz condition: for some* $C > 0$

$$\|f(t) - f(t')\|_{L_2(\Omega_t)} \leq C|t - t'| \quad \text{for all} \quad t, t' \in I.$$

Then

$$\max_{t\in I} \|u_n(t) - u(t)\|^2_{L_2(\Omega_T)} \leq \frac{C}{n}.$$

Remark 1. This result is interesting also from the numerical point of view.

Proof. Let us consider integral inequality (3.17) written for k instead of n_k, i.e.,

$$\int_0^T \langle A\tilde{u}_k(t), v(t)-\tilde{u}_k(t)\rangle_T dt + \int_0^T (U_k(t), v(t)-\tilde{u}_k(t))_T dt \le \int_0^T (f_k(t), v(t)-\tilde{u}_k(t))_T dt. \tag{4.1}$$

Putting for $k=m$,

$$v(t)=\begin{cases} u_n(t) & t\in(0,\tau),\\ u_m(t) & t\in[\tau,T),\end{cases}$$

and for $k=n$,

$$v(t)=\begin{cases} u_m(t) & t\in(0,\tau),\\ u_n(t) & t\in[\tau,T),\end{cases}$$

we obtain after adding that

$$\begin{aligned}&\int_0^\tau \langle A\tilde{u}_n(t)-A\tilde{u}_m(t), \tilde{u}_n(t)-\tilde{u}_m(t)\rangle_T dt\\ &+\int_0^\tau \Big(\frac{\partial(u_n(t)-u_m(t))}{\partial t}, \tilde{u}_n(t)-\tilde{u}_m(t)\Big)_T dt\\ &\le \int_0^\tau (f_n(t)-f_m(t), \tilde{u}_n(t)-\tilde{u}_m(t))_T dt.\end{aligned} \tag{4.2}$$

From this and (A3) we find that

$$\int_0^\tau \Big(\frac{\partial(u_n(t)-u_m(t))}{\partial t}, \tilde{u}_n(t)-\tilde{u}_m(t)\Big)_T dt \le \int_0^\tau (f_n(t)-f_m(t), \tilde{u}_n(t)-\tilde{u}_m(t))_T dt$$

and

$$\begin{aligned}&\int_0^\tau \Big(\frac{\partial(u_n(t)-u_m(t))}{\partial t}, u_n(t)-u_m(t)\Big)_T dt\\ &\le \int_0^\tau (f_n(t)-f_m(t), \tilde{u}_n(t)-\tilde{u}_m(t))_T dt\\ &+\int_0^\tau \Big(\frac{\partial(u_n(t)-u_m(t))}{\partial t}, u_n(t)-\tilde{u}_n(t)+\tilde{u}_m(t)-u_m(t)\Big)_T dt.\end{aligned} \tag{4.3}$$

It easy to see that

$$\begin{aligned}&\int_0^\tau \Big(\frac{\partial(u_n(t)-u_m(t))}{\partial t}, u_n(t)-u_m(t)\Big)_T dt\\ &=\frac{1}{2}\int_0^\tau \frac{\partial\|u_n(t)-u_m(t)\|^2_{L_2(\Omega_T)}}{\partial t}dt = \frac{1}{2}\|u_n(t)-u_m(t)\|^2_{L_2(\Omega_T)}\Big|_0^\tau = \frac{1}{2}\|u_n(\tau)-u_m(\tau)\|^2_{L_2(\Omega_T)}.\end{aligned}$$

The integrals in the right-hand side in (4.3) can be estimated as follows:

$$\int_0^\tau (f_n(t)-f_m(t), \tilde{u}_n(t)-\tilde{u}_m(t))_T dt \le \int_0^\tau \|f_n(t)-f_m(t)\|_{L_2(\Omega_T)}\, \|\tilde{u}_n(t)-\tilde{u}_m(t)\|_{L_2(\Omega_T)} dt$$

$$\leq \max_I \|f(T_n(t)) - f(T_m(t))\|_{L_2(\Omega_T)} \int_0^\tau \|\tilde{u}_n(t) - \tilde{u}_m(t)\|_{L_2(\Omega_T)} dt \leq C\,(\frac{1}{n} + \frac{1}{m}),$$

where the functions $T_n(t)$ and $T_m(t)$ are defined as

$$T_k(t) = \begin{cases} t_o & t = 0, \\ t_j & t \in (t_{j-1},\, t_j], \quad j = 1, 2, \ldots, k \end{cases}$$

with $k = n$ and $k = m$, respectively. Moreover,

$$\int_0^\tau (\frac{\partial(u_n(t) - u_m(t))}{\partial t}, u_n(t) - \tilde{u}_n(t) + \tilde{u}_m(t) - u_m(t))_T dt$$

$$\leq \int_0^\tau \|\frac{\partial(u_n(t) - u_m(t))}{\partial t}\|_{L_2(\Omega_T)} \Big[\|u_n(t) - \tilde{u}_n(t)\|_{L_2(\Omega_T)} + \|\tilde{u}_m(t) - u_m(t)\|_{L_2(\Omega_T)}\Big] dt$$

$$\leq C \max_{t \in I} \Big[\|\tilde{u}_n(t) - u_n(t)\|_{L_2(\Omega_T)} + \|\tilde{u}_m(t) - u_m(t)\|_{L_2(\Omega_T)}\Big] \leq C\,(\frac{1}{n} + \frac{1}{m}).$$

From the above considerations we conclude that

$$\|u_n(\tau) - u_m(\tau)\|^2_{L_2(\Omega_T)} \leq C\,(\frac{1}{n} + \frac{1}{m}).$$

By the limiting process in the last estimate when $m \to \infty$ we get our conclusion. □

Proposition 4.2. *Let the assumptions of Theorem* 3.1 *be satisfied except that instead of Assumptions* A3 *and* A4 *the form* $\langle Au, v\rangle_t$ *is assumed to be strongly monotone, i.e., for some* $C_0 > 0$

$$\langle Au - Av, v - u\rangle_t \geq C_0 \|u - v\|^p_{V_t}. \tag{4.4}$$

Then Rothe's sequence $\{u_n\}_{n=1}^\infty$ *strongly converges to the solution* u *in the space* $L_2(I,\, V_T)$*, i.e.,*

$$\|u_n - u\|_{L_2(I,\, V_T)} \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. Let us consider integral inequality (4.2) written for $\tau = T$, i.e.,

$$\int_0^T \langle A\tilde{u}_n(t) - A\tilde{u}_m(t), \tilde{u}_n(t) - \tilde{u}_m(t)\rangle_T dt$$

$$+ \int_0^T (\frac{\partial(u_n(t) - u_m(t))}{\partial t}, \tilde{u}_n(t) - \tilde{u}_m(t))_T dt \leq \int_0^T (f_n(t) - f_m(t), \tilde{u}_n(t) - \tilde{u}_m(t))_T dt.$$

From this and from (4.4) we get that

$$C \int_0^T \|\tilde{u}_n(t) - \tilde{u}_m(t)\|^2_{V_T}\, dt \leq \int_0^T (f_n(t) - f_m(t), \tilde{u}_n(t) - \tilde{u}_m(t))_T dt$$

$$- \int_0^T (\frac{\partial(u_n(t) - u_m(t))}{\partial t}, \tilde{u}_n(t) - \tilde{u}_m(t))_T dt.$$

The integrals in the right-hand side of this inequality tend to zero as $n, m \to \infty$, which follows from (A6) and from the fact that the Rothe sequence $\{u_n\}_{n=1}^\infty$ converges uniformly to the solution u and that the derivatives of these functions are bounded in $L_2(I,\, L_2(\Omega_T))$. Hence, we have that

$$\int_0^T \|\tilde{u}_n(t) - \tilde{u}_m(t)\|^2_{V_T}\, dt \leq C(\frac{1}{n} + \frac{1}{m}),$$

which implies that the Rothe sequence is a fundamental sequence in the space $L_2(I,\, V_T)$. By the limiting procedure in the last estimate as $n, m \to \infty$ we obtain the conclusion. □

Finally, we will discuss what the variational inequality really means for some particularly chosen operators A and sets $K_t\,(t \in I)$. Let us consider problem (1.1) – (1.2).

- If the set $K_t = V_t$, then variational problem (1.1) – (1.2) is equivalent to the following parabolic boundary value problem:

$$\frac{\partial u}{\partial t} + Au = f \quad \text{in} \quad Q,$$

$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = ... = \frac{\partial^{k-1} u}{\partial \nu^{k-1}}(x,t) = 0 \qquad 0 < t < T, \qquad x \in \partial\Omega_t,$$

$$u(x,0) = 0 \qquad x \in \Omega_0.$$

  Moreover, if the assumptions hold, then, according to Theorem 3.1, this problem has exactly one solution in the sense of Definition 1. In this sense the result of the previous section in fact generalizes the results in [9] and [10].

- Let A be defined by

$$Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{i,j}(x)\frac{\partial u}{\partial x_j}\Big) + a_0(x)u,$$

  where

$$a_0,\, a_{i,j} \in L_\infty(\Omega_T), \quad a_{i,j}(x) = a_{j,i}(x),$$

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\,\xi_j \geq \alpha|\xi|^2, \quad \text{a.e.} \quad \text{in} \quad \Omega_T,$$

$$a_0(x) \geq \alpha_0 > 0, \quad \text{a.e.} \quad \text{in} \quad \Omega_T,$$

  and let

$$K_t = \{v\,|v \in V_t = W_0^{1,2}(\Omega_t), \quad |\mathtt{grad}\, v(x)| \leq 1 \quad \text{a.e.} \quad \text{in} \quad \Omega_t\}.$$

  Then, by Theorem 3.1, the corresponding parabolic variational inequality has exactly one solution, which is also a weak solution of the following boundary value problem:

$$\frac{\partial u}{\partial t} + Au = f \quad \text{in} \quad Q',$$

$$|\mathtt{grad}_x u(x,t)| = 1 \quad \text{in} \quad Q \setminus Q',$$

$$u(x,t) = 0 \qquad 0 < t < T, \qquad x \in \partial\Omega_t,$$

$$u(x,0) = 0 \qquad x \in \Omega_0,$$

  where $Q' = \{(x,t) \in Q, \quad |\mathtt{grad}_x u(x,t)| < 1\}$.

- Let the operator A be defined by

$$Au = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p-2}\frac{\partial u}{\partial x_i}\Big) + |u|^{p-2}u$$

  and let

$$K_t = \{v \in V_t = W_0^{1,p}(\Omega_t), \quad v(x) \geq 0, \quad \text{a.e.} \quad \text{in} \quad \Omega_t\}.$$

Then, in view of Theorem 3.1, the corresponding parabolic variational inequality has exactly one solution, which is also weak solution of the following boundary value problem:

$$\frac{\partial u}{\partial t} + Au = f \quad \text{in} \quad Q,$$

$$u(x,t) \geq 0 \quad \text{in} \quad Q,$$

$$u(x,t) = 0 \qquad 0 < t < T, \qquad x \in \partial\Omega_t,$$

$$u(x,0) = 0 \qquad x \in \Omega_0.$$

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