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SOME WEAK GEOMETRIC INEQUALITIES FOR THE RIESZ POTENTIAL

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Key words: convolution operators, Riesz potential, Rayleigh-Faber-Krahn inequality, Hong-Krahn-Szegö inequality, homogeneous Lie group.

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Abstract. In the present paper, we prove that the first eigenvalue of the Riesz potential is weakly maximised in a quasi-ball among all Haar measurable sets on homogeneous Lie groups. It is an analogue of the classical Rayleigh-Faber-Krahn inequality for the Riesz potential. We also prove a weak version of the Hong-Krahn-Szegö inequality for the Riesz potential on homogeneous Lie groups.

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1 Introduction

The first eigenvalue of the Laplacian with the Dirichlet boundary condition is minimised on a ball among all domains of the same measure. This fact is called the (classical) Rayleigh-Faber-Krahn inequality (see, e.g. [2]). This also means that the norm of the inverse Dirichlet Laplacian is maximised on the ball among all domains with the same measure. However, the minimum of the second Dirichlet Laplacian eigenvalue is achieved not on one ball, but on the union of two identical balls. This fact is called the Hong-Krahn-Szegö inequality. There is a number of extentions of inequalities such type in spectral geometry to the cases of differential operators. In [14], the Rayleigh-Faber-Krahn and Hong-Krahn-Szegö inequalities were established for the Euclidean Riesz potential (that is, in the case of nonlocal operators), and then they were extended to more general convolution type positive integral operators in [16] and [17]. In [18], the Rayleigh-Faber-Krahn and Hong-Krahn-Szegö inequalities for the logarithmic potential were considered (which is an example of non-positive operator). Then, in our papers [4, 5, 8], and [10], the Rayleigh-Faber-Krahn and Hong-Krahn-Szegö type inequalities were obtained for various non-self-adjoint operators. All of these results were obtained in the Euclidean space. In this paper, we prove weak Rayleigh-Faber-Krahn and Hong-Krahn-Szegö inequalities on the so-called homogeneous groups, which are the non-compact, connected, semi-simple Lie groups with finite centers. In [11], the author proved the weak Riesz inequality on the non-compact, connected, semi-simple Lie groups with finite centers. Note that the weak Riesz inequality is related to the Kunze-Stein phenomenon (see, e.g. [1] and [11]). Thus, the weak Riesz inequality plays a key role in our proofs.

Summarising our results for the Riesz operator $\mathcal{R}_\Omega$ on the homogeneous Lie groups, in this paper, we show the following facts:

- Weak Rayleigh-Faber-Krahn type inequality: the first eigenvalue of $\mathcal{R}_\Omega$ is maximised on the quasi-ball among all sets of a given Haar measure on the homogeneous Lie groups;
- Weak Hong-Krahn-Szegö type inequality: the supremum of the second eigenvalue of (positive) $\mathcal{R}_\Omega$ on the homogeneous Lie groups among bounded open sets with a given Haar measure in

$\mathbb{G}$ is attained on the union of two identical quasi-balls with the distance between them going to infinity.

In Section 2 we present preliminary results of this paper. The proofs of the stated facts will be given in Section 3.

2 Preliminaries

Let us recall that a Lie group (on $\mathbb{R}^N$) $\mathbb{G}$ with the dilation

$$D_\lambda(x) := (\lambda^{\nu_1}x_1, \ldots, \lambda^{\nu_N}x_N),\ \nu_1, \ldots, \nu_N > 0,\ D_\lambda : \mathbb{R}^N \to \mathbb{R}^N,$$

which is an automorphism (of the group $\mathbb{G}$) for all $\lambda > 0$, is called a *homogeneous (Lie) group*. We refer to [3] for the original appearance of such groups, and to [15] for a recent comprehensive treatment (see, also recent paper [6], [7] and [9]). For simplicity, throughout this paper we use the notation λx for the dilation D_λ. The homogeneous dimension of the homogeneous group $\mathbb{G}$ is denoted by $Q := \nu_1 + \ldots + \nu_N$. Also, in this note, we denote a homogeneous quasi-norm on $\mathbb{G}$ by $|x|$, which is a continuous non-negative function

$$\mathbb{G} \ni x \mapsto |x| \in [0, \infty), \tag{2.1}$$

with the properties

i) $|x| = |x^{-1}|$ for all $x \in \mathbb{G}$,

ii) $|\lambda x| = \lambda |x|$ for all $x \in \mathbb{G}$ and $\lambda > 0$,

iii) $|x| = 0$ if and only if $x = 0$.

The quasi-ball centred at $x \in \mathbb{G}$ with radius $R > 0$ can be defined by

$$B(x, R) := \{y \in \mathbb{G} : |x^{-1}y| < R\}. \tag{2.2}$$

For brevity $B(x, R) = B$ if $x = e$ (the identity element of $\mathbb{G}$).
Recall the Riesz inequality on the Lie groups.

Theorem 2.1. [11] *Assume that $\mathbb{G}$ is a noncompact, connected, semi-simple Lie group with a finite center and real rank one. Let u^*, v^*, w^* be the symmetric rearrangements of functions u, v, w, respectively. Then, we have the weak Riesz inequality:*

$$\int_{\mathbb{G}}\int_{\mathbb{G}} u(x)g(yx^{-1})w(y)dxdy \leq C_{\mathbb{G}} \int_{\mathbb{G}}\int_{\mathbb{G}} u^*(x)g^*(yx^{-1})w^*(y)dxdy, \tag{2.3}$$

where $C_{\mathbb{G}}$ is a constant depending only on $\mathbb{G}$.

By [13], in the Euclidean space, (2.3) holds with $C_{\mathbb{R}^N} = 1$, which gives the classical Riesz inequality and also, (2.3) holds for nilpotent groups (by [15], we have that the homogeneous Lie group is a nilpotent group).

3 Main results

Assume that $\Omega \subset \mathbb{G}$ is an open Haar measurable set, and we consider the Riesz potential on $L^2(\Omega)$ of the form

$$\mathcal{R}_\Omega u(x) = \int_\Omega \frac{u(y)}{|y^{-1}x|^{Q-\alpha}} dy, \quad u \in L^2(\Omega), \;\; 0 < \alpha < Q. \tag{3.1}$$

The eigenvalues of $\mathcal{R}_\Omega$ may be enumerated in the descending order of their moduli,

$$|\lambda_1| \geq |\lambda_2| \geq \dots, \tag{3.2}$$

where $\lambda_j = \lambda_j(\Omega)$ is repeated in this series according to its multiplicity. Let us denote by $|\Omega|$ the Haar measure of Ω.

First, we present the weak Rayleigh-Faber-Krahn inequality for the Riesz potential on the homogeneous Lie groups.

Theorem 3.1. *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension Q. Then the first eigenvalue of the operator $\mathcal{R}_\Omega$ is weakly maximised in the quasi-ball B, that is,*

$$\lambda_1(\Omega) \leq C_{\mathbb{G}} \lambda_1(B), \tag{3.3}$$

for all Haar measurable sets $\Omega \subset \mathbb{G}$ with $|\Omega| = |B|$.

Remark 1. In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), so we recover the (classical) Rayleigh-Faber-Krahn inequality for the Riesz potential from [14].

Remark 2. In (3.3), the constant $C_{\mathbb{G}}$ is a particular case of the constant in (2.3).

Lemma 3.1. *The first eigenvalue λ_1 (with the largest modulus) of the operator $\mathcal{R}_\Omega$ is positive and simple. Also, the corresponding eigenfunction u_1 can be chosen to be positive.*

Proof of Lemma 3.1. All the eigenvalues are real numbers since the operator $\mathcal{R}_\Omega$ is compact and self-adjoint. Moreover, since the kernel is real, the eigenfunctions of the operator $\mathcal{R}_\Omega$ may be chosen to be real. First of all, we show that the first eigenfunction u_1 cannot change sign in $\Omega \subset \mathbb{G}$, i.e.,

$$u_1(x)u_1(y) = |u_1(x)u_1(y)|, \;\; x, y \in \Omega \subset \mathbb{G}.$$

Otherwise, by the continuity of the function u_1, there exist neighborhoods $U(x_0, r) \subset \Omega$ and $U(y_0, r) \subset \Omega$ such that

$$|u_1(x)u_1(y)| > u_1(x)u_1(y), \;\; x \in U(x_0, r) \subset \Omega, \;\; y \in U(y_0, r) \subset \Omega,$$

and from

$$\int_\Omega |z^{-1}x|^{\alpha-Q} |y^{-1}z|^{\alpha-Q} dz > 0 \tag{3.4}$$

we obtain

$$\begin{aligned} \frac{\langle \mathcal{R}_\Omega^2 |u_1|, |u_1| \rangle}{\|u_1\|^2} &= \frac{1}{\|u_1\|^2} \int_\Omega \int_\Omega \int_\Omega |z^{-1}x|^{\alpha-Q} |y^{-1}z|^{\alpha-Q} dz |u_1(x)||u_1(y)| dx dy \\ &> \frac{1}{\|u_1\|^2} \int_\Omega \int_\Omega \int_\Omega |z^{-1}x|^{\alpha-Q} |y^{-1}z|^{\alpha-Q} dz u_1(x)u_1(y) dx dy = \lambda_1^2. \end{aligned} \tag{3.5}$$

We have that the λ_1^2 is the largest eigenvalue of $\mathcal{R}_\Omega^2$ and u_1 is the eigenfunction corresponding to λ_1^2, i.e.

$$\lambda_1^2 u_1 = \mathcal{R}_\Omega^2 u_1.$$

Then, by the variational principle for the Rayleigh quotient we have

$$\lambda_1^2 = \sup_{f\in L^2(\Omega), f\not\equiv 0} \frac{\langle \mathcal{R}_\Omega^2 f, f\rangle}{\|f\|^2_{L^2(\Omega)}}. \tag{3.6}$$

This means that strict inequality (3.5) contradicts variational principle (3.6).

Then we will show that the first eigenfunction $u_1(x)$ can not become zero in Ω and thus can be chosen positive in Ω. Contrariwise, there is a point $x_0 \in \Omega$ such that

$$0 = \lambda_1^2 u_1(x_0) = \int_\Omega \int_\Omega |z^{-1}x_0|^{\alpha-Q} |y^{-1}z|^{\alpha-Q} dz\, u_1(y) dy,$$

from which, in view of condition (3.4), the contradiction follows: $u_1(y) = 0$ for almost all $y \in \Omega$.

The positivity of u_1 implies that λ_1 is simple. If an eigenfunction $\widetilde{u}_1$ (corresponding to λ_1) is linearly independent of u_1, then for all numbers $c \in \mathbb{R}$ any linear combination $u_1 + c\widetilde{u}_1$ is an eigenfunction corresponding to λ_1, hence, it cannot be equal to zero at any point in Ω. However, this is impossible since c is an arbitrary number. Since u_1 and the kernel are positive, it follows that λ_1 is positive. □

Here and below by * we denote the symmetric nonincreasing rearrangement of a function [11]. Note that for any nonnegative measurable function f one has

$$\|f\|_{L^p(\mathbb{G})} = \|f^*\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty. \tag{3.7}$$

Proof of Theorem 3.1. Let Ω be a bounded Haar measurable set in $\mathbb{G}$ and its symmetric rearrangement B is an quasi-ball centred at e (the identity element of $\mathbb{G}$) with the same measure of Ω, i.e. $|B| = |\Omega|$. Let u be a nonnegative measurable function in $\Omega \subset \mathbb{G}$.

By using Lemma 3.1 the first eigenvalue λ_1 of the operator $\mathcal{R}_\Omega$ is simple and the corresponding eigenfunction u_1 can be chosen positive in $\Omega \subset \mathbb{G}$. By using the weak Riesz inequality, we get

$$\int_\Omega \int_\Omega u_1(y) |y^{-1}x|^{\alpha-Q} u_1(x) dy dx \le C_{\mathbb{G}} \int_B \int_B u_1^*(y) |y^{-1}x|^{\alpha-Q} u_1^*(x) dy dx. \tag{3.8}$$

By using the variational principle for $\lambda_1(B)$ and (3.7), we obtain

$$\begin{aligned}\lambda_1(\Omega) = \frac{\int_\Omega \int_\Omega u_1(y)|y^{-1}x|^{\alpha-Q} u_1(x) dy dx}{\int_\Omega |u_1(x)|^2 dx} &\le C_{\mathbb{G}} \frac{\int_B \int_B u_1^*(y)|y^{-1}x|^{\alpha-Q} u_1^*(x) dy dx}{\int_B |u_1^*(x)|^2 dx} \\ \le C_{\mathbb{G}} \sup_{v\in L^2(B), v\not\equiv 0} \frac{\int_B \int_B v(y)|y^{-1}x|^{\alpha-Q} v(x) dy dx}{\int_B |v(x)|^2 dx} &= C_{\mathbb{G}} \lambda_1(B).\end{aligned}$$

□

Now we give a spectral geometry estimate for the second eigenvalue of $\mathcal{R}_\Omega$, the so-called Hong-Krahn-Szegö inequality.

Theorem 3.2. *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension Q. Let the second eigenvalue $\lambda_2(\Omega)$ of $\mathcal{R}_\Omega$ be positive. Then the weak supremum of $\lambda_2(\Omega)$ among all Haar measurable sets $\Omega \subset \mathbb{G}$ with a given measure is attained on the union of two identical quasi-balls with the distance between them going to infinity.*

Remark 3. In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), so we recover the Hong-Krahn-Szegö inequality for the Riesz potential from [14].

Proof Theorem 3.2. Suppose that

$$\Omega^+ := \{x : u_2(x) > 0\},\ \Omega^- := \{x : u_2(x) < 0\}.$$

In proofs we will use the notations

$$u_2(x) \gtrless 0,\ \forall x \in \Omega^\pm \subset \Omega \subset \mathbb{G},\ \Omega^\pm \neq \{\emptyset\},$$

and it follows from Lemma 3.1 that both sets Ω^- and Ω^+ have a positive Haar measure. By using

$$u_2^\pm(x) := \begin{cases} u_2(x), \text{ in } \Omega^\pm, \\ 0, \text{ otherwise}, \end{cases} \tag{3.9}$$

we get

$$\lambda_2(\Omega)u_2(x) = \int_{\Omega^+} |y^{-1}x|^{\alpha-Q} u_2^+(y)dy + \int_{\Omega^-} |y^{-1}x|^{\alpha-Q} u_2^-(y)dy,\ x \in \Omega.$$

By using above fact with multiplying by $u_2^+(x)$ and integrating over Ω^+ we obtain

$$\begin{aligned}\lambda_2(\Omega)\int_{\Omega^+} |u_2^+(x)|^2 dx = \int_{\Omega^+} u_2^+(x) \int_{\Omega^+} |y^{-1}x|^{\alpha-Q} u_2^+(y)dydx \\ + \int_{\Omega^+} u_2^+(x) \int_{\Omega^-} |y^{-1}x|^{\alpha-Q} u_2^-(y)dydx,\ x \in \Omega.\end{aligned}$$

By using (3.9), we have

$$\int_{\Omega^+} u_2^+(x) \int_{\Omega^-} |y^{-1}x|^{\alpha-Q} u_2^-(y)dydx \leq 0$$

Thus,

$$\lambda_2(\Omega)\int_{\Omega^+} |u_2^+(x)|^2 dx \leq \int_{\Omega^+} u_2^+(x) \int_{\Omega^+} |y^{-1}x|^{\alpha-Q} u_2^+(y)dydx,$$

that is,

$$\frac{\int_{\Omega^+} u_2^+(x) \int_{\Omega^+} |y^{-1}x|^{\alpha-Q} u_2^+(y)dydx}{\int_{\Omega^+} |u_2^+(x)|^2 dx} \geq \lambda_2(\Omega).$$

By the variational principle,

$$\begin{aligned}\lambda_1(\Omega^+) &= \sup_{v \in L^2(\Omega^+), v \not\equiv 0} \frac{\int_{\Omega^+} v(x) \int_{\Omega^+} |y^{-1}x|^{\alpha-Q} v(y)dydx}{\int_{\Omega^+} |v(x)|^2 dx} \\ &\geq \frac{\int_{\Omega^+} u_2^+(x) \int_{\Omega^+} |y^{-1}x|^{\alpha-Q} u_2^+(y)dydx}{\int_{\Omega^+} |u_2^+(x)|^2 dx} \geq \lambda_2(\Omega).\end{aligned}$$

Also with previous case, we obtain

$$\lambda_1(\Omega^-) \geq \lambda_2(\Omega).$$

Then we get

$$\min\{\lambda_1(\Omega^+), \lambda_1(\Omega^-)\} \geq \lambda_2(\Omega). \tag{3.10}$$

Suppose that B^+ and B^-, the geodesic quasi-balls of the same Haar measure as Ω^+ and Ω^-, respectively. From the Rayleigh-Faber-Krahn inequality in the Theorem 3.1, we have

$$C_{\mathbb{G}}\lambda_1(B^+) \geq \lambda_1(\Omega^+),\ C_{\mathbb{G}}\lambda_1(B^-) \geq \lambda_1(\Omega^-). \tag{3.11}$$

From (3.10) and (3.11), we get

$$C_{\mathbb{G}} \min\{\lambda_1(B^+),\ \lambda_1(B^-)\} \geq \lambda_2(\Omega). \tag{3.12}$$

Let us consider the set $B^+ \cup B^-$, with the quasi-balls B^+ and B^- placed at the distance l, that is,

$$l = \operatorname{dist}(B^+, B^-),$$

Assume that $u_1^{\circledast}$ be the first normalised eigenfunction of $\mathcal{R}_{B^+\cup B^-}$. Denote by u_+ and u_- the first normalised eigenfunctions of operators $\mathcal{R}_{B^\pm}$, respectively. We introduce the function $v^{\circledast} \in L^2(B^+ \cup B^-)$ in the following form:

$$v^{\circledast}(x) := \begin{cases} u_+(x), \text{ in } B^+, \\ \gamma u_-(x), \text{in } B^-. \end{cases} \tag{3.13}$$

We have that the functions $u_+, u_-, u_1^{\circledast}$ are positive and we can find a real number γ such that $v^{\circledast}$ is orthogonal to $u_1^{\circledast}$. Note that

$$\int_{B^+\cup B^-}\int_{B^+\cup B^-} v^{\circledast}(x)v^{\circledast}(y)|y^{-1}x|^{\alpha-Q}dxdy = \sum_{i=1}^{4}\mathcal{A}_i, \tag{3.14}$$

where

$$\mathcal{A}_1 := \int_{B^+}\int_{B^+} u_+(x)u_+(y)|y^{-1}x|^{\alpha-Q}dxdy,$$
$$\mathcal{A}_2 := \gamma\int_{B^+}\int_{B^-} u_+(x)u_-(y)|y^{-1}x|^{\alpha-Q}dxdy,$$
$$\mathcal{A}_3 := \gamma\int_{B^-}\int_{B^+} u_-(x)u_+(y)|y^{-1}x|^{\alpha-Q}dxdy,$$
$$\mathcal{A}_4 := \gamma^2\int_{B^-}\int_{B^-} u_-(x)u_-(y)|y^{-1}x|^{\alpha-Q}dxdy.$$

By the Rayleigh quotient for the second eigenvalue, we have

$$\lambda_2(B^+\cup B^-) = \sup_{v\in L^2(B^+\bigcup B^-),\, v\perp u_1,\, \|v\|=1}\int_{B^+\cup B^-}\int_{B^+\cup B^-} v(x)v(y)|y^{-1}x|^{\alpha-Q}dxdy.$$

Since $v^{\circledast}$ is orthogonal to u_1, we obtain

$$\lambda_2(B^+\cup B^-) \geq \int_{B^+\cup B^-}\int_{B^+\cup B^-} v^{\circledast}(x)v^{\circledast}(y)|y^{-1}x|^{\alpha-Q}dxdy = \sum_{i=1}^{4}\mathcal{A}_i.$$

On the other hand, since $u_\pm$ are the first normalised eigenfunctions (by Lemma 3.1 both are positive everywhere) for each quasi-ball $B^\pm$, we have

$$\lambda_1(B^\pm) = \int_{B^\pm}\int_{B^\pm} u_\pm(x)u_\pm(y)|y^{-1}x|^{\alpha-Q}dxdy.$$

Summarising the above facts, we get

$$\lambda_2(B^+ \cup B^-) \geq \sum_{i=1}^{4} \mathcal{A}_i \geq \frac{\sum_{i=1}^{4} \mathcal{A}_i}{1+\gamma^2} = \frac{\mathcal{A}_1 + \mathcal{A}_4 + \mathcal{A}_2 + \mathcal{A}_3}{\lambda_1(B^+)^{-1}\mathcal{A}_1 + \lambda_1(B^-)^{-1}\mathcal{A}_4}. \tag{3.15}$$

Taking into account that $0 < \alpha < Q$, the kernel $|y^{-1}x|^{\alpha-Q}$ tends to zero as $x \in B^{\pm},\ y^{-1} \in B^{\mp}$ and $l \to \infty$, we note that

$$\lim_{l\to\infty} \mathcal{A}_2 = \lim_{l\to\infty} \mathcal{A}_3 = 0,$$

therefore

$$\lim_{l\to\infty} \lambda_2(B^+ \cup B^-) \geq \max\{\lambda_1(B^+),\, \lambda_1(B^-)\}, \tag{3.16}$$

where $l = \mathrm{dist}(B^+, B^-)$. Inequalities (3.12) and (3.16) imply that the optimal set for λ_2 does not exist. Also by denoting $\Omega \equiv B^+ \bigcup B^-$ with $l = \mathrm{dist}(B^+, B^-) \to \infty$, and B^+ and B^- being identical, from inequalities (3.12) and (3.16) we get

$$\begin{aligned} \lim_{l\to\infty} \lambda_2(B^+ \bigcup B^-) &\leq C_{\mathbb{G}} \min\{\lambda_1(B^+),\, \lambda_1(B^-)\} \\ &\leq C_{\mathbb{G}} \max\{\lambda_1(B^+),\, \lambda_1(B^-)\} \leq C_{\mathbb{G}} \lim_{l\to\infty} \lambda_2(B^+ \cup B^-), \end{aligned} \tag{3.17}$$

and this implies that the maximising sequence for λ_2 is given by the union of two identical quasi-balls with the distance between them going to ∞. □

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