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CHARACTERIZATION OF POLYGROUPS BY IP-SUBSETS

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Communicated by U.U. Umirbaev

Key words: hypergroup, polygroup, IP-subset, single polygroup.

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Abstract. In this paper, we define the concept of IP-subsets of a polygroup and single polygroups. Indeed, if $\langle P, \circ, 1, ^{-1} \rangle$ is a polygroup of order n, then a non-empty subset Q of P is an IP-subset if $\langle Q, *, e, ^{I} \rangle$ is a polygroup, where for every $x, y \in Q$, $x * y = (x \circ y) \cap Q$. If P has no IP-subset of order $n-1$, then it is single. We show that every non-single polygroup of order n can be constructed from a polygroup of order $n-1$. In particular, we prove that there exist exactly 7 single polygroups of order less than 5.

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1 Introduction and preliminaries

The theory of algebraic hyperstructures which is a generalization of the concept of ordinary algebraic structures first was introduced by Marty [16]. Since then many researchers have worked on algebraic hyperstructures and developed it. A short review of this theory appears in [6, 7, 8, 9]. A *hypergroupoid* $(H, \circ)$ is a non-empty set H with a *hyperoperation* $\circ$ defined on H, i.e., a mapping of $H \times H$ into the family of all non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$. If A, B are non-empty subsets of H, then $A \circ B$ is given by

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b.$$

$x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. The hypergroupoid $(H, \circ)$ is called a *semihypergroup* if $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v,$$

and is called a *quasihypergroup* if for every $x \in H$, we have $x \circ H = H = H \circ x$. This condition is called the *reproduction axiom*. The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup. Application of hypergroups have mainly appeared in special subclasses. For example, polygroups which are certain subclasses of hypergroups are studied in [14] by Ioulidis and are used to study the color algebra [4]. Quasi-canonical hypergroups (called polygroups by Comer) were introduced in [2], as a generalization of canonical hypergroups. In [12] Heidari et al. studied the concept of topological polygroups as a generalization of topological groups. Ahabozorgi et al. introduced solvable polygroups [1]. The working draft [5] is a hand-written draft circulated many years ago. It enumerated all 4 element polygroups (integral relation algebras) and determined their color scheme representations. After this draft, several works are done. For example, Maddux [15] using a computer, enumerated those of order 5. A polygroup is a completely regular, reversible in itself multigroup in the sense of Dresher and Ore [11].

Definition 1. [3, 10] A *polygroup* is a system $\langle P, \circ, 1, ^{-1} \rangle$, where $1 \in P$, $^{-1}$ is a unitary operation on P, $\circ$ maps $P \times P$ into the family of non-empty subsets of P, and the following axioms hold for all $x, y, z \in P$:

(P1) $x \circ (y \circ z) = (x \circ y) \circ z$,

(P2) $1 \circ x = x = x \circ 1$,

(P3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

Clearly, every group is a polygroup. The following elementary facts about polygroups follow easily from the axioms: $1 \in x \circ x^{-1} \cap x^{-1} \circ x$, $1^{-1} = 1$, $(x^{-1})^{-1} = x$, and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$, where $A^{-1} = \{a^{-1} \mid a \in A\}$. A polygroup in which every element has order 2 (i.e., $x^{-1} = x$ for all x) is called symmetric. There exist several kinds of homomorphism of polygroups [10]. In this paper we consider a strong homomorphism. Let $\langle P_1, \cdot, e_1, ^{-1} \rangle$ and $\langle P_2, *, e_2, ^{-I} \rangle$ be two polygroups. Let f be a mapping from P_1 into P_2 such that $f(e_1) = e_2$. Then, f is called a *strong homomorphism* if

$$f(x \cdot y) = f(x) * f(y), \text{ for all } x, y \in P_1.$$

Clearly, a strong homomorphism f is an *isomorphism* if f is one to one and onto.

In [3], an extension of polygroups by polygroups have been introduced in the following way. Suppose that $\mathbf{P}$ and $\mathbf{Q}$ are polygroups whose elements have been renamed so that $P \cap Q = \{1\}$, where 1 is the identity of both $\mathbf{P}$ and $\mathbf{Q}$. A new system $\mathbf{P}[\mathbf{Q}] = (R, *, 1, ^I)$ called the extension of $\mathbf{P}$ by $\mathbf{Q}$, is formed in the following way. Set $R = P \cup Q$ and let $1^I = 1$, $x^I = x^{-1}$ (in the appropriate system), $1 * x = x * 1 = x$ for all $x \in R$, and for all $x, y \in R^* = R \setminus \{1\}$:

$$x * y = \begin{cases} x.y & \text{if } x, y \in P \\ x & \text{if } x \in Q, y \in P \\ y & \text{if } x \in P, y \in Q \\ x \circ y & \text{if } x, y \in Q, y \neq x^{-1} \\ x \circ y \cup P & \text{if } x, y \in Q, y = x^{-1}. \end{cases}$$

The extension construction $\mathbf{P}[\mathbf{Q}]$ will always yields a polygroup.

Let $\mathbf{P}$ and $\mathbf{Q}$ be finite polygroups with n and m elements, respectively. Then by considering $P = \{1, 2, 3, \ldots, n\}$ and $Q = \{1, n+1, n+2, \ldots, n+m-1\}$, the extension of $\mathbf{P}$ by $\mathbf{Q}$ is a polygroup with $n+m-1$ elements and underlying set $\{1, 2, 3, \ldots, n, n+1, \ldots, n+m-1\}$.

In [13] an extension of a polygroup by a non-empty set have been introduced in the following way. Let $\langle P, \circ, 1, ^{-1} \rangle$ be a polygroup and S be a non-empty set such that $P \cap S = \emptyset$. Put $R = P \cup S$, $x * 1 = 1 * x = x$ for all $x \in R$ and for all $x, y \in R^*$ define

$$x^{-I} = \begin{cases} x^{-1} & \text{if } x \in P \\ x & \text{if } x \in S \end{cases} \quad \text{and } x \uplus y = \begin{cases} x \circ y \cup S & \text{if } x, y \in P \\ P \cup S & \text{if } x = y \in S \\ R^* & \text{otherwise.} \end{cases}$$

The new system $\langle R, \uplus, 1, ^{-I} \rangle$ is called the extension of the polygroup $\mathbf{P}$ by the set S and denoted by $\mathbf{P}\{S\}$.

Theorem 1.1. [10] *There exist two non-isomorphic polygroups of order two.*

The cyclic group $\mathbb{Z}_2$ and

$$\mathbb{P}_2 = \begin{bmatrix} 1 & 2 \\ 2 & \{1, 2\} \end{bmatrix}$$

are two non-isomorphic polygroups of order 2.

Theorem 1.2. [5, 15] *There are 10 non-isomorphic polygroups of order three.*

All 10 non-isomorphic polygroups of order three are as follows, where $P = \{1, 2, 3\}$:

$$\mathrm{P}_3^1 = \mathbb{Z}_2[\mathbb{Z}_2],\ \mathrm{P}_3^2 = \mathbb{Z}_2[\mathbb{P}_2],\ \mathrm{P}_3^3 = \mathbb{P}_2[\mathbb{Z}_2],\ \mathrm{P}_3^4 = \mathbb{P}_2[\mathbb{P}_2],\ \mathrm{P}_3^5 = \mathbb{Z}_3,$$

$$\mathrm{P}_3^6 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & \{1,3\} & \{2,3\} \\ 3 & \{2,3\} & \{1,2\} \end{bmatrix},\ \mathrm{P}_3^7 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & \{1,3\} & \{2,3\} \\ 3 & \{2,3\} & P \end{bmatrix},$$

$$\mathrm{P}_3^8 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & P & \{2,3\} \\ 3 & \{2,3\} & P \end{bmatrix},\ \mathrm{P}_3^9 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & P \\ 3 & P & 3 \end{bmatrix},\mathrm{P}_3^{10} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & \{2,3\} & P \\ 3 & P & \{2,3\} \end{bmatrix}.$$

Theorem 1.3. *[5, 15] There exist 102 polygroups of order 4.*

2 IP-subsets

In this section, we introduce the notion of IP-subset of a polygroup and by using this concept, we define single polygroups. Then, we classify all single polygroups of order less than 5.

Definition 2. Let $\langle P, \circ, 1, ^{-1} \rangle$ and $Q \subseteq P$. Then Q is called an *IP-subset* of P if $\langle Q, *, e, ^{I} \rangle$ is a polygroup where for every $x, y \in Q$

$$x * y = (x \circ y) \cap Q.$$

The set of all IP-subsets of P is denoted by $\mathcal{IP}(P)$.

Example 1. Every sub-polygroup of a polygroup is an IP-subset. $\{1\}$ and P are trivial IP-subsets.

Definition 3. An IP-subset of a polygroup is called *pure* if it is not a sub-polygroup.

Example 2. Consider the polygroup $\langle P = \{1, 2, 3, 4\}, \circ, 1, ^{-1} \rangle$, where

$\circ$	1	2	3	4
1	1	2	3	4
2	2	$\{1,2\}$	3	4
3	3	3	$\{3,4\}$	P
4	4	4	P	$\{3,4\}$

Then $Q = \{1, 2\}, R = \{1, 3, 4\}$ and $S = \{2, 3, 4\}$ are non-trivial IP-subsets of P. Note that Q is a sub-polygroup of P also, R and S are pure IP-subsets.

Notation. We denote the $\{1, x, x^{-1}\}$ by $\hat{x}$.

Lemma 2.1. *Let $\langle P, \circ, 1, ^{-1} \rangle$ be a polygroup and $x \in P$. Then $\hat{x}$ is an IP-subset of P if and only if $\hat{x} \cap x^2 \neq \emptyset$.*

Proof. The "only if" part is straightforward. For the "if" part, suppose that $\hat{x} \cap x^2 \neq \emptyset$, then we consider two following cases:

Case 1. $1 \in x^2$. Then $\hat{x} = \{1, x\}$ and $\hat{x}$ is isomorphic to $\mathbb{Z}_2$ if $x \notin x^2$ and isomorphic to $\mathbb{P}_2$ if $x \in x^2$.

Case 2. $1 \notin x^2$. Then three sub-cases can be considered.

Sub-case 1. If $\hat{x} \cap x^2 = \{x\}$, then $x \in x \circ x^{-1}$. Thus, we have $x \circ x^{-1}$ and $x^{-1} \circ x$ contain $\hat{x}$ and $(x^{-1})^2 \cap \hat{x} = \{x^{-1}\}$. So, $\hat{x}$ is isomorphic to P_3^9.

Sub-case 2. If $\hat{x} \cap x^2 = \{x^{-1}\}$, then $x \notin x \circ x^{-1} \cap x^{-1} \circ x$ so, $x \circ x^{-1} = x^{-1} \circ x = \{1\}$ and $(x^{-1})^2 = \{x\}$. Thus, $\hat{x}$ is isomorphic to $\mathbb{Z}_3$.

Sub-case 3. If $\hat{x} \cap x^2 = \{x, x^{-1}\}$, then $x \in x \circ x^{-1} \cap x^{-1} \circ x$ so, $x \circ x^{-1} = x^{-1} \circ x \supseteq \hat{x}$ and $(x^{-1})^2 = \{x, x^{-1}\}$. Thus, $\hat{x}$ is isomorphic to $\mathbb{P}_3^{10}$. □

Example 3. In Example 2, $\hat{2} = \{1, 2\}$ and $\hat{3} = \{1, 3, 4\}$.

Corollary 2.1. *Every finite polygroup of even order contains an IP-subset.*

Proof. Every finite polygroup of even order has at least one self-inverse element, say x. Thus $1 \in \hat{x} \cap x^2$ so Lemma 2.1 implies that $\hat{x}$ is an IP-subset □

The following lemma provide a large class of IP-subsets.

Lemma 2.2. *Let P and Q be polygroups and S be a non-empty set. Then*
(1) P and Q are IP-subsets of $P[Q]$,
(2) P is an IP-subset of $P\{S\}$.

Proof. It is straightforward. □

Definition 4. Let $\langle P, \circ, 1, ^{-1} \rangle$ be a polygroup of order n. Then every IP-subset of size $n-1$ is called a *maximal IP-subset* of P. The set of all maximal IP-subsets of P is denoted by $\mathcal{M}(P)$.

Obviously, if G is a group, then $\mathcal{IP}(G)$ coincides with the set of all subgroups of G so $\mathcal{M}(G) = \emptyset$ if and only if $G \ncong \mathbb{Z}_2$. Also, $\mathcal{M}(P)$ contains pure IP-subsets if and only if $|P| > 2$.

Example 4. In Example 2, R and S are maximal IP-subsets.

Definition 5. A polygroup P is called *single* if $\mathcal{M}(P) = \emptyset$.

Any group not isomorphic to $\mathbb{Z}_2$ is a single polygroup. In the following we give examples of some single polygroups.

Example 5. The polygroup P_3^9 is single, since $\mathcal{IP}(P_3^9) = \{\{1\}, \{2\}, \{3\}, P\}$.

Example 6. Consider the Cayley table defined on $P = \{1, 2, 3, 4\}$ as follows:

$\circ$	1	2	3	4
1	1	2	3	4
2	2	$\{1,2\}$	$\{3,4\}$	$\{3,4\}$
3	3	$\{3,4\}$	2	$\{1,2\}$
4	4	$\{3,4\}$	$\{1,2\}$	2

Then, $\mathcal{IP}(P) = \{\{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, P\}$, so P is single.

Example 7. Consider the Cayley tables defined on $P = \{1, 2, 3, 4, 5\}$ as follows:

$\circ$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	$\{1,2,3\}$	4	5
3	3	$\{1,2,3\}$	3	4	5
4	4	4	4	5	$\{1,2,3\}$
5	5	5	5	$\{1,2,3\}$	4

Then, $\mathcal{IP}(P) = \{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, P\}$, so P is single.

Definition 6. Let Q be a polygroup. The set of all polygroups P such that Q is a maximal IP-subset of P denoted by $\mathcal{Q}^+$.

Example 8. We have

$$\mathbb{Z}_2^+ = \{P_3^1,\ P_3^2,\ P_3^3,\ P_3^6,\ P_3^7\}.$$

$$\mathbb{P}_2^+ = \{P_3^2,\ P_3^3,\ P_3^4,\ P_3^7,\ P_3^8\}.$$

Theorem 2.1. *If 1 is the unique self-inverse element of a polygroup P, then P is single.*

Proof. Assume by contradiction that $(P, \circ, 1, ^{-1})$ is not single and $(Q, *, e, ^{I})$ is a maximal IP-subset of P. We claim that e is self-inverse. Assume $1 \in Q$, then $1 = e * 1 \subseteq e \circ 1 = e$ so $e = 1$. If $1 \notin Q$, then $e^{-1} = e * e^{-1} \subseteq e \circ e^{-1}$. Hence, $e \in e \circ e^{-1}$. Thus, we obtain

$$\{e, e^{-1}\} \subseteq e \circ e^{-1} \cap Q = e * e^{-1} = \{e^{-1}\}.$$

Therefore, in any case $e = e^{-1}$ as required. Thus, since 1 is the only self-inverse element, $1 \in Q$. So, for every $x \in P$ we have $x \in Q$ if and only if $x^{-1} \in Q$. Hence, $Q = P$ or $|Q| < |P| - 1$. So Q is not a maximal IP-subset, which is a contradiction. □

In the proof of Theorem 2.1 we show that the identity of a maximal IP-subset Q of a polygroup P is a self-inverse element in P. So, we state this important fact as

Lemma 2.3. *If $(Q, *, e, ^{I})$ is a maximal IP-subset of a polygroup $(P, \circ, 1, ^{-1})$, then $e = e^{-1}$.*

Theorem 2.2. *Let P be a single polygroup of order 3. Then P is isomorphic to $\mathbb{Z}_3$, P_3^9 or P_3^{10}.*

Proof. Let P be a polygroup of order 3. Then every element of P is self-inverse or 1 is the only self-inverse element of P. In the former case by Corollary 2.1, for every $x \in P$, $\hat{x}$ is a maximal IP-subset of P so it is not single. In the latter case, Theorem 2.1 implies that P is single. Hence by Theorem 1.2, P is isomorphic to $\mathbb{Z}_3$, P_3^9 or P_3^{10}. □

Lemma 2.4. *Let $\langle P, \circ, 1, ^{-1}\rangle$ be a polygroup such that $\{1, a\}$ be the set of all self-inverse elements of P. If $b \circ b = a$ for some $b \in P$, then P is single.*

Proof. Assume by contradiction that Q is a maximal IP-subset of P. Then two cases can be considered:

Case 1. If $1 \in Q$, then $x \in Q$ if and only if $x^{-1} \in Q$. Since if $x \in Q$, then $1 \in x * x^I \subseteq x \circ x^I$ so $x^{-1} = x^I \in Q$. Therefore, we have $a \notin Q$ because non self-inverse elements occur in pairs and just one cannot be omitted. Hence, $b * b = \emptyset$ that is impossible.

Case 2. If $1 \notin Q$, then by Lemma 2.3, a is the identity element of Q. So, we obtain $a = a * a = a \circ a \cap Q$. Therefore, $a \circ a = \{1, a\}$. On the other hand

$$\begin{aligned} b \circ b = a \quad &\Rightarrow b^{-1} \circ b^{-1} = a \\ &\Rightarrow b^{-1} \in a \circ b \\ &\Rightarrow b^{-1} \circ b \subseteq (a \circ b) \circ b = a \circ (b \circ b) = \{1, a\} \\ &\Rightarrow b^{-1} \circ b = \{1, a\} \\ &\Rightarrow b^{-1} * b = a \\ &\Rightarrow b^I = b^{-1}. \end{aligned}$$

Since $b^I = b$, so b is a self-inverse in P, is a contradiction. Therefore, P is single. □

Theorem 2.3. *There exist exactly 4 single polygroups of order 4.*

Proof. The groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are single polygroups of order 4. Let P_4 and P'_4 be the polygroups such that their Cayley tables are defined as follows:

P_4	1	2	3	4
1	1	2	3	4
2	2	$\{1,2\}$	$\{3,4\}$	$\{3,4\}$
3	3	$\{3,4\}$	2	$\{1,2\}$
4	4	$\{3,4\}$	$\{1,2\}$	2

P'_4	1	2	3	4
1	1	2	3	4
2	2	$\{1,2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$
3	3	$\{2,3,4\}$	2	$\{1,2\}$
4	4	$\{2,3,4\}$	$\{1,2\}$	2

By Lemma 2.4 the polygroups P_4 and P'_4 are single. On the other hand by a simple computer programing one can see that

$$\mathcal{P}_3^+ = \bigcup_{Q \in \mathcal{P}_3} \mathcal{Q}^+$$

contains 98 non-single polygroups of order 4, up to isomorphism, where $\mathcal{P}_3$ is the set of all polygroups of order 3. Now, Theorem 1.3 completes the proof. $\square$

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