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CALCULATION OF THE CALDERÓN – LOZANOVSKII CONSTRUCTION FOR A COUPLE OF LOCAL MORREY SPACES

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Communicated by T.V. Tararykova

Key words: Calderón – Lozanovskii construction, local Morrey spaces, approximation in local Morrey spaces, Banach ideal spaces, sublinear operators, interpolation theorems in local Morrey spaces.

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Abstract. The calculation of the Calderón – Lozanovskii construction for a couple of local Morrey spaces is reduced to the calculation of the Calderón – Lozanovskii construction for two couples of ideal spaces of functions and sequences, that are the parameters in the definition of local Morrey spaces. These results allows us to obtain new interpolation theorems for operators on local Morrey spaces.

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1 Introduction

The role of the Calderón – Lozanovskii construction [13], [19] is well known. This construction for a couple of ideal spaces (X_0, X_1) and a concave function $\varphi(.,.)$ allows to construct a new ideal space $\varphi(X_0, X_1)$. The Calderón – Lozanovskii construction plays an important role in the interpolation theory of ideal function spaces and other issues of harmonic analysis. The construction of $\varphi(X_0, X_1)$ is ideologically close to the method of constructing Orlicz spaces and any Orlicz space can be obtained from the couple (L^1, L^∞) with the appropriate choice of the $\varphi : \mathbb{R}^2_+ \to \mathbb{R}_+$.

On the other hand, the spaces M_{λ,L^p}, introduced by Morrey [21], and their generalizations [8], [9] play an important role in harmonic analysis and in the study of partial differential equations.

The parameters of the generalized Morrey spaces $M^\tau_{l,X}$ are actually two ideal spaces: a function space X and a sequence space l. Therefore, a natural question arises how to express Calderón – Lozanovskii construction $\varphi(M^\tau_{l_0,X_0}, M^\tau_{l_1,X_1})$ via the couples of spaces (l_0, l_1), (X_0, X_1). It turns out that under certain assumptions holds the equality

$$\varphi(M_{l_0,X_0}, M_{l_1,X_1}) = M_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}.$$

In this equality the function $\varphi^*(.,.)$ is constructed by using $\varphi(.,.)$. This equality shows that the couples of spaces (X_0, X_1) and (l_0, l_1) enter unequally in the answer. However, for the power function $\varphi(.,.)$ the function $\varphi^*(.,.)$ coincides with $\varphi(.,.)$.

The obtained explicit description of the Calderón – Lozanovskii construction for a couple of local Morrey spaces allows us to calculate the Gustavsson – Peetre – Ovchinnikov interpolation functor [7], [14], [22] and obtain the corresponding interpolation theorem.

Note that the computation of any interpolation functors for a couple of Morrey spaces can be reduced [1] to the computation of this functor on a couple of vector function spaces, but the explicit form of the resulting spaces is not always known [1], [11] – [12]. Moreover, sometimes "unusual" answers are obtained [6], [23].

2 Preliminary information and technical lemmas

Let Ω be a measurable subset in $\mathbb{R}^n$, let μ be the Lebesgue measure in $\mathbb{R}^n$, let $S(\mu, \Omega)$ be the space of all measurable functions $x : \Omega \to \mathbb{R}$ and let $\chi(D)$ stand for the characteristic function of a set D. Along with the Lebesgue spaces $L^p \equiv L^p(\Omega)$, $p \in [1, \infty]$ ideal and symmetric spaces X are often used in harmonic analysis. Recall their definitions (see, for example, [15], [16], [18]).

A Banach space X of measurable functions on Ω is said to be ideal if it follows from the condition $x \in X$, the measurability of y and the validity of the inequality $|y(t)| \leq |x(t)|$ for almost all $t \in \Omega$ that $y \in X$ and $\|y|X\| \leq \|x|X\|$ (the symbol $\|x|X\|$ denotes the norm of an element x in the space X).

When $x : \Omega \to \mathbb{R}$ we denote by $\lambda(f, \gamma)$, $(\gamma > 0)$ the distribution function of x, namely, $\lambda(x, \gamma) = \mu\{t \in \Omega : |x(t)| \leq \lambda\}$, and by x^* the rearrangement of x in nonincreasing order. An ideal space X is said to be symmetric if it follows from the condition $x \in X$, the measurability of y and the validity of the inequality $\lambda(y, \gamma) \leq \lambda(x, \gamma)$ for all $\gamma \in \mathbb{R}_+$ that $y \in X$ and $\|y|X\| \leq \|x|X\|$. Examples of symmetric spaces are Orlicz, Lorentz and Marcinkiewicz spaces. Details can be found in [18], [20].

Let C_{cv} denote the set of all functions $\varphi : \mathbb{R}^2_+ \to \mathbb{R}_+$ concave, positively homogeneous of degree one, nondecreasing continuous in each variable and such that $\varphi(0, 0) = 0$.

The class C_{cv} is a cone with respect to the operations of addition and multiplication by a non-negative number.

For each function $\varphi \in C_{cv}$ we define the dilation function $\overline{\varphi}(1, .)$ by

$$\overline{\varphi}(1, \lambda) = \sup_{t>0} \frac{\varphi(1, \lambda t)}{\varphi(1, t)}, \quad \lambda \geq 0.$$

The homogeneity condition for $\varphi \in C_{cv}$ implies, for $\lambda > 0$, the equality

$$\overline{\varphi}(1, \lambda) = \sup_{t>0} \frac{\varphi(1, \lambda t)}{\varphi(1, t)} = \lambda \sup_{t>0} \frac{\varphi(\frac{1}{\lambda t}, 1)}{\varphi(\frac{1}{t}, 1)} = \lambda \sup_{t>0} \frac{\varphi(\frac{t}{\lambda}, 1)}{\varphi(t, 1)} = \lambda \overline{\varphi}(\lambda^{-1}, 1).$$

Let for any $\lambda_1, \lambda_2 > 0$

$$\overline{\varphi}(\lambda_1, \lambda_2) = \lambda_1 \overline{\varphi}(1, {\lambda_1}^{-1} \lambda_2) = \lambda_2 \overline{\varphi}({\lambda_2}^{-1} \lambda_1, 1).$$

This function is positively homogeneous of degree one, $\overline{\varphi}(1, \lambda)$ does not decrease with respect to λ and $\lambda^{-1} \overline{\varphi}(1, \lambda)$ does not increase with respect to λ, i.e. $\overline{\varphi}$ is quasi-concave. It is well known [16] that every quasi-concave function ϕ is equivalent to its smallest concave majorant $\widetilde{\phi}$ with an equivalence constant equal to 2. Therefore, for each $\varphi \in C_{cv}$, the function $\overline{\varphi}$ is equivalent to the function $\widetilde{\overline{\varphi}} \in C_{cv}$, which we will denote by φ^*.

For each $\varphi \in C_{cv}$ for all $(t, s) \in \mathbb{R}^2_+$ and any $\lambda > 0$ inequality holds

$$\varphi(t, s) \leq \overline{\varphi}(1, \lambda) \varphi(t, \frac{s}{\lambda}). \tag{2.1}$$

Indeed, inequality (2.1) is equivalent to the validity for each $\tau > 0$ of the inequality

$$\varphi(1, \tau) \leq \varphi(1, \frac{\tau}{\lambda}) \overline{\varphi}(1, \lambda)$$

or

$$\frac{\varphi(1, \lambda \tau)}{\varphi(1, \tau)} \leq \overline{\varphi}(1, \lambda).$$

The last inequality follows directly from the definition of the function $\overline{\varphi}$.

It is also easy to see that for all $\lambda > 0$ follow the inequalities

$$\overline{\varphi}(1, \lambda) \geq \frac{\varphi(1, \lambda)}{\varphi(1, 1)}, \qquad \overline{\varphi}(\lambda, 1) \geq \frac{\varphi(\lambda, 1)}{\varphi(1, 1)}. \tag{2.2}$$

Definition 1. Let $\varphi \in C_{cv}$ be given. We say that $\varphi \in C_{cv}^m$ if for any $t_0 > 0, t_1 > 0, s_0 > 0, s_1 > 0$ from the inequality

$$\varphi(t_0, t_1) \leq \varphi(s_0, s_1) \tag{2.3}$$

follows the inequality

$$\varphi(\frac{t_0}{s_0}, \frac{t_1}{s_1}) \leq c < \infty \tag{2.4}$$

with the constant c, independent of t_0, t_1, s_0, s_1.

The smallest value of the constant c in (2.4) is denoted by c_φ.

We now reformulate the membership condition $\varphi \in C_{cv}^m$ in terms of the function $\overline{\varphi}$.

Lemma 2.1. *A function $\varphi \in C_{cv}$ belongs to C_{cv}^m if and only if the inequality holds*

$$\sup_{t>0} \varphi(1, t)\overline{\varphi}(1, \frac{1}{t}) = c_\varphi < \infty. \tag{2.5}$$

Proof. Let $\varphi \in C_{cv}^m$. Without loss of generality, we can assume that equality in (2.3) holds: $\varphi(t_0, t_1) = \varphi(s_0, s_1)$, or

$$\frac{t_0}{s_0} = \frac{\varphi(1, \frac{s_1}{s_0})}{\varphi(1, \frac{t_1}{t_0})}.$$

Then condition (2.4) can be written in the equivalent form

$$\frac{\varphi(1, \frac{s_1}{s_0})}{\varphi(1, \frac{t_1}{t_0})}\varphi(1, \frac{s_0}{s_1}\frac{t_1}{t_0}) \leq c_\varphi. \tag{2.6}$$

Put $\tau_0 = \frac{s_1}{s_0}$, $\tau_1 = \frac{t_1}{t_0}$. Then (2.6) can be rewritten as follows

$$\frac{\varphi(1, \tau_0)}{\varphi(1, \tau_1)}\varphi(1, \frac{\tau_1}{\tau_0}) \leq c_\varphi. \tag{2.7}$$

Since τ_0, τ_1 can take any value from $(0, \infty)$, then (2.7) is equivalent to the following inequality

$$\varphi(1, \tau_0) \sup_{\tau_1>0} \frac{\varphi(1, \frac{\tau_1}{\tau_0})}{\varphi(1, \tau_1)} \leq c_\varphi,$$

or

$$\sup_{\tau_0>0} \varphi(1, \tau_0)\overline{\varphi}(1, \frac{1}{\tau_0}) \leq c_\varphi,$$

which follows from (2.5).

The reverse implication is proved similarly. □

Fix $0 \leq \theta \leq 1$ and $\varphi_\theta(t_0, t_1) = t_0^\theta \cdot t_1^{1-\theta}$. It is not difficult to see that $\varphi_\theta \in C_{cv}^m$, and we can put $c_\varphi = 1$. Now we will give a more difficult example. Let be given $\theta_0, \theta_1 \in (0, 1)$. We define the function $\varphi(1, .)$ by the equality $\varphi(1, t) = \min\{t^{\theta_0}, t^{\theta_1}\}$. It is not difficult to see that $\overline{\varphi}(1, \lambda) = \max\{\lambda^{\theta_0}, \lambda^{\theta_1}\}$. From here we obtain that the equality $\varphi(1, t)\overline{\varphi}(1, \frac{1}{t}) = 1$ holds and, therefore, $\varphi \in C_{cv}^m$.

We recall the definition of the construction of Calderón – Lozanovskii.

Definition 2. Let a couple of ideal spaces (X_0, X_1) on Ω and $\varphi \in C_{cv}$ be given. The space $\varphi(X_0, X_1)$ consists of all measurable functions x, for which there is a pair of functions $x_0 \in X_0$, $x_1 \in X_1$ such that almost everywhere holds the inequality

$$|x(t)| \leq \varphi(x_0(t), x_1(t)). \tag{2.8}$$

On the space $\varphi(X_0, X_1)$ the norm is introduced by the equality

$$\|x|\varphi(X_0, X_1)\| =$$
$$\inf\{\lambda > 0 : |x(t)| \le \lambda\varphi(x_0(t), x_1(t)) \ \text{(for a. a. } t \in \Omega),$$
$$x_i \in X_i, \|x_i|X_i\| \le 1; (i = 0, 1)\}. \tag{2.9}$$

The space $\varphi(X_0, X_1)$ is an ideal Banach space equipped with this norm.

If $\varphi(t, s) \equiv \varphi_\theta(t, s)$, then the definition of the space $\varphi_\theta(X_0, X_1)$, which is usually denoted by $X_0^\theta \cdot X_1^{1-\theta}$, was proposed by A.P. Calderón [13]; for an arbitrary $\varphi \in C_{cv}$ the space $\varphi(X_0, X_1)$ was defined by G.Ya. Lozanovskii [19]. The Calderón – Lozanovskii construction of $\varphi(X_0, X_1)$ has found many applications in the theory of ideal spaces [20], in the theory of interpolation of linear operators [7], [22], in the geometric theory of Banach spaces [5].

In cases in which exact estimates of constants are important, on the space $\varphi(X_0, X_1)$ we can introduce norms different from (2.9) as follows. Let $\psi(a_1, a_2) : \mathbb{R}^2 \to \mathbb{R}_+$ be a norm on $\mathbb{R}^2$. Then on $\varphi(X_0, X_1)$ the norm is defined by the equality

$$\|x|\varphi(X_0, X_1)\|_\psi =$$
$$\inf\{\psi(a_1, a_2) : |x(t)| \le \varphi(x_0(t), x_1(t)) \ \text{(for a. a. } t \in \Omega),$$
$$x_i \in X_i, \|x_i|X_i\| = a_i; (i = 0, 1)\}. \tag{2.10}$$

The space $\varphi(X_0, X_1)$ is an ideal Banach space equipped with the norm $\| \,.\, |\varphi(X_0, X_1)\|_\psi$.

Of course all the norms on $\varphi(X_0, X_1)$, defined by equation (2.10), are equivalent. If we put $\psi_\infty(a_1, a_2) = \ max\{|a_1|, |a_2|\}$, then the norm on the space $\{\varphi(X_0, X_1), \psi_\infty\}$ coincides with the norm defined in (2.9). For example [3], using the introduced norms one can to define the exact dual space $\{\varphi(X_0, X_1), \psi\}'$ and the exact dual norm on this space $\{\varphi(X_0, X_1), \psi\}$.

Directly from inequality (2.2) it follows that the continuous embedding

$$\{\varphi^*(X_0, X_1), \psi\} \subseteq \{\varphi(X_0, X_1), \psi\} \tag{2.11}$$

holds with the embedding constant equal to 1.

Inequality (2.1) implies the following estimate for the norms in the space $\{\varphi(X_0, X_1), \psi\}$.

Lemma 2.2. *Let a couple of ideal spaces X_0, X_1 on Ω and $\varphi \in C_{cv}$ be given. Let for some $x \in \{\varphi(X_0, X_1), \psi\}$ almost everywhere on Ω the inequality*

$$|x(t)| \le \varphi(x_0(t), x_1(t))$$

holds.

Then for any $\lambda > 0$ almost everywhere on Ω the inequality

$$|x(t)| \le \varphi(\overline{\varphi}(1, \lambda)x_0(t), \frac{\overline{\varphi}(1, \lambda)}{\lambda} x_1(t)) \tag{2.12}$$

holds and, therefore, the inequality

$$\|x|\{\varphi(X_0, X_1), \psi\}\| \le \inf_\lambda \psi(\overline{\varphi}(1, \lambda)\|x_0|X_0\|, \frac{\overline{\varphi}(1, \lambda)}{\lambda}\|x_1|X_1\|) \tag{2.13}$$

holds.

In the case of $\psi(a_0, a_1) = \psi_\infty(a_0, a_1)$ Lemma 2.2 can be improved.

Lemma 2.3. *Let a couple of ideal spaces X_0, X_1 on Ω and $\varphi \in C_{cv}$ be given. Then*

$$\|x|\{\varphi(X_0, X_1), \psi_\infty\}\| =$$
$$\inf\{\overline{\varphi}(a_1, a_2) : |x(t)| \le \varphi(x_0(t), x_1(t)) \text{ a. e. on } \Omega,$$
$$x_i \in X_i, \|x_i|X_i\| = a_i; (i = 0, 1)\}.$$

Proof. We first prove that the function $\overline{\varphi}$ is continuous at the point $(1, 1)$. From $\varphi \in C_{cv}$ it follows that for $\lambda > 1$ for each $t > 0$ the following inequalities are satisfied

$$\varphi(\lambda t, 1) \le \lambda\varphi(t, 1), \quad \varphi(1, \lambda t) \le \lambda\varphi(1, t).$$

From these inequalities it follows that $\overline{\varphi}(\lambda, 1) \le \lambda$ and $\overline{\varphi}(1, \lambda) \le \lambda$. Therefore, for $\lambda > 1$ the following relations are satisfied

$$|\overline{\varphi}(\lambda, 1) - \overline{\varphi}(1, 1)| \le |\lambda - 1|;$$
$$|\overline{\varphi}(1, 1) - \overline{\varphi}(\frac{1}{\lambda}, 1)| \le |\overline{\varphi}(1, 1) - \frac{1}{\lambda}\overline{\varphi}(1, 1)| + |\frac{1}{\lambda}\overline{\varphi}(1, 1) - \frac{1}{\lambda}\overline{\varphi}(1, \lambda)| \le$$
$$(1 - \frac{1}{\lambda})\overline{\varphi}(1, 1) + \frac{\lambda - 1}{\lambda}\overline{\varphi}(1, 1) = 2(1 - \frac{1}{\lambda})\overline{\varphi}(1, 1).$$

Therefore the function $\overline{\varphi}(\lambda_1, \lambda_1)$ is continuous at the point $(1, 1)$.

Denote

$$m_1 = \|x|\{\varphi(X_0, X_1), \psi_\infty\}\|,$$
$$m_2 = \inf\{\overline{\varphi}(a_1, a_2) : |x(t)| \le \varphi(x_0(t), x_1(t)) \quad \text{a. e. on } \Omega, \quad x_i \in X_i, \|x_i|X_i\| = a_i; (i = 0, 1)\}.$$

The inequality $m_2 \le m_1$ follows from the equality $\overline{\varphi}(1, 1) = 1$ and the continuity of the function $\overline{\varphi}$ at the point $(1, 1)$.

We prove the reverse inequality. Let

$$|x(t)| \le \varphi(x_0(t), x_1(t)) \quad \text{a. e. on } \Omega, \quad x_i \in X_i, \|x_i|X_i\| = a_i; (i = 0, 1).$$

Then from Lemma 2.2 it follows that almost everywhere on Ω the inequalities (2.12) - (2.13) are fulfilled. Putting $\lambda = a_1/a_0$, we obtain the inequality

$$\|x|\{\varphi(X_0, X_1), \psi_\infty\}\| \le \overline{\varphi}(\|x_0|X_0\|, \|x_1|X_1\|).$$

Therefore the inequality $m_1 \le m_2$ is true.

□

This lemma implies the well-known equality

$$\|x|\{X_0^\theta \cdot X_1^{1-\theta}, \psi_\infty\}\| =$$
$$\inf\{\|x_0|X_0\|^\theta \cdot \|x_1|X_1\|^{1-\theta} : |x(t)| \le x_0^\theta(t) \cdot x_1^{1-\theta}(t) \quad \text{a .e. on } \Omega\}.$$

The following lemma will play an important role in the describing properties of the space $\varphi(X_0, X_1)$

Lemma 2.4. *Let be given a function $\varphi \in C_{cv}$, and the functions $\overline{\varphi}, \varphi^*$ be defined as above. Suppose that the function φ^* belongs to C_{cv}^m and for some $a_0 > 0, a_1 > 0, b_0 > 0, b_1 > 0$ the inequality holds*

$$\varphi^*(a_0, a_1) \le \varphi^*(b_0, b_1).$$

Then for all $t > 0, s > 0$ the inequality

$$\varphi(\frac{t}{a_0}b_0, \frac{s}{a_1}b_1) \ge \frac{1}{c_{\varphi^*}}\varphi(t, s), \tag{2.14}$$

is satisfied, where c_{φ^*} is the constant entering Definition 1.

Proof. Inequality (2.14) is equivalent to the inequality

$$\frac{b_1}{a_1}\varphi(\frac{\tau b_0 a_1}{a_0 b_1},1) \geq \frac{1}{c_{\varphi^*}}\varphi(\tau,1)$$

or

$$c_{\varphi^*} \geq \sup_{\tau>0} \frac{\varphi(\tau,1)}{\frac{b_1}{a_1}\varphi(\frac{\tau b_0 a_1}{a_0 b_1},1)}$$

or

$$c_{\varphi^*} \geq \frac{a_1}{b_1}\overline{\varphi}(\frac{a_0 b_1}{b_0 a_1},1) = \overline{\varphi}(\frac{a_0}{b_0},\frac{a_1}{b_1}).$$

The last inequality follows from the assumption $\varphi^* \in C^m_{cv}$.

□

Along with function spaces we need ideal spaces of sequences. Let $e^i = \{...,0,1,0,...\}$, ($i \in \mathbb{Z}$, the unit stands in the i-th place) be the standard basis in the space of two-side sequences. We denote by the symbol l an ideal space of sequences $x = \sum_{i=-\infty}^{\infty} x_i e^i$ ($x_i \in \mathbb{R}$) with the norm $\|x|l\|$. All the properties listed above for function spaces are preserved for sequence spaces. For details concerning the theory of sequence spaces, see [17].

The classical Morrey space M_{λ,L^p}, ($\lambda \in \mathbb{R}$) (see [21]), consists of all functions $f \in L^{1,loc}(\mathbb{R}^n)$ for which the following norm is finite:

$$\|f|M_{\lambda,L_p}\| = \sup_{x\in\mathbb{R}^n}\sup_{r>0} r^{-\lambda}\|f\chi(B(0,r))|L^p\|.$$

We note that if $\lambda = 0$, then $M_{\lambda,L_p} = L_p$, if $\lambda = \frac{n}{p}$, then $M_{\lambda,L_p} = L_\infty$, if $\lambda < 0$ or $\lambda > \frac{n}{p}$, then M_{λ,L_p} consists only of functions equivalent to zero.

If we now replace the Lebesgue space L^p in the definition of the classical Morrey space by an ideal space X, we obtain the Morrey space $M_{\lambda,X}$ constructed from the ideal space X in which the norm is defined by the equality

$$\|f|M_{\lambda,X}\| = \sup_{x\in\mathbb{R}^n}\sup_{r>0} r^{-\lambda}\|f\chi(B(0,r))|X\|.$$

The next step in the extension of Morrey spaces consists of the replacement of the outer sup-norm by the norm in an ideal space L and the replacement of the balls $B(0,r)$ by homothetic sets $U(0,r) \subset \mathbb{R}^n$. Below, we always assume that $0 \in U(0,1)$ and $\mu(U(0,1))) \in (0,\infty)$. Moreover, we often assume that $U(0,1)$ is star-shaped with respect to the point 0, that is, if $t \in U(0,1)$, then $\gamma t \in U(0,1)$ for all $\gamma \in (0,1)$. In general, the star-shapedness assumption is not necessary, but sometimes is useful. Thus, the natural definition of local Morrey space constructed from the family of $U(0,r)$ with continuously changing parameter has the following form.

Definition 3. [1]. Let an ideal space X on $\mathbb{R}^n$, an ideal space L on $\mathbb{R}_+$ and a set $U(0,1) \subset \mathbb{R}^n$ for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$ be given. The Morrey space $M_{L,X}$ consists of all $f \in L^{1,loc}(R^n)$ for each of which the following norm is finite:

$$\|f|M_{L,X}\| = \|\|f\chi(U(0,.))|X\||L\|.$$

The spaces introduced in Definition 3 we call local continuous Morrey spaces.

Along with the space $M_{L,X}$ we need local Morrey spaces constructed from a family of sets $\{U(0,r_i)\}$ with discretely varying parameter.

We denote by Υ the set of non-negative number sequences $\tau = \{\tau_i\}$ each of which satisfies the conditions

$$\forall i: \qquad \tau_i < \tau_{i+1}, \qquad \bigcup_i (\tau_i, \tau_{i+1}] = \mathbb{R}_+.$$

When $\tau_{i+1} = \infty$, we assume that $(\tau_i, \infty] = (\tau_i, \infty)$. For every sequence $\tau = \{\tau_i\}$ we construct a family of sets $U(0, \tau_i)$ and a family of disjoint annuli $D_i = U(0, \tau_i) \backslash U(0, \tau_{i-1})$.

Definition 4. [1]. Let an ideal space X on $\mathbb{R}^n$, an ideal space l of two-sided sequences with the standard basis $\{e^i\}$ and a sequence $\tau \in \Upsilon$ be given. By the Morrey space $M^{\tau}_{l,X}$ we mean the set of all functions $f \in L^{1,loc}(\mathbb{R}^n)$ for each of which the following norm is finite:

$$\|f|M^{\tau}_{l,X}\| = \|\sum_{i=-\infty}^{\infty} e^i \|f\chi(U(0,\tau_i))|X\| |l\|.$$

The spaces introduced in Definition 4 are called local discrete Morrey spaces.

Definition 5. Let an ideal space X on $\mathbb{R}^n$, an ideal space l of two-sided sequences with the standard basis $\{e^i\}$ and a sequence $\tau \in \Upsilon$ be given. By the approximation local Morrey space $\overline{M^{\tau}_{l,X}}$ we mean the set of all functions $f \in L^{1,loc}(\mathbb{R}^n)$ for each of which the following norm is finite:

$$\|f|\overline{M^{\tau}_{l,X}}\| = \|\sum_{i=-\infty}^{\infty} e^i \|f\chi(D_i))|X\| |l\|.$$

The approximation local Morrey spaces $\overline{M^{\tau}_{l,X}}$ are also defined in [1], here we introduce the natural name for $\overline{M^{\tau}_{l,X}}$.

Discussion of interconnections of the spaces $M_{L,X}$, $M^{\tau}_{l,X}$, $\overline{M^{\tau}_{l,X}}$ and their examples are given in [1].

We only note that the embedding $M^{\tau}_{l,X} \subseteq \overline{M^{\tau}_{l,X}}$ is obvious and the reverse embedding, which plays a key role in the theory of discrete Morrey spaces, is given in the following theorem.

Theorem B. [1]. *Let an ideal space X on $\mathbb{R}^n$, an ideal space L on $\mathbb{R}_+$ and a set $U(0,1) \subset \mathbb{R}^n$ for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$ and a sequence $\tau \in \Upsilon$ be given. Let the spaces $M^{\tau}_{l,X}$ and $\overline{M^{\tau}_{l,X}}$ be constructed from the spaces X and l, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$. We introduce the operator $T: l \to l$ by the equality*

$$T(\sum_{i=-\infty}^{\infty} e^i x_i) = \sum_{k=-\infty}^{\infty} e^k y_k, \text{ where } \quad y_k = \sum_{i=-\infty}^{k} x_i. \tag{2.15}$$

When $\|T|l \to l\| = c_0 < \infty$, the spaces $M^{\tau}_{l,X}$ and $\overline{M^{\tau}_{l,X}}$ have the same set of elements and the following inequalities hold:

$$\|f|\overline{M^{\tau}_{l,X}}\| \le \|f|M^{\tau}_{l,X}\| \le c_0 \|f|\overline{M^{\tau}_{l,X}}\|.$$

Note that the coincidence conditions do not contain restrictions on the space X and the sequence $\tau \in \Upsilon$. There is only a restriction on the sequence space l.

Everywhere below c, possibly with indices, we will denote constants whose exact value are not important.

3 Main results

The following two lemmas are not only key for calculating the Calderón – Lozanovskii construction for a couple of the Morrey spaces, but are also of independent interest.

Lemma 3.1. *Let a couple of ideal spaces X_i on $\mathbb{R}^n$, a couple of ideal spaces of sequences l_i, $(i = 0, 1)$, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $\overline{M^\tau_{l_i,X_i}}$ be constructed from the spaces X_i, l_i, $(i = 0, 1)$, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$. Let $\varphi \in C_{cv}$ be given, and the functions $\overline{\varphi}, \varphi^*$ be constructed as above. Suppose that $\varphi^* \in C^m_{cv}$.*

Then continuous embedding holds

$$\overline{M^\tau_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}} \subseteq \varphi(\overline{M^\tau_{l_0,X_0}}, \overline{M^\tau_{l_1,X_1}}). \tag{3.1}$$

Proof. Let $x \in \overline{M^\tau_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}}$ be given, for which holds the inequality $\|x|\overline{M^\tau_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}}\| < 1$. Then from Lemma 2.3 it follows that for every $i \in \mathbb{Z}$ the following presentation holds

$$x\chi(D_i)(t) = \varphi(x^0_i(t), x^1_i(t)) \quad \text{for a. a. } t \in D_i,$$

where for $a^0_i = \|x^0_i|X_0\|$, $a^1_i = \|x^1_i|X_1\|$ the following inequalities hold

$$\|\sum_{i=-\infty}^{\infty} e^i\overline{\varphi}(a^0_i, a^1_i)|\varphi^*(l_0,l_1)\| < 1. \tag{3.2}$$

Choose the sequences $b^0 = \{b^0_i\}^\infty_{-\infty}$, $b^1 = \{b^1_i\}^\infty_{-\infty}$ so that the following relations are satisfied

$$\|\sum_{i=-\infty}^{\infty} e^i b^0_i|l_0\| < 1, \quad \|\sum_{i=-\infty}^{\infty} e^i b^1_i|l_1\| < 1; \quad \frac{1}{2}\varphi^*(a^0_i, a^1_i) \le \overline{\varphi}(a^0_i, a^1_i) \le \varphi^*(b^0_i, b^1_i), \quad \forall i \in \mathbb{Z}. \tag{3.3}$$

The possibility of such a choice follows from the inequality (3.2).

Put

$$y^0(t) = \sum_{i=-\infty}^{\infty} x^0_i(\cdot)\frac{b^0_i}{a^0_i}\chi(D_i); \quad y^1(t) = \sum_{i=-\infty}^{\infty} x^1_i(\cdot)\frac{b^1_i}{a^1_i}\chi(D_i). \tag{3.4}$$

Then the following inequalities hold

$$\|y^0|M^\tau_{l_0,X_0}\| \le \|\sum_{i=-\infty}^{\infty} e^i b^0_i|l_0\| \le 1, \tag{3.5}$$

$$\|y^1|M^\tau_{l_1,X_1}\| \le \|\sum_{i=-\infty}^{\infty} e^i b^1_i|l_1\| \le 1. \tag{3.6}$$

From the conditions (3.3) - (3.4), $\varphi^* \in C^m_{cv}$ and Lemma 2.4 it follows that for all $i \in \mathbb{Z}$ for almost all $t \in D_i$ the following inequalities hold

$$\varphi(y^0(t), y^1(t))\chi(D_i) = \varphi(x^0(t)\frac{b^0_i}{a^0_i}, x^1(t)\frac{b^1_i}{a^1_i})\chi(D_i) \ge$$

$$\frac{1}{2c_{\varphi^*}}\varphi(x^0(t), x^1(t))\chi(D_i) \ge \frac{1}{2c_{\varphi^*}}x(t)\chi(D_i). \tag{3.7}$$

From (3.5) - (3.7) it follows that $x \in \varphi(\overline{M^\tau_{l_0,X_0}}, \overline{M^\tau_{l_1,X_1})}$ and $\|x|\varphi(\overline{M^\tau_{l_0,X_0}}, \overline{M^\tau_{l_1,X_1})})\| \le 2c_{\varphi^*}$.

□

Corollary 3.1. *Let a couple of ideal spaces X_i on $\mathbb{R}^n$, a couple of ideal spaces of sequences l_i, $(i = 0, 1)$, a set $U(0, 1) \subset \mathbb{R}^n$, for which $0 \in U(0, 1)$ and $\mu(U(0, 1)) \in (0, \infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $M^{\tau}_{l_i,X_i}$ be constructed from the spaces X_i, l_i, $(i = 0, 1)$, the set $U(0, 1)$ and the sequence $\tau \in \Upsilon$. Let the operator T be defined by the equality (2.15). Let $\varphi \in C_{cv}$ be given, and φ^* belong to C^m_{cv}.*

If the operator $T : l_i \to l_i$ is bounded, then the continuous embedding holds

$$M^{\tau}_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)} \subseteq \varphi(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}). \tag{3.8}$$

Proof. From the boundedness of the operator T in the spaces l_i, $(i = 0, 1)$ and Theorem B follows that the following couples of spaces coinside

$$\overline{M^{\tau}_{l_0,X_0}} = M^{\tau}_{l_0,X_0};\quad \overline{M^{\tau}_{l_1,X_1}} = M^{\tau}_{l_1,X_1};\ \varphi(\overline{M^{\tau}_{l_0,X_0}}, \overline{M^{\tau}_{l_1,X_1}}) = \varphi(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}) \tag{3.9}$$

and the norms in these couples of spaces are equivalent.

The interpolation theorem for positive operators (see, for example, [2]) and the boundedness of the operator T in the couple of spaces $T : l_i \to l_i$, (i = 0.1) implies that for any $\phi \in C_{cv}$ the operator T is bounded from $\phi(l_0, l_1)$ to $\phi(l_0, l_1)$ and the inequality holds $\|T|\phi(l_0, l_1) \to \phi(l_0, l_1)\| \leq max\{\|T|l_0 \to l_0\|, \|T|l_1 \to l_1\|\}$. Applying Theorem B again, we obtain that the following spaces coinside

$$\overline{M^{\tau}_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}} = M^{\tau}_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)} \tag{3.10}$$

and the norms in these spaces are equivalent.

From (3.9) - (3.10) follows (3.8).

□

Lemma 3.2. *Let a couple of ideal space X_i on $\mathbb{R}^n$, a couple of ideal space of sequences l_i, $(i = 0, 1)$, a set $U(0, 1) \subset \mathbb{R}^n$, for which $0 \in U(0, 1)$ and $\mu(U(0, 1)) \in (0, \infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $M^{\tau}_{l_i,X_i}$ be constructed from the spaces X_i, l_i, $(i = 0, 1)$, the set $U(0, 1)$ and the sequence $\tau \in \Upsilon$. Let $\varphi \in C_{cv}$ be given, and the functions $\overline{\varphi}, \varphi^*$ be constructed.*

Then the continuous embedding holds

$$\varphi(\overline{M^{\tau}_{l_0,X_0}}, \overline{M^{\tau}_{l_1,X_1}}) \subseteq \overline{M^{\tau}_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}}. \tag{3.11}$$

Proof. It suffices to prove (3.11) for non-negative functions. Let x a non-negative function from $\varphi(\overline{M^{\tau}_{l_0,X_0}}, \overline{M^{\tau}_{l_1,X_1}})$ be given and $\|x|\varphi(\overline{M^{\tau}_{l_0,X_0}}, \overline{M^{\tau}_{l_1,X_1}})\| < 1$. This assumption means that there are a pair of functions $y^0 \in X_0$, $y^1 \in X_1$ for which the following inequalities are satisfied

$$\|y^0|\overline{M^{\tau}_{l_0,X_0}}\| \leq 1;\qquad \|y^1|\overline{M^{\tau}_{l_1,X_1}}\| \leq 1. \tag{3.12}$$

$$x(t) \leq \varphi(y^0(t), y^1(t)), (\text{a. e. on } D). \tag{3.13}$$

Put

$$y^0_i(t) = y^0\chi(D_i)(t),\quad a^0_i = \|y^0_i|X_0\|;\quad y^1_i(t) = y^1\chi(D_i)(t),\quad a^1_i = \|y^1_i|X_0\|. \tag{3.14}$$

Then inequalities (3.12) can be written as

$$\|\sum_{-\infty}^{\infty} e^i a^0_i|l_0\| \leq 1;\quad \|\sum_{-\infty}^{\infty} e^i a^1_i|l_1\| \leq 1. \tag{3.15}$$

From (3.13) – (3.14) it follows that for every $i \in \mathbb{Z}$ the following inequalities are satisfied

$$x(t)\chi(D_i) \leq \varphi(y^0_i(t), y^1_i(t)) \text{ a. e. on } D_i, \tag{3.16}$$

$$\|x\chi(D_i)|\{\varphi(X_0, X_1), \psi_\infty\}\| \leq \overline{\varphi}(a^0_i, a^1_i) \leq \varphi^*(a^0_i, a^1_i). \tag{3.17}$$

Embedding of spaces (3.11) follows from (3.15) - (3.17). □

Using Lemma 3.2 by analogy with proof of Corollary 3.1 we can prove

Corollary 3.2. *Let a couple of ideal spaces X_i on $\mathbb{R}^n$, a couple of ideal spaces of sequences l_i, $(i = 0,1)$, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $M^{\tau}_{l_i,X_i}$ be constructed from the spaces X_i, l_i, $(i = 0,1)$, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$. Let the operator T be defined by the equality (2.15). Let $\varphi \in C_{cv}$ be given, and the functions $\overline{\varphi}, \varphi^*$ be constructed.*

If the operator $T : l_i \to l_i$, $(i = 0,1)$ is bounded, then the continuous embedding holds

$$\varphi(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}) \subseteq M^{\tau}_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}.$$

Combining Lemmas 3.1 and 3.2, we can compute the Calderón – Lozanovskii construction on a couple of approximation Morrey spaces.

Theorem 3.1. *Let a couple of ideal spaces X_i on $\mathbb{R}^n$, a couple of ideal spaces of sequences l_i, $(i = 0,1)$, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $\overline{M^{\tau}_{l_i,X_i}}$ be constructed from the spaces X_i, l_i, $(i = 0,1)$, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$. Let $\varphi \in C_{cv}$ be given, and the functions $\overline{\varphi}, \varphi^*$ be constructed. Suppose that φ^* belong C^m_{cv}.*

Then the following spaces coincide

$$\varphi(\overline{M^{\tau}_{l_0,X_0}}, \overline{M^{\tau}_{l_1,X_1}}) = \overline{M^{\tau}_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}}.$$

and the norms in these spaces are equivalent.

By analogy, using Corollaries 3.1 and 3.2, we can compute the Calderón – Lozanovskii construction on a couple of local Morrey spaces.

Theorem 3.2. *Let a couple of ideal spaces X_i on $\mathbb{R}^n$, a couple of ideal spaces of sequences l_i, $(i = 0,1)$, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $M^{\tau}_{l_i,X_i}$ be constructed from the spaces X_i, l_i, $(i = 0,1)$, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$. Let the operator T be defined by equality (2.15). Let $\varphi \in C_{cv}$ be given, and the functions $\overline{\varphi}, \varphi^*$ be constructed. Suppose that φ^* belong C^m_{cv}.*

If the operator $T : l_i \to l_i$ is bounded, then the following spaces coincide

$$\varphi(M^{\tau}_{l_0,X_0}, M^{\tau}_{l_1,X_1}) = M^{\tau}_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}.$$

and the norms in these spaces are equivalent.

To obtain interpolation theorems for the Calderón – Lozanovskii construction constructed over a couple of local Morrey spaces, we need to study some geometric properties of the spaces under consideration.

Recall that a ideal space of functions $X \in S(\mu,\Omega)$ has an absolutely continuous norm (see, for example, [15], [16]) if for every $x \in X$ the following two conditions are satisfied:

$$\lim_{\delta\to 0} \sup_{\{D;\mu(D)\le\delta\}} \|x\chi(D)|X\| = 0, \quad \lim_{R\to\infty} \|x\chi(\Omega \setminus B(R,0)|X\| = 0.$$

For the spaces of two-sided sequences the definition is as follows. A discrete ideal space l has an absolutely continuous norm if for each $x \in l$ the following two conditions are satisfied

$$\lim_{k\to\infty} \|\Sigma^{-k}_{-\infty} e^i x_i|l\| = 0, \quad \lim_{k\to\infty} \|\Sigma^{\infty}_{k} e^i x_i|l\| = 0.$$

Recall the definition of the Fatou property. It is said (see, for example, [15], [16], [17]) that an ideal space X has the Fatou property if from $0 \le x_n \uparrow x$; $x_n \in X$ and $\sup_n \|x_n|X\| < \infty$ it follows that $x \in X$ and $\|x|X\| = \sup_n \|x_n|X\|$.

Lemma 3.3. *Let a ideal space X on $\mathbb{R}^n$, a ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $\overline{M^{\tau}_{l,X}}$ be constructed from the spaces X, l, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$.*

The space $\overline{M^{\tau}_{l,X}}$ has an absolutely continuous norm, if and only if the ideal space of functions X and the ideal space of sequences l have absolutely continuous norm.

Proof. Let $x \in \overline{M^{\tau}_{l,X}}$ be given, for which

$$\|x|\overline{M^{\tau}_{l,X}}\| = \|\Sigma^{\infty}_{-\infty}\|x\chi(D_i)|X\|e^i|l\| = 1.$$

Let $V \subset \Omega$. Fix $k \in \mathbb{N}$ and define

$$s_1 = \|\Sigma^{-k}_{-\infty}\|x\chi(D_i)\chi(V)|X\|e^i|l\|, \quad s_2 = \|\Sigma^{k}_{-k}\|x\chi(D_i)\chi(V)|X\|e^i|l\|,$$
$$s_3 = \|\Sigma^{\infty}_{k}\|x\chi(D_i)\chi(V)|X\|e^i|l\|.$$

Then the following inequalities are true

$$max\{s_1, s_2, s_3\} \leq \|x\chi(V)|\overline{M^{\tau}_{l,X}}\| \leq s_1 + s_2 + s_3. \tag{3.18}$$

Let the ideal spaces X and l have absolutely continuous norm. Since the space l has absolutely continuous norm, then $\lim_{k\to\infty} s_1 + s_3 = 0$. Since the space X has absolutely continuous norm, then for each fixed $k \in \mathbb{N}$ $\lim_{\mu(V)\to 0} s_2 = 0$. This implies the sufficiency of the conditions of the lemma.

Let the ideal spaces $\overline{M^{\tau}_{l,X}}$ have absolutely continuous norm. The proof that both spaces X and l have absolutely continuous norm is analogous to the proof of sufficiency, one only needs to use the left inequality in (3.18). □

Lemma 3.4. *Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $\overline{M^{\tau}_{l,X}}$ be constructed from the spaces X, l, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$.*

If the spaces X and l have the Fatou property, then the spaces $\overline{M^{\tau}_{l,X}}$ also have the Fatou property.

Proof. Let $x_n \in \overline{M^{\tau}_{l,X}}$, $\|x_n|\overline{M^{\tau}_{l,X}}\| \leq 1$ and $x_n \uparrow x$ be given. Then for any $i \in \mathbb{Z}$ the relation $x_n\chi(D_i) \uparrow x\chi(D_i)$ holds. Since X has the Fatou property, then $\|x_n\chi(D_i)|X\| \uparrow \|x\chi(D_i)|X\|$. Therefore, the condition

$$\Sigma^{\infty}_{-\infty}\|x_n\chi(D_i)|X\|e^i \uparrow \Sigma^{\infty}_{-\infty}\|x\chi(D_i)|X\|e^i.$$

Since l has the Fatou property, then

$$\lim_{n\to\infty} \|\Sigma^{\infty}_{-\infty}\|x_n\chi(D_i)|X\|e^i|l\| = \|\Sigma^{\infty}_{-\infty}\|x\chi(D_i)|X\|e^i|l\|.$$

From this follows the lemma. □

The following theorem is not a very general fact for the Calderón – Lozanovskii construction on a couple of Morrey spaces. The question of when the space $\varphi(X_0, X_1)$ has absolutely continuous norm or has the Fatou property depends on the properties of the couple of ideal spaces (X_0, X_1) and the function φ. This is discussed in more detail in [4]. Directly from Lemmas 3.3, 3.4 and [4] we obtain the following theorem.

Theorem 3.3. *Let a couple of ideal space X_i on $\mathbb{R}^n$, a couple of ideal space of sequences l_i, $(i = 0,1)$, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the spaces $\overline{M^{\tau}_{l_i,X_i}}$ be constructed from the spaces X_i, l_i, $(i = 0,1)$, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$. Let $\varphi \in C_{cv}$ be given, and the functions $\overline{\varphi}, \varphi^*$ be constructed.*

If each of the spaces X_0, X_1, l_0, l_1 have absolutely continuous norms, then the space $M^{\tau}_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}$ also has an absolutely continuous norm.

If each of the spaces X_0, X_1, l_0, l_1 have the Fatou property, then the space $M^{\tau}_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}$ also has the Fatou property.

We apply Theorems 3.2 and 3.3 to obtain interpolation theorems. Namely, we write out the conditions for the coincidence of the Calderón – Lozanovskii construction on a couple of Morrey spaces with the value of the Gustavsson – Peetre – Ovchinnikov interpolation functors on a couple of Morrey spaces.

We recall (see [7], [16]) that a couple of normed spaces (A_0, A_1) is referred to as an interpolation couple if both spaces are embedded in a separable topological linear space V.

Let $\varphi \in C_{cv}$ and an interpolation couple (A_0, A_1) be given. Denote by $(A_0, A_1)_\varphi$ the Gustavsson – Peetre – Ovchinnikov interpolation functor [7], [14], [22] calculated on the couple (A_0, A_1). It is well known [7], [14], [22] that if a couple of ideal Banach spaces (X_0, X_1) have absolutely continuous norms or have the Fatou property, then for any φ the spaces $\{\varphi(X_0, X_1), \psi\}$ and $(A_0, A_1)_\varphi$ coincide. Therefore the space $\{\varphi(X_0, X_1), \psi\}$ is an interpolation space.

From Theorems 3.2 and 3.3 we obtain the following interpolation theorem.

Theorem 3.4. *Let a couple of ideal space X_i on $\mathbb{R}^n$, a couple of ideal space of sequences l_i, $(i = 0, 1)$, a set $U(0, 1) \subset \mathbb{R}^n$, for which $0 \in U(0, 1)$ and $\mu(U(0, 1)) \in (0, \infty)$, and a sequence $\tau \in \Upsilon$ be given. Let all spaces X_0, X_1, l_0, l_1 have absolutely continuous norms or have the Fatou property. Let the spaces $M^\tau_{l_i,X_i}$ be constructed from the spaces X_i, l_i, $(i = 0, 1)$, the set $U(0, 1)$ and the sequence $\tau \in \Upsilon$. Let an interpolation couple (A_0, A_1) be given. Let $\varphi \in C_{cv}$ be given, and the functions $\overline{\varphi}, \varphi^*$ be constructed. Suppose that φ^* belongs to C^m_{cv}.*

Let the operator $T : l_i \to l_i$, $(i = 0, 1)$, defined by equality (2.15), be bounded.

If a linear operator S is bounded as an operator $S : A_i \to M^\tau_{l_i,X_i}$, (i = 0,1), then S is bounded from $(A_0, A_1)_\varphi$ to $M^\tau_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}$.

If a linear operator P is bounded as an operator $P : M^\tau_{l_i,X_i} \to A_i$, (i = 0,1), then S is bounded from $M^\tau_{\varphi^*(l_0,l_1),\varphi(X_0,X_1)}$ to $(A_0, A_1)_\varphi$.

Remark 1. In this article we considered the Morrey spaces of functions defined on $\mathbb{R}^n$. If we consider the Morrey spaces of functions on a subset $\Omega \subset \mathbb{R}^n$, $(0 \in \Omega)$, then in Definitions 3 - 5 is necessary to replace $U(0, \tau)$ by $U(0, \tau) \cap \Omega$. All results will remain true.

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