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A NONLOCAL MULTIPOINT PROBLEM FOR A SYSTEM OF FOURTH-ORDER PARTIAL DIFFERENTIAL EQUATIONS

A.T. Assanova, Z.S. Tokmurzin

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Key words: system of fourth-order partial differential equations, nonlocal multipoint problem, family of multipoint boundary value problems with integral conditions, system of ordinary integro-differential equations, solvability.

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Abstract. A nonlocal multipoint problem for a system of fourth-order partial differential equations is investigated. Based on the results obtained for a family of multipoint boundary value problems with an integral condition for a system of ordinary integro-differential equations, conditions for the existence of classical solutions of a nonlocal multipoint problem for a fourth-order partial differential equation system are established.

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1 Introduction

In this paper, in the domain $\bar{\Omega} = [0,T] \times [0,\omega]$ we consider the following nonlocal multipoint problem for a system of fourth-order partial differential equations

$$\frac{\partial^4 u}{\partial x^3 \partial t} = \sum_{i=1}^{3} \left\{ A_i(t,x) \frac{\partial^{4-i} u}{\partial x^{4-i}} + B_i(t,x) \frac{\partial^{4-i} u}{\partial x^{3-i} \partial t} \right\} + C(t,x)u + f(t,x), \tag{1.1}$$

$$\sum_{j=0}^{m} \sum_{i=0}^{3} M_{i,j}(x) \frac{\partial^i u(t_j,x)}{\partial x^i} = \varphi(x), \qquad x \in [0,\omega], \tag{1.2}$$

$$u(t,0) = \psi_0(t), \qquad \frac{\partial u(t,x)}{\partial x}\Big|_{x=0} = \psi_1(t), \qquad \frac{\partial^2 u(t,x)}{\partial x^2}\Big|_{x=0} = \psi_2(t), \qquad t \in [0,T], \tag{1.3}$$

where $u(t,x) = col(u_1(t,x), u_2(t,x), ..., u_n(t,x))$ is the unknown vector function, the $(n \times n)$ matrices $A_i(t,x)$, $B_i(t,x)$, $i = 1,2,3$, $C(t,x)$, and the n vector function $f(t,x)$ are continuous on $\bar{\Omega}$, the $(n \times n)$ matrices $M_{i,j}(x)$, and the n vector function $\varphi(x)$ are continuous on $[0,\omega]$, $i = \overline{0,3}$, $j = \overline{0,m}$, the n vector functions $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t)$ are continuously differentiable on $[0,T]$.

Let $C(\bar{\Omega}, \mathbb{R}^n)$ $(C(\Omega, \mathbb{R}^n))$ be the space of all continuous on $\bar{\Omega}$ (Ω) vector functions $u(t,x)$ with the norm

$$||u||_0 = \max_{(t,x)\in\bar{\Omega}} ||u(t,x)|| \ (||u||_0 = \sup_{(t,x)\in\Omega} ||u(t,x)||), \quad ||u(t,x)|| = \max_{i=\overline{1,n}} |u_i(t,x)|;$$

$C([0,\omega], \mathbb{R}^n)$ be the space of all continuous on $[0,\omega]$ vector functions $\varphi(x)$ with the norm

$$||\varphi||_0 = \max_{x\in[0,\omega]} ||\varphi(x)||;$$

$C^1([0,T],\mathbb{R}^n)$ be the space of all continuously differentiable on $[0,T]$ vector functions $\psi(t)$ with the norm

$$||\psi||_1 = \max\Big(\max_{t\in[0,T]}||\psi(t)||, \max_{t\in[0,T]}||\dot{\psi}(t)||\Big).$$

A function $u(t,x)\in C(\bar{\Omega},\mathbb{R}^n)$ having partial derivatives $\frac{\partial^{s+p}u(t,x)}{\partial x^p\partial t^s}\in C(\bar{\Omega},\mathbb{R}^n)$, $s=\overline{0,1}$, $p=\overline{0,3}$, $s+p<4$, $\frac{\partial^4 u(t,x)}{\partial x^3\partial t}\in C(\Omega,\mathbb{R}^n)$ is called a *classical solution* to problem (1.1)–(1.3) if it satisfies system (1.1) for all $(t,x)\in\Omega$ and boundary conditions (1.2), (1.3).

In recent decades, various problems for fourth-order partial differential equations are of great interest to specialists. Many problems for equation (1.1) arise in the study liquid filtration in fissured media, moisture transfer in soil, impulse radial wave propagation, various biological processes, and in the inverse problem theory [9, 11-18]. Note that system of equations (1.1) is a generalization of many model equations describing physical processes, for example, the generalized moisture transfer equation, heat transfer equation, telegraph equation, string vibration equation, etc. [13, 16, 21]. As noted in [16] the solutions of the generalized Hallaire moisture equation

$$\frac{\partial^3 u}{\partial x^2\partial t} = a_1(t,x)\frac{\partial^2 u}{\partial x^2} + a_2(t,x)\frac{\partial^2 u}{\partial x\partial t} + a_3(t,x)\frac{\partial u}{\partial x} + a_4(t,x)\frac{\partial u}{\partial t} + a_5(t,x)u + g(t,x)$$

can be smooth solutions of equation (1.1) when choosing the appropriate coefficients.

In [2], a linear multipoint boundary value problem for a system of hyperbolic equations was investigated by the method of introducing a functional parameter [3-5]. The necessary and sufficient conditions for the well-posedness of a linear multipoint boundary value problem for a system of hyperbolic equations with a mixed derivative were established in terms of the initial data. This method and these results were applied to a multipoint boundary value problem for a system of quasilinear hyperbolic equations with a mixed derivative in [6].

In this article, we study the problem of existence of a classical solution of a nonlocal multipoint problem for a system of fourth-order partial differential equations (1.1)–(1.3) and methods for constructing their approximate solutions. The results and methods [2, 6] are extended to the nonlocal multipoint problem for a system of fourth-order partial differential equations in two variables. Introducing a new unknown function, we reduce the original problem (1.1)–(1.3) to an equivalent problem for a system of ordinary integro-differential equations of the first order containing a parameter. We establish sufficient conditions for the unique solvability of nonlocal multipoint problem (1.1)–(1.3) in terms of the unique solvability of a family of multipoint boundary value problems with integral conditions of a system of ordinary first-order integro-differential equations. The results can be used in the numerical methods of solving applied problems.

2 Equivalent problem and its solvability

In this section, we introduce a new unknown function $v(t,x)=\dfrac{\partial^3 u(t,x)}{\partial x^3}$.
Taking into account conditions (1.3), we have:

$$\frac{\partial^2 u(t,x)}{\partial x^2} = \psi_2(t) + \int_0^x v(t,\xi)d\xi, \tag{2.1}$$

$$\frac{\partial u(t,x)}{\partial x} = \psi_1(t) + \psi_2(t)x + \int_0^x (x-\xi)v(t,\xi)d\xi, \tag{2.2}$$

$$u(t,x) = \psi_0(t) + \psi_1(t)x + \psi_2(t)\frac{x^2}{2!} + \int_0^x \frac{(x-\xi)^2}{2!}v(t,\xi)d\xi. \tag{2.3}$$

From (2.1)–(2.3) we can find their partial derivatives in t:

$$\frac{\partial^3 u(t,x)}{\partial x^2 \partial t} = \dot{\psi}_2(t) + \int_0^x \frac{\partial v(t,\xi)}{\partial t} d\xi, \tag{2.4}$$

$$\frac{\partial^2 u(t,x)}{\partial x \partial t} = \dot{\psi}_1(t) + \dot{\psi}_2(t)x + \int_0^x (x-\xi)\frac{\partial v(t,\xi)}{\partial t} d\xi, \tag{2.5}$$

$$\frac{\partial u(t,x)}{\partial t} = \dot{\psi}_0(t) + \dot{\psi}_1(t)x + \dot{\psi}_2(t)\frac{x^2}{2!} + \int_0^x \frac{(x-\xi)^2}{2!}\frac{\partial v(t,\xi)}{\partial t} d\xi. \tag{2.6}$$

Using representations (2.1)–(2.6), we reduce problem (1.1)–(1.3) to the following equivalent problem:

$$\frac{\partial v}{\partial t} = A_1(t,x)v + \int_0^x \Big\{ K_1(t,x,\xi)\frac{\partial v(t,\xi)}{\partial t} + K_2(t,x,\xi)v(t,\xi)\Big\} d\xi + F(t,x), \tag{2.7}$$

$$\sum_{j=0}^m M_{3,j}(x)v(t_j,x) + \sum_{j=0}^m \int_0^x L_j(x,\xi)v(t_j,\xi)d\xi = \Phi(x), \tag{2.8}$$

where

$$K_1(t,x,\xi) = B_1(t,x) + B_2(t,x)(x-\xi) + B_3(t,x)\frac{(x-\xi)^2}{2!},$$

$$K_2(t,x,\xi) = A_2(t,x) + A_3(t,x)(x-\xi) + C(t,x)\frac{(x-\xi)^2}{2!},$$

$$F(t,x) = A_2(t,x)\psi_2(t) + A_3(t,x)[\psi_1(t) + \psi_2(t)x] + C(t,x)\big[\psi_0(t) + \psi_1(t)x + \psi_2(t)\frac{x^2}{2!}\big]$$

$$+B_1(t,x)\dot{\psi}_2(t) + B_2(t,x)[\dot{\psi}_1(t) + \dot{\psi}_2(t)x] + B_3(t,x)\big[\dot{\psi}_0(t) + \dot{\psi}_1(t)x + \dot{\psi}_2(t)\frac{x^2}{2!}\big] + f(t,x),$$

$$L_j(x,\xi) = M_{2,j}(x) + M_{1,j}(x)(x-\xi) + M_{0,j}(x)\frac{(x-\xi)^2}{2!},$$

$$\Phi(x) = \varphi(x) - \sum_{j=0}^m \Big\{ M_{2,j}(x)\psi_2(t_j) + M_{1,j}(x)[\psi_1(t_j) + \psi_2(t_j)x] + M_{0,j}(x)\big[\psi_0(t_j) + \psi_1(t_j)x + \psi_2(t_j)\frac{x^2}{2!}\big]\Big\}.$$

A continuous function $v : \bar{\Omega} \to \mathbb{R}^n$ having a continuous derivative with respect to t on Ω is called a solution to the family of multipoint boundary value problems for ordinary integro-differential equations (2.7), (2.8) if it satisfies system (2.7) and condition (2.8) for all $(t,x) \in \Omega$ and $x \in [0,\omega]$, respectively.

Let $u^*(t,x)$ be a classical solution to problem (1.1)–(1.3). Then the function $v^*(t,x)$ defined by equality $v^*(t,x) = \frac{\partial^3 u^*(t,x)}{\partial x^3}$, is a solution to problem (2.7), (2.8). Conversely, if a function $\widetilde{v}(t,x)$ is a solution to problem (2.7), (2.8), then $\widetilde{u}(t,x)$ defined by equality

$$\widetilde{u}(t,x) = \psi_0(t) + \psi_1(t)x + \psi_2(t)\frac{x^2}{2!} + \int_0^x \frac{(x-\xi)^2}{2!}\widetilde{v}(t,\xi)d\xi$$

is a classical solution to problem (1.1)–(1.3).

Problem (2.7), (2.8) is a family of multipoint boundary value problems with integral conditions for the system of ordinary first-order integro-differential equations. Problem (2.7), (2.8) can be interpreted as a multipoint boundary value problem for a system of parametrically loaded differential equations [16]. The variable x plays the role of a parameter and changes on $[0,\omega]$.

For a fixed $x \in [0,\omega]$ problem (2.7), (2.8) is a linear multipoint boundary value problem with integral condition for a system of ordinary integro-differential equations. Suppose that the x takes values in the interval $[0,\omega]$, then we obtain a family of multipoint boundary value problems with an integral condition for a system of ordinary integro-differential equations. System (2.7) depends on the variable x, the integrals of the desired function, and its derivative with respect to this variable.

Various boundary value problems for the system of ordinary integro-differential equations (2.7) have been studied by numerous authors (see [1, 8, 10, 19, 20] and their bibliography). Having found the function $v(t,x)$ from problem (2.7), (2.8), we determine the function $u(t,x)$ from integral relation (2.3), which is a classical solution to problem (1.1)–(1.3).

Consider a family of multipoint boundary value problems with integral conditions for system of ordinary integro-differential equations (2.7), (2.8). The following theorem provides conditions for the unique solvability of problem (2.7), (2.8) in terms of the fundamental matrix of the system $\frac{\partial v}{\partial t} = A_1(t,x)v$.

In its proof the following particular case of the Grönwall -Bellman inequality will be used.
Let $\alpha,\beta\in\mathbb{R}$, $\beta\geq 0$ and w be a continuous function defined on $[0,\omega]$.

If w satisfies the integral inequality

$$w(x)\leq\alpha+\beta\int_0^x w(s)ds,\qquad \forall x\in[0,\omega],$$

then

$$w(x)\leq\alpha e^{\beta x},\qquad \forall x\in[0,\omega].$$

Theorem 1. *Problem (2.7), (2.8) is uniquely solvable and for its solution $v^*(t,x)$ we have the estimate*

$$\max_{t\in[0,T]}||v^*(t,x)||\leq\widetilde{C}\max\Big\{\max_{t\in[0,T]}||F(t,x)||,||\Phi(x)||\Big\}$$

for all $x\in[0,\omega]$, for some $\widetilde{C}>0$ independent of x, v^*, F and Φ, if the $n\times n$ matrix $Q(x)=\sum\limits_{j=0}^{m}M_{3,j}(x)X(t_j,x)$ is invertible for every $x\in[0,\omega]$, where X is a solution to the Cauchy problem

$$\frac{\partial X}{\partial t}=A_1(t,x)X,\qquad X(0,x)=I,$$

and I is the identity matrix.

Proof. To find a solution to problem (2.7), (2.8) we use an iterative method.
Suppose that $\frac{\partial v(t,\xi)}{\partial t}=v(t,\xi)=0$ for all $(t,\xi)\in\bar{\Omega}$ in integral terms of the right-hand side of system (2.7) and condition (2.8). Then, we get the following problem:

$$\frac{\partial v}{\partial t}=A_1(t,x)v+F(t,x),\tag{2.9}$$

$$\sum_{j=0}^{m}M_{3,j}(x)v(t_j,x)=\Phi(x),\tag{2.10}$$

Let X be a solution to the Cauchy problem

$$\frac{\partial X}{\partial t}=A_1(t,x)X,\qquad X(0,x)=I.$$

By the Cauchy formula [7, p. 48], the vector function

$$v(t,x)=X(t,x)c(x)+X(t,x)\int_0^t X^{-1}(\tau,x)F(\tau,x)d\tau\tag{2.11}$$

is a solution to system (2.9) for each $c(x) \in C([0,\omega],\mathbb{R}^n)$. Conversely, for each solution of this system there exists $c(x) \in C([0,\omega],\mathbb{R}^n)$ such that representation (2.11) holds.

Substituting representation (2.11) into (2.10), we have:

$$\sum_{j=0}^{m} M_{3,j}(x)X(t_j,x)c(x) + \sum_{j=0}^{m} M_{3,j}(x)X(t_j,x)\int_0^{t_j} X^{-1}(\tau,x)F(\tau,x)d\tau = \Phi(x),$$

where $x \in [0,\omega]$. This implies

$$Q(x)c(x) = \Phi(x) - \sum_{j=0}^{m} M_{3,j}(x)X(t_j,x)\int_0^{t_j} X^{-1}(\tau,x)F(\tau,x)d\tau, \quad x \in [0,\omega]. \tag{2.12}$$

If the $n \times n$ matrix $Q(x)$ is invertible for all $x \in [0,\omega]$, then the system of functional equations (2.12) has a unique solution

$$c^{(0)}(x) = Q^{-1}(x)\Big\{\Phi(x) - \sum_{j=0}^{m} M_{3,j}(x)X(t_j,x)\int_0^{t_j} X^{-1}(\tau,x)F(\tau,x)d\tau\Big\}, \quad x \in [0,\omega]. \tag{2.13}$$

Replacing $c(x)$ by $c^*(x)$ in (2.11), we obtain the following representation of the unique solution to the family of problems (2.9), (2.10)

$$v^{(0)}(t,x) = X(t,x)Q^{-1}(x)\Big\{\Phi(x) - \sum_{j=0}^{m} M_{3,j}(x)X(t_j,x)\int_0^{t_j} X^{-1}(\tau,x)F(\tau,x)d\tau\Big\}$$

$$+X(t,x)\int_0^t X^{-1}(\tau,x)F(\tau,x)d\tau. \tag{2.14}$$

The solution $v^{(0)}$ satisfies the following estimate

$$\max_{t\in[0,T]} ||v^{(0)}(t,x)|| \le C_0 \max\Big(\max_{t\in[0,T]} ||F(t,x)||, ||\Phi(x)||\Big), \tag{2.15}$$

where the constant C_0 does not depend on F, Φ and $x \in [0,\omega]$.

The following estimate is also valid:

$$\max\Big(\max_{t\in[0,T]}\Big|\Big|\frac{\partial v^{(0)}(t,x)}{\partial t}\Big|\Big|, \max_{t\in[0,T]} ||v^{(0)}(t,x)||\Big)$$

$$\le \max(\alpha_1 C_0 + 1, C_0)\max\Big(\max_{t\in[0,T]} ||F(t,x)||, ||\Phi(x)||\Big), \tag{2.16}$$

where $\alpha_1 = \max\limits_{(t,x)\in\bar{\Omega}} ||A_1(t,x)||$.

Therefore, provided that the matrix $Q(x)$ is invertible for all $x \in [0,\omega]$, the family of problems (2.9), (2.10) is uniquely solvable and for its solution $v^{(0)}$ the estimate (2.14) holds, i.e. problem (2.9), (2.10) is well-posed.

Further, we assume that $\frac{\partial v(t,\xi)}{\partial t} = \frac{\partial v^{(0)}(t,\xi)}{\partial t}$, $v(t,\xi) = v^{(0)}(t,\xi)$ for all $(t,\xi) \in \bar{\Omega}$ in integral terms of the right-hand side of system (2.7) and condition (2.8). We have:

$$\frac{\partial v}{\partial t} = A_1(t,x)v + F^{(0)}(t,x), \tag{2.17}$$

$$\sum_{j=0}^{m} M_{3,j}(x)v(t_j,x) = \Phi^{(0)}(x), \tag{2.18}$$

where $F^{(0)}(t,x) = \int_0^x \Big\{K_1(t,x,\xi)\dfrac{\partial v^{(0)}(t,\xi)}{\partial t} + K_2(t,x,\xi)v^{(0)}(t,\xi)\Big\}d\xi + F(t,x)$,

$$\Phi^{(0)}(x) = \Phi(x) - \sum_{j=0}^{m}\int_0^x L_j(x,\xi)v^{(0)}(t_j,\xi)d\xi.$$

From the family of multipoint problems (2.17), (2.18) we find $v^{(1)}(t,x) \in C(\bar{\Omega},\mathbb{R}^n)$ and its derivative $\frac{\partial v^{(1)}(t,x)}{\partial t} \in C(\bar{\Omega},\mathbb{R}^n)$. Similarly to (2.12), we obtain

$$Q(x)c(x) = \Phi^{(0)}(x) - \sum_{j=0}^{m} M_{3,j}(x)X(t_j,x)\int_0^{t_j} X^{-1}(\tau,x)F^{(0)}(\tau,x)d\tau, \quad x \in [0,\omega]. \tag{2.19}$$

If the $n\times n$ matrix $Q(x)$ is invertible for all $x \in [0,\omega]$, then the system of functional equations (2.19) has a unique solution

$$c^{(1)}(x) = Q^{-1}(x)\Big\{\Phi^{(0)}(x) - \sum_{j=0}^{m} M_{3,j}(x)X(t_j,x)\int_0^{t_j} X^{-1}(\tau,x)F^{(0)}(\tau,x)d\tau\Big\}, \quad x \in [0,\omega]. \tag{2.20}$$

Thus, we obtain the following representation of the unique solution to the family of problems (2.17), (2.18)

$$v^{(1)}(t,x) = X(t,x)Q^{-1}(x)\Big\{\Phi^{(0)}(x) - \sum_{j=0}^{m} M_{3,j}(x)X(t_j,x)\int_0^{t_j} X^{-1}(\tau,x)F^{(0)}(\tau,x)d\tau\Big\}$$

$$+X(t,x)\int_0^t X^{-1}(\tau,x)F^{(0)}(\tau,x)d\tau. \tag{2.21}$$

Solution $v^{(1)}(t,x)$ and its derivative $\frac{\partial v^{(1)}(t,x)}{\partial t}$ satisfy the following estimates:

$$\max_{t\in[0,T]}||v^{(1)}(t,x)|| \le C_0 \max\Big(\max_{t\in[0,T]}||F^{(0)}(t,x)||, ||\Phi^{(0)}(x)||\Big), \tag{2.22}$$

$$\max\Big(\max_{t\in[0,T]}\Big|\Big|\frac{\partial v^{(1)}(t,x)}{\partial t}\Big|\Big|, \max_{t\in[0,T]}||v^{(1)}(t,x)||\Big)$$

$$\le \max(\alpha_1 C_0 + 1, C_0)\max\Big(\max_{t\in[0,T]}||F^{(0)}(t,x)||, ||\Phi^{(0)}(x)||\Big). \tag{2.23}$$

From this, taking into account the representations $F^{(0)}(t,x)$ and $\Phi^{(0)}(x)$, and estimate (2.16), we obtain:

$$\max_{t\in[0,T]}||v^{(1)}(t,x)|| \le C_0\Big\{C_1\int_0^x \max\Big(\max_{t\in[0,T]}||F(t,\xi)||, ||\Phi(\xi)||\Big)d\xi$$

$$+\max\Big(\max_{t\in[0,T]}||F(t,x)||, ||\Phi(x)||\Big)\Big\}, \tag{2.24}$$

$$\max\Big(\max_{t\in[0,T]}\Big|\Big|\frac{\partial v^{(1)}(t,x)}{\partial t}\Big|\Big|, \max_{t\in[0,T]}||v^{(1)}(t,x)||\Big)$$

$$\le \max(\alpha_1 C_0 + 1, C_0)\Big\{C_1\int_0^x \max\Big(\max_{t\in[0,T]}||F(t,\xi)||, ||\Phi(\xi)||\Big)d\xi$$

$$+\max\Big(\max_{t\in[0,T]}||F(t,x)||,||\Phi(x)||\Big)\Big\}, \tag{2.25}$$

where $C_1=\max\Big([k_1+k_2]\max(\alpha_1C_0+1,C_0),\sum\limits_{j=0}^{m}l_jC_0\Big)$,

$k_i=\max\limits_{(t,x,\xi)\in\bar{\Omega}\times[0,\omega]}||K_i(t,x,\xi)||,\ \ i=1,2,\ \ l_j=\max\limits_{x\in[0,\omega]}||L_j(x)||,\ \ j=\overline{0,m}.$

Continuing this process, at the kth step, $k=1,2,...$, to find the function $v^{(k)}(t,x)$, we get the following problem

$$\frac{\partial v}{\partial t}=A_1(t,x)v+F^{(k-1)}(t,x), \tag{2.26}$$

$$\sum_{j=0}^{m}M_{3,j}(x)v(t_j,x)=\Phi^{(k-1)}(x), \tag{2.27}$$

where $F^{(k-1)}(t,x)=\displaystyle\int_0^x\Big\{K_1(t,x,\xi)\frac{\partial v^{(k-1)}(t,\xi)}{\partial t}+K_2(t,x,\xi)v^{(k-1)}(t,\xi)\Big\}d\xi+F(t,x)$,

$\Phi^{(k-1)}(x)=\Phi(x)-\displaystyle\sum_{j=0}^{m}\int_0^x L_j(x,\xi)v^{(k-1)}(t_j,\xi)d\xi.$

The unique solution to the family of problems (2.26), (2.27) has the following form:

$$v^{(k)}(t,x)=X(t,x)Q^{-1}(x)\Phi^{(k-1)}(x)$$

$$-X(t,x)Q^{-1}(x)\sum_{j=0}^{m}M_{3,j}(x)X(t_j,x)\int_0^{t_j}X^{-1}(\tau,x)F^{(k-1)}(\tau,x)d\tau$$

$$+X(t,x)\int_0^{t}X^{-1}(\tau,x)F^{(k-1)}(\tau,x)d\tau. \tag{2.28}$$

For $v^{(k)}(t,x)$ the following estimates hold:

$$\max_{t\in[0,T]}||v^{(k)}(t,x)||\le C_0\max\Big(\max_{t\in[0,T]}||F^{(k-1)}(t,x)||,||\Phi^{(k-1)}(x)||\Big), \tag{2.29}$$

$$\max\Big(\max_{t\in[0,T]}\Big|\Big|\frac{\partial v^{(k)}(t,x)}{\partial t}\Big|\Big|,\max_{t\in[0,T]}||v^{(k)}(t,x)||\Big)$$

$$\le\max(\alpha_1C_0+1,C_0)\max\Big(\max_{t\in[0,T]}||F^{(k-1)}(t,x)||,||\Phi^{(k-1)}(x)||\Big). \tag{2.30}$$

Hence, we obtain:

$$\max_{t\in[0,T]}||v^{(k)}(t,x)||\le C_0\cdot\Big\{C_1\int_0^x\max\Big(\max_{t\in[0,T]}\Big|\Big|\frac{\partial v^{(k-1)}(t,\xi)}{\partial t}\Big|\Big|,\max_{t\in[0,T]}||v^{(k-1)}(t,\xi)||\Big)d\xi$$

$$+\max\Big(\max_{t\in[0,T]}||F(t,x)||,||\Phi(x)||\Big)\Big\}, \tag{2.31}$$

$$\max\Big(\max_{t\in[0,T]}\Big|\Big|\frac{\partial v^{(k)}(t,x)}{\partial t}\Big|\Big|,\max_{t\in[0,T]}||v^{(k)}(t,x)||\Big)$$

$$\le\max(\alpha_1C_0+1,C_0)\Big\{C_1\int_0^x\max\Big(\max_{t\in[0,T]}\Big|\Big|\frac{\partial v^{(k-1)}(t,\xi)}{\partial t}\Big|\Big|,\max_{t\in[0,T]}||v^{(k-1)}(t,\xi)||\Big)d\xi$$

$$+\max\Big(\max_{t\in[0,T]}||F(t,x)||,||\Phi(x)||\Big)\Big\}. \tag{2.32}$$

Consider the following problem:

$$\frac{\partial v}{\partial t}=A_1(t,x)v+F^{(k)}(t,x), \tag{2.33}$$

$$\sum_{j=0}^{m}M_{3,j}(x)v(t_j,x)=\Phi^{(k)}(x). \tag{2.34}$$

From problem (2.33), (2.34) we find the function $v^{(k+1)}(t,x)$ and its derivative $\frac{\partial v^{(k+1)}(t,x)}{\partial t}$ for all $(t,x)\in\bar{\Omega}$. Using the solutions $v^{(k)}(t,x)$ and $v^{(k+1)}(t,x)$ of problems (2.26), (2.27) and (2.33), (2.34), respectively, we compose the differences $\Delta z^{(k+1)}(t,x)=\frac{\partial v^{(k+1)}(t,x)}{\partial t}-\frac{\partial v^{(k)}(t,x)}{\partial t}$ and $\Delta v^{(k+1)}(t,x)=v^{(k+1)}(t,x)-v^{(k)}(t,x)$ for all $(t,x)\in\bar{\Omega}$. Then the function $\Delta v^{(k+1)}(t,x)$ is a solution to the following problem

$$\frac{\partial \Delta v}{\partial t}=A_1(t,x)\Delta v+F^{(k)}(t,x)-F^{(k-1)}(t,x), \tag{2.35}$$

$$\sum_{j=0}^{m}M_{3,j}(x)\Delta v(t_j,x)=\Phi^{(k)}(x)-\Phi^{(k-1)}(x). \tag{2.36}$$

Taking into account inequalities (2.29)–(2.32) and the representations of the functions $F^{(k)}(t,x)$, $\Phi^{(k)}(x)$ for $k=1,2,3,...$, we have the following estimates:

$$\max_{t\in[0,T]}||\Delta v^{(k+1)}(t,x)||\le C_0C_1\int_0^x\max\Big\{\max_{t\in[0,T]}||\Delta z^{(k)}(t,\xi)||,\max_{t\in[0,T]}||\Delta v^{(k)}(t,\xi)||\Big\}d\xi, \tag{2.37}$$

$$\max\Big\{\max_{t\in[0,T]}||\Delta z^{(k+1)}(t,x)||,\max_{t\in[0,T]}||\Delta v^{(k+1)}(t,x)||\Big\}$$

$$\le\max(\alpha_1C_0+1,C_0)C_1\int_0^x\max\Big\{\max_{t\in[0,T]}||\Delta z^{(k)}(t,\xi)||,\max_{t\in[0,T]}||\Delta v^{(k)}(t,\xi)||\Big\}d\xi. \tag{2.38}$$

It follows from (2.38) that the sequences $\{v^{(k)}(t,x)\}$ and $\{z^{(k)}(t,x)\}$ converge to $v^*(t,x)$ and $z^*(t,x)$, respectively, as $k\to\infty$ for all $(t,x)\in\bar{\Omega}$. In this case, the limit functions $v^*(t,x)$ and $z^*(t,x)$ are continuous on $\bar{\Omega}$. Moreover, we have the equality $z^*(t,x)=\frac{\partial v^*(t,x)}{\partial t}$ for all $(t,x)\in\bar{\Omega}$.

Passing to the limit in relations (2.32), (2.31) as $k\to\infty$ and using the Grönwall - Bellman inequality, we obtain the following estimates:

$$\max\Big\{\max_{t\in[0,T]}\Big|\Big|\frac{\partial v^*(t,x)}{\partial t}\Big|\Big|,\max_{t\in[0,T]}||v^*(t,x)||\Big\}\le C_2e^{C_2x}\max\Big\{\max_{t\in[0,T]}||F(t,x)||,||\Phi(x)||\Big\}. \tag{2.39}$$

$$\max_{t\in[0,T]}||v^*(t,x)||\le C_0\big(C_1C_2e^{C_2x}+1\big)\max\Big\{\max_{t\in[0,T]}||F(t,x)||,||\Phi(x)||\Big\}, \tag{2.40}$$

where $C_2=\max(\alpha_1C_0+1,C_0)C_1$.

So, we found a solution to the family of problems (2.7), (2.8).

Let us show the uniqueness of the solution to problem (2.7), (2.8). Let the functions $v^*(t,x)$ and $v^{**}(t,x)$ be solutions to problem (2.7), (2.8).

Suppose that $z^*(t,x)=\frac{\partial v^*(t,x)}{\partial t}$ and $z^{**}(t,x)=\frac{\partial v^{**}(t,x)}{\partial t}$.

Using inequalities (2.31), (2.32) for the differences $v^*(t,x)-v^{**}(t,x)$, $z^*(t,x)-z^{**}(t,x)$, we obtain

$$\max\Big\{\max_{t\in[0,T]}||z^*(t,x)-z^{**}(t,x)||,\max_{t\in[0,T]}||v^*(t,x)-v^{**}(t,x)||\Big\}$$

$$\leq C_2 \int_0^x \max\Big\{\max_{t\in[0,T]} ||z^*(t,\xi) - z^{**}(t,\xi)||, \max_{t\in[0,T]} ||v^*(t,\xi) - v^{**}(t,\xi)||\Big\} d\xi. \tag{2.41}$$

Using again the Grönwall - Bellman inequality, get that for any $0 \leq x \leq \omega$

$$\max\Big\{\max_{t\in[0,T]} ||z^*(t,x) - z^{**}(t,x)||, \max_{t\in[0,T]} ||v^*(t,x) - v^{**}(t,x)||\Big\} \leq 0 \cdot e^{C_2 x} = 0. \tag{2.42}$$

From (2.42) it follows that $z^*(t,x) \equiv z^{**}(t,x)$ and $v^*(t,x) \equiv v^{**}(t,x)$ for all $(t,x) \in \Omega$. This contradicts our assumption that problem (2.7), (2.8) has two solutions, the functions $v^*(t,x)$, $z^*(t,x)$ and $v^{**}(t,x)$, $z^{**}(t,x)$. Therefore, the solution to problem (2.7), (2.8) is unique. Finally, inequality (2.40) implies the inequality stated in the theorem with $\widetilde{C} = C_0\big(C_1 C_2 e^{C_2\omega} + 1\big)$. □

3 Unique solvability of problem (1.1)–(1.3). Main result

We consider problem (2.7), (2.8), which is equivalent to problem (1.1)–(1.3).

If we know $v(t,x)$, the solution to problem (2.7), (2.8), then from the integral relation

$$u(t,x) = \psi_0(t) + \psi_1(t)x + \psi_2(t)\frac{x^2}{2!} + \int_0^x \frac{(x-\xi)^2}{2!} v(t,\xi)d\xi \tag{3.1}$$

we find $u(t,x)$.

The following statement provides sufficient conditions for the unique solvability of problem (1.1)–(1.3).

Theorem 2. *Assume that the $n \times n$ matrix $Q(x) = \sum_{j=0}^{m} M_{3,j}(x)X(t_j,x)$ is invertible for all $x \in [0,\omega]$, where X is the solution to the Cauchy problem*

$$\frac{\partial X}{\partial t} = A_1(t,x)X, \qquad X(0,x) = I,$$

and I is the identity matrix.

Then problem (1.1)–(1.3) has a unique classical solution and for its solution $u^*(t,x)$ the following estimate holds:

$$\max\Big\{\Big|\Big|\frac{\partial^4 u^*}{\partial x^3 \partial t}\Big|\Big|_0, \Big|\Big|\frac{\partial^3 u^*}{\partial x^3}\Big|\Big|_0, \Big|\Big|\frac{\partial^3 u^*}{\partial x^2 \partial t}\Big|\Big|_0, \Big|\Big|\frac{\partial^2 u^*}{\partial x^2}\Big|\Big|_0, \Big|\Big|\frac{\partial^2 u^*}{\partial x \partial t}\Big|\Big|_0, \Big|\Big|\frac{\partial u^*}{\partial x}\Big|\Big|_0, \Big|\Big|\frac{\partial u^*}{\partial t}\Big|\Big|_0, ||u^*||_0\Big\}$$

$$\leq \widehat{C} \max\Big\{||\psi_0||_1, ||\psi_1||_1, ||\psi_2||_1, ||\varphi||_0, ||f||_0\Big\}$$

for some $\widehat{C} > 0$ independent of u^*, ψ_0, ψ_1, ψ_2, φ and f.

Proof. Let the assumptions of the theorem be satisfied. Then, by Theorem 1, the family of multipoint boundary value problems (2.7), (2.8) with the integral condition is uniquely solvable, and inequalities (2.39), (2.40) hold for its solution $v^*(t,x)$. Using the integral relation (3.1) we define the function $u^*(t,x)$:

$$u^*(t,x) = \psi_0(t) + \psi_1(t)x + \psi_2(t)\frac{x^2}{2!} + \int_0^x \frac{(x-\xi)^2}{2!} v^*(t,\xi)d\xi. \tag{3.2}$$

This implies the following relations:

$$\frac{\partial u^*(t,x)}{\partial x} = \psi_1(t) + \psi_2(t)x + \int_0^x (x-\xi)v^*(t,\xi)d\xi, \tag{3.3}$$

$$\frac{\partial u^*(t,x)}{\partial t} = \dot{\psi}_0(t) + \dot{\psi}_1(t)x + \dot{\psi}_2(t)\frac{x^2}{2!} + \int_0^x \frac{(x-\xi)^2}{2!}\frac{\partial v^*(t,\xi)}{\partial t}d\xi, \tag{3.4}$$

$$\frac{\partial^2 u^*(t,x)}{\partial x^2} = \psi_2(t) + \int_0^x v^*(t,\xi)d\xi, \tag{3.5}$$

$$\frac{\partial^2 u^*(t,x)}{\partial x\partial t} = \dot{\psi}_1(t) + \dot{\psi}_2(t)x + \int_0^x (x-\xi)\frac{\partial v^*(t,\xi)}{\partial t}d\xi, \tag{3.6}$$

$$\frac{\partial^3 u^*(t,x)}{\partial x^2\partial t} = \dot{\psi}_2(t) + \int_0^x \frac{\partial v^*(t,\xi)}{\partial t}d\xi, \tag{3.7}$$

$$\frac{\partial^3 u^*(t,x)}{\partial x^3} = v^*(t,x), \tag{3.8}$$

$$\frac{\partial^4 u^*(t,x)}{\partial x^3\partial t} = \frac{\partial v^*(t,x)}{\partial t}. \tag{3.9}$$

In problem (2.7), (2.8) instead of the matrices $K_1(t,x,\xi)$, $K_2(t,x,\xi)$, $L_j(x,\xi)$, $j=\overline{0,m}$, and vectors $F(t,x)$, $\Phi(x)$ substituting the corresponding expressions, we obtain

$$\begin{aligned}\frac{\partial v^*}{\partial t} &= A_1(t,x)v^* + B_1(t,x)\left[\dot{\psi}_2(t) + \int_0^x \frac{\partial v^*(t,\xi)}{\partial t}d\xi\right] + A_2(t,x)\left[\psi_2(t) + \int_0^x v^*(t,\xi)d\xi\right] \\ &+B_2(t,x)\left[\dot{\psi}_1(t) + \dot{\psi}_2(t)x + \int_0^x (x-\xi)\frac{\partial v^*(t,\xi)}{\partial t}d\xi\right] + A_3(t,x)\Big[\psi_1(t) + \psi_2(t)x \\ &+ \int_0^x (x-\xi)v^*(t,\xi)d\xi\Big] + B_3(t,x)\left[\dot{\psi}_0(t) + \dot{\psi}_1(t)x + \dot{\psi}_2(t)\frac{x^2}{2!} + \int_0^x \frac{(x-\xi)^2}{2!}\frac{\partial v^*(t,\xi)}{\partial t}d\xi\right] \\ &+C(t,x)\left[\psi_0(t) + \psi_1(t)x + \psi_2(t)\frac{x^2}{2!} + \int_0^x \frac{(x-\xi)^2}{2!}v^*(t,\xi)d\xi\right] + f(t,x),\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\sum_{j=0}^m \Big\{ M_{3,j}(x)v^*(t_j,x) + M_{2,j}(x)\left[\psi_2(t_j) + \int_0^x v^*(t_j,\xi)d\xi\right] \\ &+M_{1,j}(x)\left[\psi_1(t_j) + \psi_2(t_j)x + \int_0^x (x-\xi)v^*(t_j,\xi)d\xi\right] \\ &+M_{0,j}(x)\left[\psi_0(t_j) + \psi_1(t_j)x + \psi_2(t_j)\frac{x^2}{2!} + \int_0^x \frac{(x-\xi)^2}{2!}v^*(t_j,\xi)d\xi\right]\Big\} = \varphi(x).\end{aligned} \tag{3.11}$$

In (3.10), (3.11), replacing the expressions in square brackets by the corresponding expressions from (3.2)–(3.9), we obtain

$$\frac{\partial^4 u^*}{\partial x^3\partial t} = \sum_{i=1}^3 \left\{ A_i(t,x)\frac{\partial^{4-i}u^*}{\partial x^{4-i}} + B_i(t,x)\frac{\partial^{4-i}u^*}{\partial x^{3-i}\partial t}\right\} + C(t,x)u^* + f(t,x), \tag{3.12}$$

$$\sum_{j=0}^m \sum_{i=0}^3 M_{i,j}(x)\frac{\partial^i u^*(t_j,x)}{\partial x^i} = \varphi(x), \qquad x\in[0,\omega]. \tag{3.13}$$

From (3.2), (3.3), and (3.5) for $x=0$ we set

$$u^*(t,0) = \psi_0(t), \qquad \frac{\partial u^*(t,x)}{\partial x}\Big|_{x=0} = \psi_1(t), \qquad \frac{\partial^2 u^*(t,x)}{\partial x^2}\Big|_{x=0} = \psi_2(t), \quad t\in[0,T]. \tag{3.14}$$

It follows that the function $u^*(t,x)$ is a classical solution to problem (1.1)–(1.3).

Taking into account inequality (2.39) and relations (3.3)–(3.9), we obtain the following estimate:

$$\max\Big\{\Big|\Big|\frac{\partial^4 u^*}{\partial x^3\partial t}\Big|\Big|_0, \Big|\Big|\frac{\partial^3 u^*}{\partial x^3}\Big|\Big|_0, \Big|\Big|\frac{\partial^3 u^*}{\partial x^2\partial t}\Big|\Big|_0, \Big|\Big|\frac{\partial^2 u^*}{\partial x^2}\Big|\Big|_0, \Big|\Big|\frac{\partial^2 u^*}{\partial x\partial t}\Big|\Big|_0, \Big|\Big|\frac{\partial u^*}{\partial x}\Big|\Big|_0, \Big|\Big|\frac{\partial u^*}{\partial t}\Big|\Big|_0, ||u^*||_0\Big\}$$

$$\leq C_3 \max\Big\{||\psi_0||_1, ||\psi_1||_1, ||\psi_2||_1, ||\varphi||_0, ||f||_0\Big\}, \tag{3.15}$$

where $C_3 = 1 + \max(1,\omega) + \max\big(1,\omega,\frac{\omega^2}{2}\big) + C_2\big[1 + \omega\max\big(1,\omega,\frac{\omega^2}{2}\big)\big]e^{C_2\omega}$.

The unique solvability of problem (2.7), (2.8) implies the uniqueness of a solution to problem (1.1)–(1.3). Therefore, the classical solution $u^*(t,x)$ of problem (1.1)–(1.3) is unique, and it satisfies estimate (3.2) for $\widehat{C} = C_3$. □

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