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MIKHAIL L'VOVICH GOLDMAN

(to the 75th birthday)

Mikhail L'vovich Goldman was born on April 13, 1945 in Moscow. In 1963 he graduated from school in Moscow and entered the Physical Faculty of the M.V. Lomonosov Moscow State University (MSU) from which he graduated in 1969 and became a PhD student (1969–1972) at the Mathematical Department of this Faculty. In 1972 he has defended the PhD thesis, and in 1988 his DSc thesis "The study of spaces of differentiable functions of many variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor in Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he is a Professor of the S.M. Nikol'skii Mathematical Institute at the Peoples Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman are: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators. Main achievements: optimal embeddings of spaces with generalized smoothness, sharp conditions of the convergence of spectral expansions, descriptions of integral and differential properties of the generalized Bessel and Riesz-type potentials, sharp estimates for operators on cones and optimal envelopes for the cones of functions with properties of monotonicity. Professor M.L. Goldman has over 140 scientific publications in leading mathematical journals.

Under his scientific supervision, 8 candidate theses in Russia and 1 thesis in Kazakhstan were successfully defended. Some of his former students are now professors in Ethiopia, Columbia, Mongolia.

Participation in scientific and organizational activities of M.L. Goldman is well known. He is a member of the DSc Councils at RUDN and MSU, of the PhD Council in the Lulea Technical University (Sweden), a member of the Editorial Board of the Eurasian Mathematical Journal, an invited lector and visiting professor at universities of Russia, Germany, Sweden, UK etc., an invited speaker at many international conferences.

The mathematical community, friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Mikhail L'vovich Goldman on the occasions of his 75th birthday and wish him good health, happiness, and new achievements in mathematics and mathematical education.

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ON SOLVABILITY OF PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES (II)

A.M. Selitskii

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Abstract. In this paper, a parabolic functional differential equation is considered in the spaces $C(0,T;H_p^s(Q))$ for s close to 1 and p close to 2. The transformations of the space argument are supposed to be bounded in the spaces $H_p^s(Q)$ with small smoothness exponent and p close to 2. The corresponding resolvent estimate of the elliptic part of the operator is obtained in order to show that it generates a strongly continuous semigroup.

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1 Introduction and main result

The target of this article is to generalize the result of paper [1] on solvability of parabolic equations in the spaces $H_2^s(Q)$ to the spaces $H_p^s(Q)$, defined by equality (1.7), for indices s close to 1 and p close to 2. This problem was studied for elliptic systems by M. Agranovich in paper [2], where he considered functional coefficients; in contrast, we consider operators as the coefficients. The use of operator-valued coefficients allows one to study functional differential equations in bounded domains. Note, that in $\mathbb{R}^n$, parabolic differential-difference equations were fully studied by A. Muravnik in work [3], while parabolic functional differential equations with contractions of variables were studied by L. Rossovskii and A. Khanalyev in article [4]. In survey paper [5], an example of parabolic problem with rotation of the variables is considered in Section 10 and more literature on parabolic functional differential equations is cited in Section 11.

Let $H^1(Q)$ be the Sobolev space of complex-valued functions belonging to $L_2(Q)$ having all generalized derivatives of the first order belonging to $L_2(Q)$. We consider the sesquilinear form $\Phi(v,\, w)$ in $L_2(Q)$ with the domain $H^1(Q)$ defined by the formula

$$\Phi(v,\, w) = \sum_{i,j=1}^{n} \left(A_{ij} v_{x_j},\, w_{x_i}\right)_{L_2(Q)} + \sum_{i=1}^{n} \left(B_i v_{x_i},\, w\right)_{L_2(Q)} + \left(Cv,\, w\right)_{L_2(Q)}. \tag{1.1}$$

By assumption, the operators $A_{ij},\, B_i,\, C\colon L_2(Q) \to L_2(Q)$ are bounded. Therefore, it follows that there exists a constant $c_0 > 0$ such that

$$|\Phi(v,\, w)| \le c_0 \|v\|_{H^1(Q)} \|w\|_{H^1(Q)} \qquad (v, w \in H^1(Q)). \tag{1.2}$$

Since the sesquilinear form $\Phi(v,\, w)$ is continuous with respect to w in $H^1(Q)$, there exists a linear bounded operator $A\colon H^1(Q) \to [H^1(Q)]' = \widetilde{H}^{-1}(Q)$, such that

$$\langle Av,\, \overline{w}\rangle = \Phi(v,\, w) \qquad (v, w \in H^1(Q)), \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing with respect to the scalar product in $L_2(Q)$.

We consider the following problem

$$u_t + Au = f(x,\, t), \tag{1.4}$$

$$u|_{t=0} = \varphi(x). \tag{1.5}$$

We suppose that the form $\Phi(v,\, w)$ is coercive, i.e., there exist numbers $c_1 > 0$ and $c_2 \geq 0$ such that

$$\operatorname{Re} \Phi(v,\, v) \geq c_1\|v\|^2_{H^1(Q)} - c_2\|v\|^2_{L_2(Q)} \qquad (v \in H^1(Q)). \tag{1.6}$$

We can assume that $c_2 = 0$ in inequality (1.6). Otherwise, we set $u = z\, e^{c_2 t}$ that transforms operator A to $A + c_2 I$.

The space $H^s_p(\mathbb{R}^n)$ can be defined for $1 < p < \infty$ and $s \in \mathbb{R}$ as the space of all distributions in S' with the finite norm

$$\|u\|_{H^s_p(\mathbb{R}^n)} = \|\Lambda^s u\|_{L_p(\mathbb{R}^n)}, \tag{1.7}$$

where $\Lambda^s = F^{-1}(1+|\xi|^2)^{s/2}F$. Here F is the Fourier transform in the sense of distributions. The space $H^s_p(Q)$ is defined as the restriction of the space $H^s_p(\mathbb{R}^n)$ on Q with the inf-norm. For details see book [6], Section 14.5.

In paper [1], the generalization of the operator A was considered: $A_p\colon H^1_p(Q) \to \widetilde{H}^{-1}_p(Q) = \left[H^1_q(Q)\right]'$. In this paper we will extend it to more general spaces $H^s_p(Q)$.

Suppose that $v \in H^{1/2+\sigma+1/p}_p(Q)$ and $w \in H^{1/2-\sigma+1/q}_q(Q)$ for $1 < p < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If the operators A_{ij} are bounded in $H^s_p(Q)$ (with small s and p close to 2) and the operators B_i and C are bounded in $L_p(Q)$ (with p close to 2), then the form $\Phi(v,w)$ is bounded on $H^{1/2+\sigma+1/p}_p(Q) \times H^{1/2-\sigma+1/q}_q(Q)$:

$$\begin{aligned} |\Phi(v,w)| &\leq c_3 \sum_{i,j=1}^n |\langle v_{x_i},\, \overline{w_{x_j}}\rangle| \leq c_4 \sum_{i,j=1}^n \|v_{x_i}\|_{\widetilde{H}^{-1/2+\sigma+1/p}_p(Q)} \|w_{x_j}\|_{H^{1/2-\sigma-1/p}_q(Q)} \\ &\leq c_5\|v\|_{H^{1/2+\sigma+1/p}_p(Q)}\|w\|_{H^{1/2-\sigma+1/q}_q(Q)}. \end{aligned} \tag{1.8}$$

Here we used the fact $\widetilde{H}^{-1/2+\sigma+1/p}_p(Q) = H^{-1/2+\sigma+1/p}_p(Q)$ for $|\sigma| < 1/2$.

The form $\Phi(v,\, w)$ defines the operator $A_{s,p}\colon H^{1/2+\sigma+1/p}_p(Q) \to \widetilde{H}^{-1/2+\sigma-1/q}_p(Q)$ which coincides with the operator A on $H^{1+s}_p(Q)$ ($p \leq 2$ and $s = \sigma + 1/p - 1/2 \geq 0$). For more details see [1].

Remark 1. *There is a large class of operators A_{ij} satisfying the above conditions. E.g., operators of shift or rotation are bounded in $H^s_p(Q)$, if $-1/q < s < 1/p$.*

Instead of problem (1.4)-(1.5) we consider the following problem

$$u_t + A_{s,p}u = f(x,\, t), \tag{1.9}$$

$$u|_{t=0} = \varphi(x). \tag{1.10}$$

Definition 1. A function $u \in C([0,T); H^{-1+s}_p(Q)) \cap C^1((0,T); H^{-1+s}_p(Q))$, $s = \sigma + 1/p - 1/2$, is called a classical solution of (1.9)-(1.10) if $u(t,\cdot) \in H^{1+s}_p(Q)$ for $0 < t < T$ and equalities (1.9) and (1.10) are satisfied on $[0,\, T)$.

Theorem 1.1. *Let $f \in L_1(0,\, T; H^{-1+s}_p(Q))$ be Lipschitz continuous on $[0,\, T]$ and $\varphi \in H^{1+s}_p(Q)$. Then there exist $s_0 > 0$ and $\delta > 0$ such that for $|s| < s_0$ and $\left|\dfrac{1}{2} - \dfrac{1}{p}\right| < \delta$ problem (1.9)-(1.10) has a unique classical solution.*

Proof. The statement of the theorem follows from the estimate

$$\left\|(A_{s,p}-\lambda I)^{-1}\right\| \le \frac{c_{s,p}}{1+|\lambda|}, \tag{1.11}$$

where $\lambda \in \Lambda_{\varepsilon,\alpha} = \left\{\mu \in \mathbb{C} : |\arg\mu| > \dfrac{\pi}{2} - \alpha\right\} \cup B_\varepsilon(0)$ for some $\varepsilon > 0$ and $\alpha > 0$, and $c_{s,p} > 0$ (see, e.g., Theorem 5.2 and Corollary 2.11 in [7]).

Note that the spaces $H^s_p(Q)$ form an interpolation scale: $\left[H^{s_1}_{p_1}(Q),\, H^{s_2}_{p_2}(Q)\right]_\theta = H^{(1-\theta)s_1+\theta s_2}_r(Q)$, where $\dfrac{1}{r} = \dfrac{1-\theta}{p_1} + \dfrac{\theta}{p_2}$. Therefore, if we prove estimate (1.11) for two segments $p = 2$ and $s \in [1-s_0,\, 1+s_0]$, and $s = 1$ and $p \in [2-\delta_1,\, 2+\delta_2]$ with small positive s_0, δ_1, and δ_2, then, by the interpolation property, it will be true in all the quadrilateral on the $(s,\, p)$-plane (see, e.g., Remark 5.2 (iii) and Lemma 5.3 in [8] or Theorem 3.10 in [9]).

The estimate for $p = 2$ is proved in Section 2. For the estimate in case $s = 1$ see Section 4. □

2 Resolvent estimate in $H^{1+s}_2(Q)$

Let us denote by A_s the operators the $A_{s,2}$. In paper [10], the following estimate was proved for $0 \le s < s_1 < 1/2$

$$\left\|(A_s-\lambda I)^{-1}\right\| \le \frac{c_6}{1+|\lambda|}, \tag{2.1}$$

where $c_6 > 0$ is independent of λ.

Consider the operator A^*_s defined by

$$\langle A^*_s v,\, \overline{w}\rangle = \overline{\Phi(w,\, v)} \qquad (v \in H^{1+s}_2(Q), \quad w \in H^{1-s}_2(Q)). \tag{2.2}$$

It is obvious that it satisfies estimate (2.1). If $v \in H^{1-s}_2(Q)$, $f \in \widetilde{H}^{-1+s}_2$, and $w = \left(A^*_s - \overline{\lambda} I\right)^{-1} f$ (with $s > 0$), then

$$(v,\, f) = (v,\, A^*_s w - \overline{\lambda} w) = (A_s v - \lambda v, w), \tag{2.3}$$

and

$$|(v,\, f)| \le c_7 \left\|A_s v - \lambda v\right\|_{\widetilde{H}^{-1-s}_2(Q)} \left\|w\right\|_{H^{1+s}_2(Q)} \le c_8 \left\|A_s v - \lambda v\right\|_{\widetilde{H}^{-1-s}_2(Q)} \left\|f\right\|_{\widetilde{H}^{-1+s}_2(Q)}. \tag{2.4}$$

Thus,

$$\|v\|_{H^{1-s}_2(Q)} \le c_9 \|A_s v - \lambda v\|_{\widetilde{H}^{-1-s}_2(Q)}. \tag{2.5}$$

$$\begin{aligned} |\lambda|\, \|v\|_{H^{1-s}_2(Q)} &\le \|A_s v\|_{\widetilde{H}^{-1-s}_2(Q)} + \|A_s v - \lambda v\|_{\widetilde{H}^{-1-s}_2(Q)} \\ &\le c_{10} \|v\|_{H^{1-s}_2(Q)} + \|A_s v - \lambda v\|_{\widetilde{H}^{-1-s}_2(Q)} \\ &\le c_{11} \|A_s v - \lambda v\|_{\widetilde{H}^{-1-s}_2(Q)}. \end{aligned} \tag{2.6}$$

By adding the last inequalities, we obtain (2.1) for negative s.

3 Mixed problem

Denote by $\Omega = Q \times (-1,\, 1)$. This is a Lipschitz domain. Consider the form

$$\begin{aligned} \Psi(V,\, W) = &\sum_{i,j=1}^{n} \left(A_{ij} V_{x_j},\, W_{x_i}\right)_{L_2(\Omega)} + \sum_{i=1}^{n} \left(B_i V_{x_i},\, W\right)_{L_2(\Omega)} + (CV,\, W)_{L_2(\Omega)} \\ &+ \eta \left(V_{x_{n+1}},\, W_{x_{n+1}}\right)_{L_2(\Omega)} \end{aligned} \tag{3.1}$$

on $H^1_{2,\Gamma}(\Omega) = \{V \in H^1_2(\Omega) : V|_\Gamma = 0\}$, where $\Gamma = D_+ \cup D_-$, and $D_\pm = \{x \in \overline{\Omega} : x_{n+1} = \pm 1\}$.

Note, that

$$\Psi(V, V) = \int\limits_{-1}^{1} \Phi(V, V)\, dx_{x+1} + \eta \left\| V_{x_{n+1}} \right\|^2_{L_2(\Omega)}. \tag{3.2}$$

$$\operatorname{Re} \Psi(V, V) \geq \min(c_1, \operatorname{Re}\eta) \|V\|^2_{H^1_{2,\Gamma}(\Omega)}. \tag{3.3}$$

Since the sesquilinear form $\Psi(V, W)$ is continuous with respect to W in $H^1_{2,\Gamma}(\Omega)$, there exists a linear bounded operator $L\colon H^1_{2,\Gamma}(\Omega) \to \left[H^1_{2,\Gamma}(\Omega)\right]' = \widetilde{H}^{-1}_{2,\Gamma}(\Omega)$, such that

$$\left\langle LV, \overline{W} \right\rangle = \Psi(V, W) \qquad (V, W \in H^1_{2,\Gamma}(\Omega)). \tag{3.4}$$

It is obvious, that on smooth enough functions it acts as $L = A - \dfrac{\partial^2}{\partial x^2_{n+1}}$.

Like in Section 1, the form $\Psi(V, W)$ is bounded on $H^{1/2+\sigma+1/p}_{p,\Gamma}(\Omega) \times H^{1/2-\sigma+1/q}_{q,\Gamma}(\Omega)$ and defines the operator $L_p\colon H^{1/2+\sigma+1/p}_{p,\Gamma}(\Omega) \to \widetilde{H}^{-1/2-\sigma-1/q}_{p,\Gamma}(\Omega)$. We consider the line $\sigma + 1/p = 1/2$ (or $s = 0$). By virtue of the Shneiberg theorem on extrapolation of invertibility (see, e.g., Theorem 13.7.3 in [6]), there exists a neighborhood of the point $\sigma = 0$ and $p = 2$ on this line where the operator L_p remains invertible. Therefore, there exists a positive number r_2, such that L_p^{-1} exists for $|1/p - 1/2| < r_2$. Here we used the fact that the spaces $H^{1+s}_{p,\Gamma}(\Omega)$ have the same interpolation properties as the spaces $H^{1+s}_p(\Omega)$ (see, e.g., Theorem 3.3 in [11]).

4 Resolvent estimate in $H^1_p(Q)$

In this section, we use the method suggested by M. Agranovich in paper [2].

From the previous section, we know that for $|1/p - 1/2| < r_2$ there exist a $c_{12} > 0$, such that

$$\|V\|_{H^1_{p,\Gamma}(\Omega)} \leq c_{12} \|L_p V\|_{\widetilde{H}^{-1}_{p,\Gamma}(\Omega)}. \tag{4.1}$$

Let us suppose $p < 2$. We use this estimate for functions $V(x', x_{n+1}) = v(x')\varphi_\mu(x_{n+1})$, with $\varphi(\xi) = e^{i\mu\xi}\psi(\xi)$, where $\mu > 0$ and $\psi \in C_0^\infty[-1; 1]$ is a non-negative function, such that $\psi(\xi) = 1$ for $1/2 \leq \xi \leq 1/2$. Notice that

$$\left\langle L_p V, \overline{W} \right\rangle = \int\limits_{-1}^{1} \left(\left\langle A_p v, \overline{W} \right\rangle \varphi - \eta \varphi'' \left\langle v, \overline{W} \right\rangle \right) dx_{n+1} = \Phi(v, w) - \eta \int\limits_{-1}^{1} \varphi'' \left\langle v, \overline{W} \right\rangle dx_{n+1}, \tag{4.2}$$

where $w = \int\limits_{-1}^{1} \overline{\varphi} W\, dx_{n+1}$. The last term has the form

$$-\eta \int\limits_{-1}^{1} \varphi'' \left\langle v, \overline{W} \right\rangle dx_{n+1}$$

$$= 2\eta \int\limits_{-1}^{1} e^{i\mu x_{n+1}} \psi' \left\langle v, \overline{W'} \right\rangle dx_{n+1} + \eta \int\limits_{-1}^{1} e^{i\mu x_{n+1}} \left(\mu^2 \psi - \psi'' \right) \left\langle v, \overline{W} \right\rangle dx_{n+1}. \tag{4.3}$$

Now we set $\lambda = -\eta\mu^2$

$$\left\langle \varphi A_p v - \eta\varphi'' v,\, \overline{W} \right\rangle_\Omega = \Phi(v,w) - \lambda \left\langle v,\, \overline{w} \right\rangle_Q + \eta \left\langle v,\, \int\limits_{-1}^{1} e^{i\mu x_{n+1}} \left(2\psi' \overline{W'} - \psi'' \overline{W} \right) dx_{n+1} \right\rangle. \tag{4.4}$$

Applying the Hölder inequality twice and using the fact that $p < q$, we obtain

$$\left| \left\langle \varphi A_p v - \eta\varphi'' v,\, \overline{W} \right\rangle_\Omega \right| \le c_{13} \left(\|A_p v - \lambda v\|_{\widetilde{H}_p^{-1}(Q)} \|w\|_{H_q^1(Q)} + \|v\|_{L_p(Q)} \|W\|_{H^1_{q,\Gamma}(\Omega)} \right). \tag{4.5}$$

The norm $\|w\|_{H_q^1(Q)}$ is dominated by the norm $\|W\|_{H^1_{q,\Gamma}(\Omega)}$. Dividing (4.5) by $\|W\|_{H^1_{q,\Gamma}(\Omega)}$ and passing to the least upper bound, we obtain

$$\|\varphi A_p v - \eta\varphi'' v\|_{\widetilde{H}_p^{-1}(\Omega)} \le c_{14} \left(\|A_p v - \lambda v\|_{\widetilde{H}_p^{-1}(Q)} + \|v\|_{L_p(Q)} \right). \tag{4.6}$$

From (4.1) it follows that

$$\|v\|_{H_p^1(Q)} \le c_{12} \|V\|_{H^1_{p,\Gamma}(\Omega)} \le c_{15} \|\varphi_\mu A_p v - \eta v \varphi''_\mu\|_{\widetilde{H}^{-1}_{p,\Gamma}(\Omega)}, \tag{4.7}$$

or

$$\|v\|_{H_p^1(Q)} \le c_{16} \left(\|A_p v - \lambda v\|_{\widetilde{H}_p^{-1}(Q)} + \|v\|_{L_p(Q)} \right). \tag{4.8}$$

Using the inequality

$$\|v\|_{L_p(Q)} \le \rho \|v\|_{H_p^1(Q)} + c_\rho \|v\|_{\widetilde{H}_p^{-1}(Q)}, \tag{4.9}$$

with sufficiently small ρ, we obtain

$$\|v\|_{H_p^1(Q)} \le c_{17} \left(\|A_p v - \lambda v\|_{\widetilde{H}_p^{-1}(Q)} + \|v\|_{\widetilde{H}_p^{-1}(Q)} \right). \tag{4.10}$$

Since, $\lambda v = (\lambda v - A_p v) + A_p v$, and $\|v\|_{H_p^1(Q)}$ dominates $\|A_p v\|_{\widetilde{H}_p^{-1}(Q)}$, we obtain that

$$|\lambda| \|v\|_{\widetilde{H}_p^{-1}(Q)} \le c_{18} \left(\|A_p v - \lambda v\|_{\widetilde{H}_p^{-1}(Q)} + \|v\|_{\widetilde{H}_p^{-1}(Q)} \right). \tag{4.11}$$

If $|\lambda| > 2c_{18}$, then the last inequality takes the form

$$|\lambda| \|v\|_{\widetilde{H}_p^{-1}(Q)} \le c_{19} \|A_p v - \lambda v\|_{\widetilde{H}_p^{-1}(Q)}. \tag{4.12}$$

For a compact set of λ this is also true, thus we proved (4.12) for all $\lambda = -\eta\mu^2$.

Since,

$$\|v\|_{H_p^1(Q)} \le c_{20} \|A_p v\|_{\widetilde{H}_p^{-1}(Q)} \le c_{20} \left(\|A_p v - \lambda v\|_{\widetilde{H}_p^{-1}(Q)} + |\lambda| \|v\|_{\widetilde{H}_p^{-1}(Q)} \right), \tag{4.13}$$

we obtain

$$\|v\|_{\widetilde{H}_p^{-1}(Q)} + |\lambda| \|v\|_{\widetilde{H}_p^{-1}(Q)} \le c_{21} \|A_p v - \lambda v\|_{\widetilde{H}_p^{-1}(Q)}, \tag{4.14}$$

that leads to inequality (1.11).

The statement for the case $p > 2$ can be proved using the operator A_p^*, as in Section 2.

We obtained the result for $\lambda = -\eta\mu^2$, where $|\eta| = 1$ and $|\arg\eta| < \pi/2$. From inequalities (1.2) and (1.6) it follows that the values of $\Phi(v,v)$ are contained in the sector $|\arg\lambda| < \arccos(1/C_0 C_1)$. Denote $2\alpha = \pi/2 - \arccos(1/C_0C_1)$, and consider the form $\Phi_\alpha(v,w) = e^{i\alpha}\Phi(v,w)$. It is still a coercive form corresponding to the operator $e^{i\alpha}A$. We set $\max\arg\eta = \pi/2 - \alpha/2$, and obtain estimate (4.14) for the operator $e^{i\alpha}A_p$. Then we notice that

$$\|e^{i\alpha}A_p - \lambda I\| = \|A_p - e^{-i\alpha}\lambda I\|, \tag{4.15}$$

that provides the estimate for $\arg\lambda < \pi/2$.

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