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MIKHAIL L'VOVICH GOLDMAN

(to the 75th birthday)

Mikhail L'vovich Goldman was born on April 13, 1945 in Moscow. In 1963 he graduated from school in Moscow and entered the Physical Faculty of the M.V. Lomonosov Moscow State University (MSU) from which he graduated in 1969 and became a PhD student (1969–1972) at the Mathematical Department of this Faculty. In 1972 he has defended the PhD thesis, and in 1988 his DSc thesis "The study of spaces of differentiable functions of many variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor in Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he is a Professor of the S.M. Nikol'skii Mathematical Institute at the Peoples Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman are: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators. Main achievements: optimal embeddings of spaces with generalized smoothness, sharp conditions of the convergence of spectral expansions, descriptions of integral and differential properties of the generalized Bessel and Riesz-type potentials, sharp estimates for operators on cones and optimal envelopes for the cones of functions with properties of monotonicity. Professor M.L. Goldman has over 140 scientific publications in leading mathematical journals.

Under his scientific supervision, 8 candidate theses in Russia and 1 thesis in Kazakhstan were successfully defended. Some of his former students are now professors in Ethiopia, Columbia, Mongolia.

Participation in scientific and organizational activities of M.L. Goldman is well known. He is a member of the DSc Councils at RUDN and MSU, of the PhD Council in the Lulea Technical University (Sweden), a member of the Editorial Board of the Eurasian Mathematical Journal, an invited lector and visiting professor at universities of Russia, Germany, Sweden, UK etc., an invited speaker at many international conferences.

The mathematical community, friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Mikhail L'vovich Goldman on the occasions of his 75th birthday and wish him good health, happiness, and new achievements in mathematics and mathematical education.

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APPROXIMATE IDEAL CONNES AMENABILITY OF DUAL BANACH ALGEBRAS AND IDEAL CONNES AMENABILITY OF DISCRETE BEURLING ALGEBRAS

A. Minapoor

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Key words: Banach algebra, discrete weighted group algebras, approximate ideal Connes amenable.

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Abstract. The concept of approximate Connes amenability of dual Banach algebras was introduced in [9]. In this paper we introduce approximate ideal Connes amenability for dual Banach algebras. We show that every approximate Connes amenable dual Banach algebra is approximate ideally Connes amenable. The notion of ideal Connes amenability for dual Banach algebras was introduced in [14]. In this paper we also study ideal Connes amenability for discrete Beurling algebras.

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1 Introduction

The notion of amenability for groups was defined by Von Neumann [22] for discrete groups and by Day [5] for locally compact groups. Amenability for Banach algebras was introduced and studied by Johnson in [11]. He proved that a locally compact group G is amenable if and only if the group algebra $L^1(G)$ is amenable as a Banach algebra.

After the pioneering work of Johnson, several modifications of the original concept of amenability of Banach algebras were investigated. One of the most important modifications was suggested in [12], where the authors introduced a notion of amenability more suitable for von Neumann algebras. It modifies the original definition in the sense that it takes into account the dual space structure of a von Neumann algebra. Like the amenable Banach algebras, the Connes amenable von Neumann algebras allow for an intrinsic characterization in terms of a diagonal type elements: a von Neumann algebra is Connes amenable if and only if it has a normal, virtual diagonal [7]. It is not true in general. For example, if G is an amenable [SIN]-group that fails to be compact, the dual Banach algebra $WAP(G)^*$ is Connes amenable, but has no normal, virtual diagonal [16], where $WAP(G)$ denotes the weakly almost periodic functions on G. However, the concept of Connes amenability is different from amenability. Examples of dual Banach algebras (besides von Neumann algebras) include the measure algebra $M(G)$ and the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G. In particular, Runde showed that a locally compact group G is amenable if and only if its measure algebra $M(G)$ is Connes amenable [19]. This result sounds more interesting, when compared to a deep results of Dales, Ghahramani, and Helemskii, showing that $M(G)$ is amenable if and only if G is discrete and amenable [3].

The concept of ideal amenability of Banach algebras was introduced by Gordji and Yazdanpanah in [8]. They related this notion to weak amenability of Banach algebras, and by means of some examples showed that the ideal amenability is different from the amenability and weak amenability. For example, all C^*-algebras are ideally amenable while they are amenable if and only if they are nuclear [6]. Approximate Connes amenability was introduced by Esslamzadeh and Shojaee in [9]. The concept of ideal Connes amenability of dual Banach algebras was introduced by Minapoor, Bodaghi and Ebrahimi Bagh in [14]. In this paper, we define and study the notion of an approximate ideal Connes amenability for the dual Banach algebras. We show that every approximately Connes amenable dual Banach algebra is an approximate ideally Connes amenable dual Banach algebra and every ideally Connes amenable dual Banach algebra is approximate ideally Connes amenable, but the converse is not necessarily true. We also study ideal Connes amenability of weighted discrete group algebras.

2 Main results

We first recall some definitions in the Banach algebras settings. Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \longrightarrow X$ is called a *derivation* if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathcal{A}).$$

For each $x \in X$, we define the map $D_x : \mathcal{A} \longrightarrow X$ by

$$D_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathcal{A}).$$

It is easily seen that D_x is a derivation. Derivations of this form are called *inner derivations*. $\mathcal{Z}^1(\mathcal{A}, X)$ is the space of all continuous derivations from $\mathcal{A}$ into X, $\mathcal{N}^1(\mathcal{A}, X)$ is the space of all inner derivations from $\mathcal{A}$ into X, and the first Hochschild cohomology group of $\mathcal{A}$ with coefficients in X is the quotient space

$$\mathcal{H}^1(\mathcal{A}, X) = \mathcal{Z}^1(\mathcal{A}, X)/\mathcal{N}^1(\mathcal{A}, X).$$

Let X be a $\mathcal{A}$-bimodule. In the following by $\langle f, x\rangle$ we mean $f(x)$. The dual space X^* of X is also a Banach $\mathcal{A}$-bimodule by the following module actions:

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle, \quad \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle, \qquad (a \in \mathcal{A},\ x \in X,\ f \in X^*).$$

With the above notations, a Banach algebra A is called *amenable* if $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule X, and *weakly amenable* if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. Let $n \in \mathbb{N}$. Then, $\mathcal{A}$ is called *n-weakly amenable* if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$, where $\mathcal{A}^{(n)}$ is n-th dual of $\mathcal{A}$. $\mathcal{A}$ is said to be *n-ideally amenable* if $\mathcal{H}^1(\mathcal{A}, \mathcal{I}^{(n)}) = \{0\}$ for every closed two sided ideal $\mathcal{I}$ in $\mathcal{A}$.

A Banach algebra $\mathcal{A}$ is said to be dual if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = (\mathcal{A}_*)^*$. One can see that a Banach algebra which is also a dual space is a dual Banach algebra if and only if the multiplication map is separately w^*-continuous [17]. Examples of dual Banach algebras include all von Neumann algebras, the algebra $B(E) = (E\hat{\otimes}E^*)^*$ of all bounded operators on a reflexive Banach space E, the measure algebra $M(G) = C_0(G)^*$, the Fourier-Stieljes algebra $B(G) = C^*(G)^*$, and the second dual B^{**} of an Arens regular Banach algebra B.

Let $\mathcal{A}$ be a Banach algebra. A dual Banach $\mathcal{A}$-bimodule X is called *normal* if for each $x \in X$ the maps $a \mapsto a{\cdot}x$ and $b \mapsto x{\cdot}b$ from $\mathcal{A}$ into X are w^*-continuous, and *Connes amenable* if for every normal dual Banach $\mathcal{A}$-bimodule X, every w^*-continuous derivation $D : \mathcal{A} \longrightarrow X$ is inner [17]. We denote by $\mathcal{Z}^1_{w^*}(\mathcal{A}, X^*)$ the w^*-continuous derivations from A into X^* and $\mathcal{H}^1_{w^*}(\mathcal{A}, X^*) = \mathcal{Z}^1_{w^*}(\mathcal{A}, X^*)/\mathcal{B}^1(\mathcal{A}, X^*)$.

Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{I}$ be a *weak**-closed two-sided ideal of $\mathcal{A}$. A dual Banach algebra $\mathcal{A}$ is $\mathcal{I}-$Connes amenable if $\mathcal{H}^1_{w^*}(\mathcal{A}, \mathcal{I}) = \{0\}$ and $\mathcal{A}$ is ideally Connes amenable if it is $\mathcal{I}$-Connes amenable for every *weak**-closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$ [14].

A derivation $D : \mathcal{A} \longrightarrow X$ is approximately inner if there exists a net $(x_\alpha) \subseteq X$ such that for every $a \in \mathcal{A}$; $D(a) = lim_\alpha(a.x_\alpha - x_\alpha.a)$, the limit being with respect to the norm.

Let $\mathcal{A}$ be a dual Banach algebra. $\mathcal{A}$ is approximately Connes amenable, if for each normal dual Banach $\mathcal{A}$-bimodule X, every w^*-continuous derivation $D \in \mathcal{Z}^1(\mathcal{A}, X)$ is approximately inner [9].

Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{I}$ be a *weak**-closed two-sided ideal of $\mathcal{A}$. We say that $\mathcal{A}$ is approximately $\mathcal{I}-$Connes amenable if every *weak**-continuous derivation $D \in \mathcal{Z}^1_{w^*}(\mathcal{A}, \mathcal{I})$ is approximately inner. We say that $\mathcal{A}$ is approximately ideally Connes amenable if for every *weak**-closed two-sided ideal $\mathcal{I}$ of $\mathcal{A}$ it is approximately $\mathcal{I}-$Connes amenable. Let $\mathcal{I}$ be an arbitrary *weak**-closed two-sided ideal of $\mathcal{A}$. If in the definition of the approximate Connes amenability we replace X by $\mathcal{I}$, then it is obvious that every approximate Connes amenable dual Banach algebra is approximately ideally Connes amenable. Since every inner derivation is approximately inner so it is obvious that every ideally Connes amenable dual Banach algebra is approximate ideally Connes amenable, but not every approximately ideally Connes amenable dual Banach algebra is ideally Connes amenable. In Example 2 we will introduce such a dual Banach algebra.

Definition 1. Let $(\mathcal{A}_*)^* = \mathcal{A}$ be a dual Banach algebra, and $(\mathcal{I}_*)^* = \mathcal{I}$ be a w^*-closed two-sided ideal in $\mathcal{A}$ and Z be the centre of $\mathcal{A}$. We say that $\mathcal{I}$ has the dual approximate trace extension property if for each $\lambda \in Z \cap \mathcal{I}$, there is a net $(\tau_\alpha)_\alpha \subseteq \mathcal{A}$, $(\alpha \in \Lambda)$, such that $\tau_\alpha|_{\mathcal{I}_*} = \lambda|_{\mathcal{I}_*}$, i.e., coincide as functionals on $\mathcal{I}_*$ and for each $a \in \mathcal{A}$, we have $a \cdot \tau_\alpha - \tau_\alpha \cdot a \longrightarrow 0$ with respect to the net $\alpha \in \Lambda$.

Let $\mathcal{X}$ be a Banach space, $\mathcal{M}$ be a subspace of $\mathcal{X}$, and $\mathcal{N}$ be a subspace of $\mathcal{X}^*$. Then, the annihilators $\mathcal{M}^\perp$ and ${}^\perp\mathcal{N}$ are defined as follows:

$$\mathcal{M}^\perp = \{f \in \mathcal{X}^* : \langle f, x\rangle = 0 \,\text{for all}\, x \in \mathcal{M}\}$$

$${}^\perp\mathcal{N} = \{x \in \mathcal{X} : \langle f, x\rangle = 0 \,\text{for all}\, f \in \mathcal{N}\}$$

It is well-known that $\mathcal{M}^\perp$ and ${}^\perp\mathcal{N}$ are *weak**-closed subspaces and the norm-closed subspaces of $\mathcal{X}^*$ and $\mathcal{X}$, respectively. Moreover, ${}^\perp(\mathcal{M}^\perp)$ is the norm-closure of $\mathcal{M}$ and $({}^\perp\mathcal{N})^\perp$ is the *weak**-closure of $\mathcal{N}$.

Lemma 2.1. *Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and $\mathcal{I}$ be a weak*-closed two-sided ideal of $\mathcal{A}$. Then, $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are dual Banach algebras.*

Proof. $\mathcal{I}$ is a dual Banach space with predual $\mathcal{I}_* = \mathcal{A}_*/{}^\perp\mathcal{I}$. Indeed, $\mathcal{I}$ is the *weak**-closed of $\mathcal{A}$ and so $(\mathcal{I}_*)^* = \big(\mathcal{A}_*/{}^\perp\mathcal{I}\big)^* = ({}^\perp\mathcal{I})^\perp = \mathcal{I}$. Moreover, $\mathcal{I}_*$ is a submodule of $\mathcal{A}^*/\mathcal{I}^\perp = \mathcal{I}^*$. Thus, $\mathcal{I}$ is a dual Banach algebra. Once more, ${}^\perp\mathcal{I}$ is a submodule of $\mathcal{I}^\perp = (\mathcal{A}/\mathcal{I})^*$ and $({}^\perp\mathcal{I})^* = (\mathcal{A}_*)^*/({}^\perp\mathcal{I})^\perp = \mathcal{A}/\mathcal{I}$. Thus, $\mathcal{A}/\mathcal{I}$ is a dual Banach space. On the other hand, multiplication in $\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ is separately weak*-continuous and therefore $\mathcal{A}/\mathcal{I}$ is a dual Banach algebra. □

Theorem 2.1. *Let $\mathcal{A}$ be an ideally Connes amenable dual Banach algebra, and $\mathcal{I}$ be w^*-closed two-sided ideal in $\mathcal{A}$ with the dual approximate trace extension property. Then $\mathcal{A}/\mathcal{I}$ is an approximate ideally Connes amenable dual Banach algebra.*

Proof. Let $\mathcal{J}/\mathcal{I}$ be a w^*-closed two-sided ideal in $\mathcal{A}/\mathcal{I}$. Then $\mathcal{J}$ is a w^*-closed two-sided ideal in $\mathcal{A}$, so ${}^\perp\mathcal{I}$ is predual of $\mathcal{A}/\mathcal{I}$. We know that ${}^\perp\mathcal{I}$ is a closed $\mathcal{A}$-submodule of $\mathcal{J}_*$. Let $\pi_* : \mathcal{J}_* \longrightarrow^\perp \mathcal{I}$ be the natural projection $\mathcal{A}$-bimodule morphism and $q : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{I}$ be the natural quotient map and $(\pi_*)^*$ be the adjoint of π_*. Let $D : \mathcal{A}/\mathcal{I} \longrightarrow \mathcal{J}/\mathcal{I}$ be a w^*-continuous derivation, then $d = (\pi_*)^* \circ D \circ q :$

$\mathcal{A} \longrightarrow \mathcal{J}$ is a w^*-continuous derivation, if $j_* \in \mathcal{J}_*$ then we have

$$\begin{aligned}
\langle j_*, d(ab)\rangle &= \langle j_*, (\pi_*)^*(D \circ q(ab))\rangle \\
&= \langle j_*, (\pi_*)^*(D(a+\mathcal{I})(b+\mathcal{I}))\rangle \\
&= \langle j_*, (\pi_*)^*((a+\mathcal{I}) \cdot D(b+\mathcal{I}) + D(a+\mathcal{I}) \cdot (b+\mathcal{I})\rangle \\
&= \langle \pi_*(j_*), (a+\mathcal{I}) \cdot D(b+\mathcal{I}) + D(a+\mathcal{I}) \cdot (b+\mathcal{I})\rangle \\
&= \langle \pi_*(j_*) \cdot (a+\mathcal{I}), D(b+\mathcal{I})\rangle + \langle (b+\mathcal{I}) \cdot \pi_*(j_*), D(a+\mathcal{I})\rangle \\
&= \langle \pi_*(j_*) \cdot a, D(b+\mathcal{I})\rangle + \langle b \cdot \pi_*(j_*), D(a+\mathcal{I})\rangle \\
&= \langle \pi_*(j_* \cdot a), D(b+\mathcal{I})\rangle + \langle \pi_*(b \cdot j_*), D(a+\mathcal{I})\rangle \\
&= \langle j_* \cdot a, (\pi_*)^*(D(b+\mathcal{I}))\rangle + \langle b \cdot j_*, (\pi_*)^*(D(a+\mathcal{I}))\rangle \\
&= \langle j_*, a \cdot (\pi_*)^*(D \circ q(b)) + (\pi_*)^*(D \circ q(a)) \cdot b\rangle \\
&= \langle j_*, a \cdot d(b) + d(a) \cdot b\rangle.
\end{aligned}$$

So there is an element $\lambda \in \mathcal{J}$ such that $d(a) = a \cdot \lambda - \lambda \cdot a$ $(a \in \mathcal{A})$. Let m be the restriction of λ on $\mathcal{I}_*$, then $m \in \mathcal{I}$ and for $i_* \in \mathcal{I}_*$ we have

$$\begin{aligned}
\langle i_*, a \cdot m - m \cdot a\rangle &= \langle i_* \cdot a - a \cdot i_*, m\rangle \\
&= \langle i_* \cdot a - a \cdot i_*, \lambda\rangle \\
&= \langle i_*, a \cdot \lambda - \lambda \cdot a\rangle \\
&= \langle i_*, (\pi_*)^* \circ D \circ q(a)\rangle \\
&= \langle \pi_*(i_*), D \circ q(a)\rangle \\
&= \langle \pi_*(i_*), D(a+\mathcal{I})\rangle = 0.
\end{aligned}$$

The reason for the last equality is that π_* is the projection on ${}^{\perp}\mathcal{I}$, so if $i_* \in \mathcal{I}_*$ and since $\mathcal{I}_* = \mathcal{A}_*/{}^{\perp}\mathcal{I}$, i_* is not in ${}^{\perp}\mathcal{I}$ so $\pi_*(i_*) = 0$. Therefore $a \cdot m = m \cdot a$ for each $(a \in \mathcal{A})$. Hence by the assumption, there exists a net $(\kappa_\alpha)_\alpha \subseteq \mathcal{A}$ such that for any α, we have that $\kappa_\alpha|_{\mathcal{I}_*} = m$ and $lim_\alpha a \cdot \kappa_\alpha - \kappa_\alpha \cdot a = 0$ $(a \in \mathcal{A})$. Let τ_α be the restriction of κ_α on $\mathcal{J}_*$, for every α. Then $(\tau_\alpha)_\alpha \subseteq \mathcal{J}$ and $\lambda - \tau_\alpha = 0$ on $\mathcal{I}_*$. Therefore $\lambda - \tau_\alpha \in \mathcal{J}/\mathcal{I}$.

Now let x be in $(\mathcal{J}/\mathcal{I})_*$. Then there is a $j_* \in \mathcal{J}_*$ such that $\pi_*(j_*) = x$, so we have

$$\begin{aligned}
\langle x, D(a+\mathcal{I})\rangle = \langle \pi_*(j_*), D(a+\mathcal{I})\rangle &= \langle j_*, a \cdot \lambda - lim_\alpha(a \cdot \tau_\alpha - \tau_\alpha \cdot a) - \lambda \cdot a\rangle \\
&= lim_\alpha \langle j_*, a \cdot \lambda - a \cdot \tau_\alpha + \tau_\alpha \cdot a - \lambda \cdot a\rangle \\
&= lim_\alpha \langle j_*, a \cdot (\lambda - \tau_\alpha) - (\lambda - \tau_\alpha) \cdot a\rangle \\
&= \langle j_*, lim_\alpha a \cdot (\lambda - \tau_\alpha) - (\lambda - \tau_\alpha) \cdot a\rangle.
\end{aligned}$$

If $j_* \in^{\perp} \mathcal{I}$ then by definition of π_* we have $\pi_*(j_*) = j_*$, and if j_* is not in ${}^{\perp}\mathcal{I}$ then $\pi_*(j_*) = 0$. In the first case we have

$$\begin{aligned}
\langle j_*, lim_\alpha a \cdot (\lambda - \tau_\alpha) - (\lambda - \tau_\alpha) \cdot a\rangle &= \langle \pi_*(j_*), lim_\alpha a \cdot (\lambda - \tau_\alpha) - (\lambda - \tau_\alpha) \cdot a\rangle \\
&= \langle x, lim_\alpha a \cdot (\lambda - \tau_\alpha) - (\lambda - \tau_\alpha) \cdot a\rangle.
\end{aligned}$$

Hence

$$D(a+\mathcal{I}) = lim_\alpha a \cdot (\lambda - \tau_\alpha) - (\lambda - \tau_\alpha) \cdot a.$$

This means that D is an approximate inner derivation. In the second case D is also an approximately inner derivation. So we conclude that $\mathcal{A}/\mathcal{I}$ is approximately ideally Connes amenable. □

Theorem 2.2. *Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{I}$ be a w^*-closed two-sided ideal in $\mathcal{A}$. If $\mathcal{A}$ is an approximate $\mathcal{I}$-Connes amenable then $\mathcal{H}^1_{w^*}(\mathcal{I},\mathcal{I})=\{0\}$.*

Proof. Let $D:\mathcal{I}\longrightarrow\mathcal{I}$ be a w^*-continuous derivation and $J:\mathcal{A}\longrightarrow\mathcal{I}$ be natural projection $\mathcal{A}$-bimodule morphism. Then $D\circ J:\mathcal{A}\longrightarrow\mathcal{I}$ is a w^*-continuous derivation. Since $\mathcal{A}$ is approximate $\mathcal{I}$-Connes amenable, $D\circ J$ is approximately inner, therefore D is approximately inner, thus $\mathcal{H}^1_{w^*}(\mathcal{I},\mathcal{I})=\{0\}$. □

Similarly to ([8] Theorem 1.6) we can state the following theorem for dual Banach algebras.

Theorem 2.3. *Let $\mathcal{A}$ be dual Banach algebra and $(X_*)^*=X$ be a dual Banach $\mathcal{A}$-bimodule and $(Y_*)^*=Y$ be a w^*-closed submodule of X. If every derivation from $\mathcal{A}$ to Y, and every derivation from $\mathcal{A}$ to $(X_*/Y_*)^*$ are approximately inner, then every derivation from $\mathcal{A}$ to X is approximately inner.*

Proof. Let $D:\mathcal{A}\longrightarrow X$ be a w^*-continuous derivation, and $i:Y_*\longrightarrow X_*$ be the natural embedding $\mathcal{A}$-bimodule morphism, then $i^*\circ D:\mathcal{A}\longrightarrow Y$ is a w^*-continuous derivation. Due to the assumption there is a net $(\xi_\alpha)_\alpha\subseteq Y$ such that $i^*\circ D(a)=lim_\alpha a\cdot\xi_\alpha-\xi_\alpha\cdot a$. Set $D_\alpha(a)=a\cdot\xi_\alpha-\xi_\alpha\cdot a$, therefore $i^*\circ D=lim_\alpha D_\alpha$. Without loss of generality we can suppose that $(\xi_\alpha)\subseteq X$. Now define $d:\mathcal{A}\longrightarrow(Y_*)^\perp$ by $d=D-lim_\alpha D_\alpha$, therefore d is a w^*-continuous derivation and so there is a net $(\eta_\beta)\subseteq(Y_*)^\perp=(X_*/Y_*)^*\subseteq X$, such that $d(a)=a\cdot\eta_\beta-\eta_\beta\cdot a$. Set $\xi_{\alpha,\beta}=\xi_\alpha+\eta_\beta$, such that $lim_\alpha lim_\beta\xi_{\alpha,\beta}=lim_\alpha\xi_\alpha+lim_\beta\eta_\beta$. Now by using suitable subnets and by using the iterated limit[13], there is a net $(\xi_\gamma)_\gamma\subseteq X$ such that $lim_\gamma\xi_\gamma=lim_\alpha lim_\beta\xi_{\alpha,\beta}$. Since $D=d+lim_\alpha D_\alpha$ we have

$$\begin{aligned}D(a)&=d(a)+lim_\alpha D_\alpha(a)\\&=lim_\beta a\cdot\eta_\beta-\eta_\beta\cdot a+lim_\alpha a\cdot\xi_\alpha-\xi_\alpha\cdot a\\&=lim_\alpha lim_\beta a\cdot(\eta_\beta+\xi_\alpha)-(\eta_\beta+\xi_\alpha)\cdot a\\&=lim_\alpha lim_\beta a\cdot\xi_{\alpha,\beta}-\xi_{\alpha,\beta}\cdot a\\&=lim_\gamma a\cdot\xi_\gamma-\xi_\gamma\cdot a.\end{aligned}$$

□

Corollary 2.1. *Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{I},\mathcal{J}$ be w^*-closed two-sided ideals in $\mathcal{A}$ such that $\mathcal{J}\subseteq\mathcal{I}$. If $\mathcal{A}$ is approximate $\mathcal{J}$-Connes amenable and every derivation from $\mathcal{A}$ into $(\mathcal{I}_*/J_*)^*$ is approximately inner, then $\mathcal{A}$ is approximately $\mathcal{I}$-Connes amenable.*

Corollary 2.2. *Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{I}$ be a w^*-closed two-sided ideal in $\mathcal{A}$. If every derivation from $\mathcal{A}$ into $(\mathcal{A}_*/\mathcal{I}_*)^*=(\mathcal{I}_*)^\perp$ is approximately inner, then every w^*-continuous derivation from $\mathcal{A}$ to $\mathcal{A}$ is approximately inner.*

Theorem 2.4. *Let $\mathcal{A}$ be Arens regular dual Banach algebra and $\mathcal{I}$ be a w^*-closed two-sided ideal in $\mathcal{A}$. If every derivation from $\mathcal{I}$ into $\mathcal{I}$ is approximately inner, then $\mathcal{A}$ is approximately $\mathcal{I}$-Connes amenable.*

Proof. Let $D:\mathcal{A}\longrightarrow\mathcal{I}$ be a w^*-continuous derivation and $i:\mathcal{I}\longrightarrow\mathcal{A}$ be an embedding map. Then $D\circ i:\mathcal{I}\longrightarrow\mathcal{I}$ is a w^*-continuous derivation, so there is a net $(\xi_\alpha)_\alpha\subseteq\mathcal{I}$ such that

$D\circ i(a)=lim_\alpha a\cdot\xi_\alpha-\xi_\alpha\cdot a,\qquad a\in\mathcal{A}.$

Since every derivation from $\mathcal{I}$ into $\mathcal{I}$ is approximately inner, therefore by [9], $\mathcal{I}$ has an approximate identity. Let $(e_\beta)_\beta$ be its approximate identity, and $i_*\in\mathcal{I}_*$ then $lim_\beta e_\beta\cdot i_*=i_*$ and $lim_\beta i_*\cdot e_\beta=i_*$.

So we get

$$\begin{aligned}\langle i_*, D(a)\rangle &= lim_\beta \langle i_* \cdot e_\beta, D(a)\rangle \\ &= lim_\beta \langle i_*, D(e_\beta a) - D(e_\beta)\cdot a\rangle \\ &= lim_\beta lim_\alpha (\langle i_*, e_\beta a\cdot \xi_\alpha - \xi_\alpha \cdot e_\beta a\rangle - \langle a\cdot i_*, e_\beta \cdot \xi_\alpha - \xi_\alpha \cdot e_\beta\rangle) \\ &= lim_\alpha lim_\beta (\langle i_* e_\beta a, \xi_\alpha\rangle - \langle a i_* e_\beta, \xi_\alpha\rangle) \\ &= lim_\alpha (\langle i_*, a\cdot \xi_\alpha - \xi_\alpha \cdot a\rangle).\end{aligned}$$

Since $\mathcal{A}$ is Arens regular we can change the order of limits. Therefore $D(a) = lim_\alpha(a\cdot\xi_\alpha - \xi_\alpha\cdot a)$ for all $a \in \mathcal{A}$.

Hence $\mathcal{A}$ is approximately $\mathcal{I}$-Connes amenable. □

Theorem 2.5. *Let $\mathcal{A}$ be dual Banach algebra and $\mathcal{I}$ be a w^*-closed two-sided ideal in $\mathcal{A}$ with a bounded approximate identity. Suppose $\mathcal{A}$ is approximate ideally Connes amenable, then $\mathcal{I}$ is approximate ideally Connes amenable.*

Proof. Assume that $\mathcal{J}$ is a *weak**-closed two-sided ideal of $\mathcal{I}$. Then, $\mathcal{J}$ is an ideal of $\mathcal{A}$ which is dual. Let $D : \mathcal{I} \longrightarrow \mathcal{J}$ be a w^*-w^*-continuous derivation. Let $(e_\alpha)_\alpha$ be approximate identity for $\mathcal{I}$. Due to [18, Proposition 2.1.6], there exists a unique extension of D to a derivation $\tilde{D} : \mathcal{A} \longrightarrow \mathcal{J}; a \mapsto w^* - \lim_\alpha (D(ae_\alpha) - aD(e_\alpha))$. Let $a_k \xrightarrow{w^*} 0$ in $\mathcal{A}$. For $j_* \in \mathcal{J}_*$, we get

$$\langle \tilde{D}(a_k), j_*\rangle = \lim_\alpha \langle D(a_k e_\alpha) - a_k D(e_\alpha), j_*\rangle = \lim_\alpha \langle D(a_k e_\alpha), j_*\rangle - \lim_\alpha \langle a_k D(e_\alpha), j_*\rangle$$

Since D is w^*-w^*-continuous and $a_k D(e_\alpha) \xrightarrow{w^*} 0$, the right-hand side of the last equality tends to zero. Hence, $\tilde{D}$ is w^*-continuous. Since $\mathcal{A}$ is approximately ideally Connes amenable, $\tilde{D}$ and so D is an approximately inner derivation. Therefore $\mathcal{I}$ is approximately ideally Connes amenable. □

Let $\mathcal{A}$ be a non-unital Banach algebra. Then $\mathcal{A}^{\#} = \mathcal{A}\oplus\mathbb{C}$, the unitization of $\mathcal{A}$, is a unital Banach algebra which contains $\mathcal{A}$ as a closed ideal. Our next proposition is an analogue of [8, Proposition 1.14], whose proof carries over almost verbatim.

Theorem 2.6. *Let $\mathcal{A}$ be dual Banach algebra. Then $\mathcal{A}$ is approximately ideally Connes amenable if and only if $\mathcal{A}^\sharp$ is approximately ideally Connes amenable.*

3 Ideal Connes amenability of discrete Beurling algebras

Let G be a locally compact group and ω be a Borel measurable function $\omega : G \longrightarrow [1,\infty)$ such that $\omega(s+t) \leq \omega(s).\omega(t)$ and $\omega(e) = 1$, where e is the identity element of G. Then ω is called a weight function on G. The Beurling algebra (weighted group algebra) $L^1(G,\omega)$ is defined as the set of all (equivalence classes of) measurable functions $f : G \longrightarrow \mathbb{C}$ such that

$$\|f\|_\omega = \int_G \mid f(x) \mid \omega(x)dx < \infty.$$

Let ω be a weight on discrete group G, then $M(G,\omega) \cong l^1(G,\omega)$, so $l^1(G,\omega)$ is dual Banach algebra with the predual $C_0(G, 1/\omega)$. We say that ω is diagonaly bounded if $sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$.

Theorem 3.1. [4] *Let G be a discrete group and ω be a weight on G, and let $A = l^1(G,\omega)$. Then the following are equivalent*

(i) A is Connes amenable, with respect to the predual $C_0(G, 1/\omega)$;

(ii) A is amenable.

Theorem 3.2. [10] *Let G be a discrete group and ω be a weight on G. Then $l^1(G,\omega)$ is amenable if and only if G is amenable group and ω is diagonally bounded.*

As a result of two previous theorems $l^1(G,\omega)$ is Connes amenable if and only if G is an amenable group and ω is diagonally bounded. In this section we want to study an ideal Connes amenability of $l^1(G,\omega)$.

Theorem 3.3. *Let G be a discrete group and ω be a diagonally bounded weight on G. Then every w^*-continuous derivation from $l^1(G,\omega)$ to itself is inner.*

Proof. Let $\delta_g \in l^1(G,\omega)$ be the point mass at $g \in G$. Due to ([20], Proposition 7.4) we have

$$D(\delta_g) = \mu * \delta_g - \delta_g * \mu \qquad \mu \in l^1(G,\omega).$$

Let $f \in l^1(G,\omega)$. Using ([20], Lemma 2.3) we can find a net $\{f_\alpha\}_{\alpha\in\Lambda}$ from $lin\{\delta_g : g \in G\}$, such that $f_\alpha \longrightarrow f$ in a strong operator topology. Then for each f_α

$$D(f_\alpha) = \mu * f_\alpha - f_\alpha * \mu.$$

If we show that $D(f_\alpha) \longrightarrow D(f)$ in w^*-topology and

$$\mu * f_\alpha - f_\alpha * \mu \longrightarrow \mu * f - f * \mu,$$

in w^*-topology, then we are done. Since $f_\alpha \longrightarrow f$ in a strong operator topology, so for each $h \in l^1(G,\omega)$ we have $f_\alpha * h \longrightarrow f * h$ in the norm topology, so $f_\alpha * h \longrightarrow f * h$ in w^*-topology. Let e be the identity element of G. If we replace h by δ_e, then we have $f_\alpha \longrightarrow f$ in w^*-topology. On the other hand since $f_\alpha \longrightarrow f$ in a strong operator topology, then $\mu * f_\alpha \longrightarrow \mu * f$ in the norm topology, hence $\mu * f_\alpha \longrightarrow \mu * f$ in w^*-topology. In a similar manner we can show that $f_\alpha * \mu \longrightarrow f * \mu$ in w^*-topology. So

$$\mu * f_\alpha - f_\alpha * \mu \longrightarrow \mu * f - f * \mu,$$

in w^*-topology. Since D is w^*-continuous if $f_\alpha \longrightarrow f$ in w^*-topology we conclude that $D(f_\alpha) \longrightarrow D(f)$ in w^*-topology, and the proof is completed. □

Lemma 3.1. *Let G be a discrete group, and ω be a weight on G. Suppose that the map D from $\{\delta_x\}_{x\in G}$ to $l^1(G,\omega) = (C_0(G,1/\omega))^*$ has the following properties*

$$D(\delta_{xy}) = D(\delta_x) * \delta_y + \delta_x * D(\delta_y) \ \ (x, y \in G), \tag{3.1}$$

$$\|D(\delta_x)\| \le C\omega(x) \qquad (x \in G),$$

where $C > 0$ is a constant. Then D can be extended to a bounded derivation from $l^1(G,\omega)$ to $l^1(G,\omega)$.

Proof. We first extend D to the linear span of $\{\delta_x\}_{x\in G}$ by linearity. The linear mapping $D(f * g)$ satisfies the derivation relation

$$D(f * g) = D(f) * g + f * D(g)$$

for f, g from the generating set $\{\delta_x\}_{x\in G}$ by (3.1). So the relation still holds for all $f, g \in lin\{\delta_x\}_{x\in G}$. By Theorem 3.3 we conclude that $lin\{\delta_x\}_{x\in G}$ is w^*-dense in $l^1(G,\omega)$, so we can extend D to $\overline{D}$ on $l^1(G,\omega)$, which is still a derivation. □

Theorem 3.4. *Let G be a discrete group, and ω be a weight on G. If there exist a function, $\Psi : G \longrightarrow \mathbb{R}$, and $x_0 \in G$ such that ω is bounded away from zero on the conjugacy class $\{y^{-1}x_0^{-1}y\}_{y\in G}$ with the following properties*

$$\sum_{x,y\in G} |\Psi(yx^{-1}) - \Psi(x^{-1}y)| < \infty \tag{3.2}$$

$$sup_{y\in G}|\Psi(y^{-1}x_0^{-1}y)|\omega(y^{-1}x_0^{-1}y) = \infty, \tag{3.3}$$

then $l^1(G,\omega)$ is not ideally Connes amenable.

Proof. As usual, to show that $l^1(G,\omega)$ is not ideally Connes amenable, we construct a non-inner w^*-continuous derivation $D : l^1(G,\omega) \longrightarrow l^1(G,\omega)$. We first define the operator $D : \{\delta_x\}_{x\in G} \longrightarrow l^1(G,\omega)$, in the following way

$$D(\delta_x)(y) = \Psi(yx^{-1}) - \Psi(x^{-1}y) = (\Psi * \delta_x)(y) - (\delta_x * \Psi)(y) \ \ (x, y \in G).$$

It is easy to see that D really ranges in $l^1(G,\omega)$ because

$$\begin{aligned} \parallel D(\delta_x) \parallel_{l^1(G,\omega)} &= \sum_{y\in G} |D(\delta_x)(y)|\omega(y) \\ &\leq \sum_{y\in G} |\Psi(yx^{-1}) - \Psi(x^{-1}y)|\omega(y) \\ &\leq sup_{y\in G}\omega(y) \sum_{y\in G} |\Psi(yx^{-1}) - \Psi(x^{-1}y)| < \infty. \end{aligned}$$

By Lemma 3.1, D can be extended to a bounded derivation from $l^1(G,\omega)$ to $l^1(G,\omega)$. Let δ_{x_i} be a net in $\{\delta_x\}_{x\in G}$ such that $\delta_{x_i} \longrightarrow 0$ in w^*-topology. Then for every $g \in C_0(G, 1/\omega)$ we have $g(x_i) \longrightarrow 0$. If we show that $D(\delta_{x_i}) \longrightarrow 0$ in w^*-topology, then the proof of w^*-continuity of D is completed. If $l^1(G,\omega)$ is a dual Banach algebra and multiplication in dual Banach algebra is separately w^*-continuous, and if $\delta_{x_i} \longrightarrow 0$ in w^*-topology, then $\Psi * \delta_{x_i}$ and $\delta_{x_i} * \Psi$ tend to zero in w^*-topology. Thus $D(\delta_{x_i}) \longrightarrow 0$, in w^*-topology. Now we show that D has propery (3.1). Indeed ,we have

$$\begin{aligned} D(\delta_{xy})(t) &= \Psi(ty^{-1}x^{-1}) - \Psi(y^{-1}x^{-1}t) \\ &= (\Psi(ty^{-1}x^{-1}) - \Psi(x^{-1}ty^{-1})) + (\Psi(x^{-1}ty^{-1}) - \Psi(y^{-1}x^{-1}t)) \\ &= D(\delta_x)(ty^{-1}) + D(\delta_y)(x^{-1}t) \\ &= (D(\delta_x) * \delta_y)(t) + (\delta_x * D(\delta_y))(t) \qquad (x, y, t \in G). \end{aligned}$$

So D can be extended in the desired way. To finish the proof we only need to show that the extended derivation D is not inner. Suppose the contrary, that D is inner. Then there exists a function $\phi \in l^1(G,\omega)$ such that $D(h) = \phi * h - h * \phi$, for $h \in l^1(G,\omega)$. In particular,

$D(\delta_x)(y) = (\phi * \delta_x)(y) - (\delta_x * \phi)(y) = \phi(yx^{-1}) - \phi(x^{-1}y) \quad (x, y \in G)$. On the other hand, by the definition of D we have $D(\delta_x)(y) = \Psi(yx^{-1}) - \Psi(x^{-1}y)$. Taking $x = x_0^{-1}y$, we obtain

$$\Psi(x_0^{-1}) - \Psi(y^{-1}x_0^{-1}y) = D(\delta_{x_0^{-1}y})(y) = \phi(x_0^{-1}) - \phi(y^{-1}x_0^{-1}y) \quad (y \in G),$$

which implies $\phi(y^{-1}x_0^{-1}y) = \Psi(y^{-1}x_0^{-1}y) + \phi(x_0^{-1}) - \Psi(x_0^{-1}) \quad (y \in G)$. Then using (3.3) and the fact that $inf_{y\in G}\omega(y^{-1}x_0^{-1}y)>0$, we have

$$\begin{aligned}\parallel \phi \parallel_{l^1(G,\omega)} &= \sum_{x\in G}|\phi(x)|\omega(x) \geqslant \sum_{y\in G}|\phi(y^{-1}x_0^{-1}y)|\omega(y^{-1}x_0^{-1}y)\\ &= \sum_{y\in G}|\Psi(y^{-1}x_0^{-1}y)+(\phi(x_0^{-1})-\Psi(x_0^{-1}))|\omega(y^{-1}x_0^{-1}y)\\ &> \sum_{y\in G}|\Psi(y^{-1}x_0^{-1}y)\omega(y^{-1}x_0^{-1}y)+\sum_{y\in G}|\phi(x_0^{-1})-\Psi(x_0^{-1})|\omega(y^{-1}x_0^{-1}y)=\infty,\end{aligned}$$

which contradicts to $\phi\in l^1(G,\omega)$. This proves that D is not inner, and hence $l^1(G,\omega)$ is not ideally Connes amenable. □

The idea of the following theorem is similar to ([23] Theorem 3.1).

Theorem 3.5. *Let G be a discrete Abelian group and ω be a weight on G. Then $l^1(G,\omega)$ is ideally Connes amenable if and only if there does not exist a non-zero continuous group homomorphism $\phi: G\longrightarrow \mathbb{C}$ such that $sup_{t\in G}|\phi(t)|\omega(t^{-1})<\infty$.*

Proof. If $l^1(G,\omega)$ is not ideally Connes amenable, then there is a non-zero w^*-continuous derivation $D: l^1(G,\omega)\longrightarrow l^1(G,\omega)$. Define

$$\phi(t)=\langle\delta_{t^{-1}},D(\delta_t)\rangle$$

Since $l^1(G,\omega)=(C_0(G,1/\omega)^*$, and $\delta_{t^{-1}}\in C_0(G,1/\omega)$, so $\phi(t)$ is well defined. Let e be the identity element of G, then $\phi(e)=0$, also

$$\begin{aligned}\phi(ab) &= \langle\delta_{(ab)^{-1}},D(\delta_{ab})\rangle\\ &= \langle\delta_{b^{-1}a^{-1}},D(\delta_a*\delta_b)\rangle\\ &= \langle\delta_{b^{-1}}*\delta_{a^{-1}},\delta_a*D(\delta_b)+D(\delta_a)*\delta_b\rangle\\ &= \langle\delta_{b^{-1}}*\delta_{a^{-1}},\delta_a*D(\delta_b)\rangle+\langle\delta_{b^{-1}}*\delta_{a^{-1}},D(\delta_a)*\delta_b\rangle\\ &= \langle\delta_{b^{-1}},D(\delta_b)\rangle+\langle\delta_{a^{-1}},D(\delta_a)\rangle\\ &= \phi(b)+\phi(a).\end{aligned}$$

Since D is w^*-continuous, ϕ is a continuous group homomorphism from G to $\mathbb{C}$, and

$$\begin{aligned}|\phi(t)| &= |\langle\delta_{t^{-1}},D(\delta_t)\rangle|\\ &\leq \parallel D(\delta_t)\parallel_{l^1(G,\omega)} . \parallel \delta_{t^{-1}}\parallel_{C_0(G,1/\omega)} \leq \parallel D(\delta_t)\parallel_{l^1(G,\omega)} .1/\omega(t^{-1}).\end{aligned}$$

So we have $\sup_{t\in G}|\phi(t)|\omega(t^{-1})\leq\parallel D(\delta_t)\parallel_{l^1(G,\omega)}<\infty$.

To prove the converse, we assume $\phi: G\longrightarrow\mathbb{C}$ is a continuous non-zero group homomorphism that satisfies $\sup_{t\in G}|\phi(t)|\omega(t^{-1})\leq m$, for some $m<\infty$.

Let B be a finite set containing e in G. For each $h\in l^1(G,\omega)$, we define

$$D(h)(t)=\textstyle\sum_{\xi\in B}\phi(t\xi)h(t\xi).$$

The map D ranges in $l^1(G,\omega)$,

$$\begin{aligned}\|D(h)(t)\|_{l^1(G,\omega)} &= \sum_{t\in G}\sum_{\xi\in B}\phi(t\xi)h(t\xi)\omega(t)\\ &\leq \sum_{t\in G}\sum_{\xi\in B}\phi(t\xi)h(t\xi)\omega(t\xi)\omega(\xi^{-1}).\end{aligned}$$

Since $sup_{t\in G}|\phi(t)|\omega(t^{-1}) < \infty$, we get $sup_{t\in G}|\phi(t)| \leq sup_{t\in G}|\phi(t)|\omega(t^{-1}) < m$, for some $m > 0$. Let $A = max\{\omega(s^{-1}); s \in B\}$, then we have $\|D(h)(t)\|_{l^1(G,\omega)} \leq mA\|h\|_{l^1(G,\omega)}$.

Let h_α be a net in $l^1(G,\omega)$ that tends to zero in w^*-topology of $l^1(G,\omega)$. Then for every $g \in C_0(G, 1/\omega)$, we have $g(h_\alpha) \longrightarrow 0$, so for every $t \in G$, $\delta_t(h_\alpha) \longrightarrow 0$ and so $\sum_{y\in G}\delta_t(y)(h_\alpha)(y) \longrightarrow 0$. It means that $h_\alpha(t) \longrightarrow 0$. If we prove that $D(h_\alpha) \longrightarrow 0$ in w^*-topology, then it leads to w^*-continuity of D.

Indeed, $g(D(h_\alpha))(y) = \sum_{y\in G} g(y)D(h_\alpha)(y) = \sum_{y\in G} g(y)\sum_{\xi\in B}\phi(t\xi)h_\alpha(t\xi) \longrightarrow 0$, so $D(h_\alpha) \longrightarrow 0$, in w^*-topology.

Let $f, g \in l^1(G,\omega)$, then we have

$$\begin{aligned}
D(f * g)(t) &= \sum_{\xi\in B}\phi(t\xi)(f * g)(t\xi)\\
&= \sum_{\xi\in B}\phi(t\xi)(\sum_{y\in G} f(y)g(y^{-1}t\xi)\\
&= \sum_{\xi\in B}\sum_{y\in G} f(y)(\phi(y^{-1}t\xi) + \phi(y))g(y^{-1}t\xi)\\
&= \sum_{y\in G}\sum_{\xi\in B} f(y)\phi(y^{-1}t\xi)g(y^{-1}t\xi) + \sum_{y\in G}\sum_{\xi\in B} f(y)\phi(y)g(y^{-1}t\xi)\\
&= \sum_{y\in G} f(y)D(g)(y^{-1}t) + \sum_{y\in G} D(f)(y)g(y^{-1}t)\\
&= f * D(g)(t) + D(f) * g(t)\\
&= (f * D(g) + D(f) * g)(t) \qquad (t \in G).
\end{aligned}$$

Therefore $D(f * g) = f * D(g) + D(f) * g$, for all $f, g \in l^1(G,\omega)$; i.e. $D : l^1(G,\omega) \longrightarrow l^1(G,\omega)$, is a non-zero w^*-continuous derivation. Thus $l^1(G,\omega)$ is not ideally Connes amenable. □

Example 1. Let $G = Z$ (where Z is the discrete additive group of all integers). All group homomorphisms from Z to $\mathbb{C}$ are of the form $\phi(n) = nc_0$ where $c_0 \in \mathbb{C}, n \in Z$. Therefore for any weight ω on Z, $l^1(Z,\omega)$ is ideally Connes amenable if and only if $sup_{n\in N} n\omega(-n) = \infty$.

For example $\omega_\alpha(n) = (1 + |n|)^\alpha$ is a weight on Z if $\alpha > 0$. Then

$$sup_{n\in N} n\omega(-n) = \infty.$$

Thus, we conclude that $l^1(Z,\omega_\alpha)$ is ideally Connes amenable if and only if $\alpha > 0$.

Theorem 3.6. *Let G be a discrete Abelian group and ω be a weight on G. Then $l^1(G,\omega)$ is ideally Connes amenable if and only if there does not exist a non-zero continuous group homomorphism $\phi : G \longrightarrow \mathbb{R}$ such that*

$$sup_{t\in G}|\phi(t)|\omega(t^{-1}) < \infty. \tag{3.4}$$

Proof. The necessity is trivial. For the sufficiency, suppose that $l^1(G,\omega)$ is not ideally Connes amenable. Then by theorem 3.5, there is a continuous complex-valued non-zero homomorphism ϕ such that (3.4), holds. The real part ϕ_r and the imaginary part ϕ_i of ϕ are both still continuous group homomorphisms, they satisfy the same inequality (3.4), and they are real-valued. If $\phi \neq 0$, then at least one of ϕ_r and ϕ_i is non-zero. Therefore, there exists a non-zero continuous real-valued group homomorphism such that (3.4) holds. □

Corollary 3.1. *Let G be a discrete Abelian group and ω be a weight on G. If for each $t \in G$ we have*

$$sup_{n\in N}\omega(t^{-n})n = \infty, \tag{3.5}$$

then $l^1(G,\omega)$ is ideally Connes amenable.

Proof. Let $\phi : G \longrightarrow \mathbb{R}$ be a non-zero group homomorphism and let $s \in G$ be such that $\phi(s) \neq 0$. We have $\phi(s^n) = n\phi(s)$, for $n \in N$. If (3.5) holds for $t = s$, then

$$sup_{t\in G}|\phi(t)|\omega(t^{-1}) \geq sup_{n\in N}|\phi(s^n)|\omega(s^{-n}) = sup_{n\in N}|\phi(s)|n\omega(s^{-n}) = \infty.$$

So (3.4) does not hold for any non-zero homomorphism ϕ. By Theorem 3.6, $l^1(G, \omega)$ is ideally Connes amenable. □

In this section we use notations and definitions of [1]. Let $\mathcal{A}$ be a Banach algebra and E be an $\mathcal{A}$-bimodule. ${}_{\mathcal{A}}WAP_{\mathcal{A}}(E)$ is the set of all elements $x \in E$ for which the maps $R_x^{\mathcal{A}} : \mathcal{A} \longrightarrow E$, $a \longmapsto a \cdot x$ and $L_x^{\mathcal{A}} : \mathcal{A} \longrightarrow E$, $a \longmapsto x \cdot a$, are both weakly compact. ${}_{\mathcal{A}}WAP_{\mathcal{A}}(E)$ is a closed sub-bimodule of E.

Given a Banach algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule X, we write $F_{\mathcal{A}}(X)_*$ for the $\mathcal{A}$-bimodule ${}_{\mathcal{A}}WAP_{\mathcal{A}}(X^*)$, where X^* is equiped with the usual bimodule action induced by X. We define $F_{\mathcal{A}}(X)$ to be the dual $\mathcal{A}$-bimodule $(F_{\mathcal{A}}(X)_*)^*$. In the special case where $X = \mathcal{A}$, regarded as an $\mathcal{A}$-bimodule in the canonical way, we shall usually omit the subscripts, and simply use the notation $F(\mathcal{A})$.

We denote by $\eta_X : X \longrightarrow F_{\mathcal{A}}(X)$ the map obtained by composing the canonical inclusion of X in its second dual with the adjoint of the inclusion map ${}_{\mathcal{A}}WAP_{\mathcal{A}}(X^*) \hookrightarrow X^*$. Observe that η_X is a norm-continuous $\mathcal{A}$-bimodule map, as it is the composition of two such maps. $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow F(\mathcal{A})$ is a norm-continuous homomorphism with w^*-dense range. $(F_{\mathcal{A}}(\mathcal{A})_*)^* = F(\mathcal{A})$ is actually a dual Banach algebra.

Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{I}$ be a w^*-closed two-sided ideal of $\mathcal{A}$. Then $F_{\mathcal{A}}(\mathcal{I})$ is a w^*-closed two-sided ideal of $F(\mathcal{A})$.

Lemma 3.2. *Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{I}$ be a w^*-closed two-sided ideal of $\mathcal{A}$, and $D : \mathcal{A} \longrightarrow \mathcal{I}$ be a w^*-continuous derivation. Then there exists a unique w^*-continuous derivation $\overline{D} : F(\mathcal{A}) \longrightarrow F_{\mathcal{A}}(\mathcal{I})$ such that $\overline{D}\eta_{\mathcal{A}} = \eta_{\mathcal{I}} D$.*

Proof. It is obvious by ([1] Theorem 4.4). □

Lemma 3.3. *Let $\mathcal{A}$ be dual Banach algebra. If $\mathcal{A}$ is not ideally Connes amenable, then $F(\mathcal{A})$ is not ideally Connes amenable*

Proof. Assume the contrary that $F(\mathcal{A})$ is ideally Connes amenable. Let $\mathcal{I}$ be a w^*-closed two-sided ideal of $\mathcal{A}$, and $D : \mathcal{A} \longrightarrow \mathcal{I}$ be a w^*-continuous derivation. Then by lemma 3.2 there exists a unique w^*-continuous derivation $\overline{D} : F(\mathcal{A}) \longrightarrow F_{\mathcal{A}}(\mathcal{I})$ such that $\overline{D}\eta_{\mathcal{A}} = \eta_{\mathcal{I}} D$. Since $F(\mathcal{A})$ is ideally Connes amenable then there exists $E \in F_{\mathcal{A}}(\mathcal{I})$ such that $\overline{D}(\eta_{\mathcal{A}}(a)) = \eta_{\mathcal{A}}(a)\Box E - E\Box\eta_{\mathcal{A}}(a)$. Since the range of $\eta_{\mathcal{I}}$ is w^*-dense in $F(\mathcal{I})$, there is a bounded net $(i_\alpha)_\alpha$ in $\mathcal{I}$ such that $w^* - lim\eta_{\mathcal{I}}(i_\alpha) = E$.

Let $t \in F(\mathcal{I})_*$, then by definition of $\eta_{\mathcal{A}}$ we have $\langle \eta_{\mathcal{A}}(a), t\rangle = \langle t, a\rangle$ for all $a \in \mathcal{A}$. For $a, s \in \mathcal{A}$ we have

$$\begin{aligned}\langle \eta_{\mathcal{A}}(a) \cdot t, s\rangle &= \langle \eta_{\mathcal{A}}(a), t \cdot s\rangle \\ &= \langle t \cdot s, a\rangle \\ &= \langle a \cdot t, s\rangle.\end{aligned}$$

We get

$$\begin{aligned}\langle t, D(a)\rangle = \langle \eta_{\mathcal{I}}(D(a), t\rangle &= \langle \eta_{\mathcal{A}}(a)\Box E - E\Box\eta_{\mathcal{A}}(a), t\rangle \\ &= \langle \eta_{\mathcal{A}}(a), E \cdot t\rangle - \langle E, \eta_{\mathcal{A}}(a) \cdot t\rangle \\ &= \langle E \cdot t, a\rangle - \langle E, a \cdot t\rangle \\ &= w^* - lim(\langle \eta_{\mathcal{I}}(i_\alpha) \cdot t, a\rangle - \langle \eta_{\mathcal{I}}(i_\alpha), a \cdot t\rangle) \\ &= w^* - lim(\langle i_\alpha \cdot t, a\rangle - \langle a \cdot t, i_\alpha\rangle).\end{aligned}$$

Let $w^*-\lim i_\alpha = i$. Then we have $\langle t, D(a)\rangle = \langle t, a \cdot i - i \cdot a\rangle$, so we conclude that D is inner, that is contradiction, so $F(\mathcal{A})$ is not ideally Connes amenable □

Theorem 3.7. [15] *Suppose that ω is a weight function on locally compact group G such that $\omega \geq 1$ and diagonally bounded. Then G is amenable if and only if $L^1(G, \omega)$ is approximately amenable.*

Example 2. Let ω be a diagonally bounded weight function and G be an amenable discrete group. If there is a non-zero group homomorphism $\phi : G \longrightarrow \mathbb{R}$ such that

$$sup_{x\in G}|\Phi(x)|.\omega(x^{-1}) < \infty,$$

then by Theorem 3.5, $\mathcal{A} = l^1(G, \omega)$ is not ideally Connes amenable, but it is approximate amenable by Theorem 3.7.

Now as a corollary of ([21], Lemma 2.2), $F(\mathcal{A})$ is approximately Connes amenable, so it is approximately ideally Connes amenable. Now by Lemma 3.3, $F(\mathcal{A})$ is not ideally Connes amenable.

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