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# EURASIAN MATHEMATICAL JOURNAL

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## MIKHAIL L'VOVICH GOLDMAN

(to the 75th birthday)

Mikhail L'vovich Goldman was born on April 13, 1945 in Moscow. In 1963 he graduated from school in Moscow and entered the Physical Faculty of the M.V. Lomonosov Moscow State University (MSU) from which he graduated in 1969 and became a PhD student (1969–1972) at the Mathematical Department of this Faculty. In 1972 he has defended the PhD thesis, and in 1988 his DSc thesis "The study of spaces of differentiable functions of many variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor in Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he is a Professor of the S.M. Nikol'skii Mathematical Institute at the Peoples Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman are: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators. Main achievements: optimal embeddings of spaces with generalized smoothness, sharp conditions of the convergence of spectral expansions, descriptions of integral and differential properties of the generalized Bessel and Riesz-type potentials, sharp estimates for operators on cones and optimal envelopes for the cones of functions with properties of monotonicity. Professor M.L. Goldman has over 140 scientific publications in leading mathematical journals.

Under his scientific supervision, 8 candidate theses in Russia and 1 thesis in Kazakhstan were successfully defended. Some of his former students are now professors in Ethiopia, Columbia, Mongolia.

Participation in scientific and organizational activities of M.L. Goldman is well known. He is a member of the DSc Councils at RUDN and MSU, of the PhD Council in the Lulea Technical University (Sweden), a member of the Editorial Board of the Eurasian Mathematical Journal, an invited lector and visiting professor at universities of Russia, Germany, Sweden, UK etc., an invited speaker at many international conferences.

The mathematical community, friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Mikhail L'vovich Goldman on the occasions of his 75th birthday and wish him good health, happiness, and new achievements in mathematics and mathematical education.

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# ON STABILITY OF BASES IN HILBERT SPACES

**E.A. Larionov**

Communicated by M.L. Gol'dman

**Key words:** perturbation, compact operator, orthoprojector, isotropically non-compact sequence.

**AMS Mathematics Subject Classification:** 35P15.

**Abstract.** In a Hilbert space we consider a minimal and complete system asymptotically close to an almost normed unconditional basis and find conditions under which such system also forms an unconditional basis. The proof of this statement is based on a new criterion of compactness of linear operators proposed in this paper.

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## 1 Introduction

The stability of bases in Banach and Hilbert spaces under certain perturbations was studied in [1-10, 12, 13, 15]. It is known that if biorthogonal systems $(e_n)$ and $(q_n)$ form bases in a Hilbert space $H$ then for a complete system $(e'_n)$ the convergence of the series

$$\sum_{n,k=1}^{\infty} (e_n - e'_n, e_k - e'_k)\,(q_n, q_k) \tag{1.1}$$

implies that $(e'_n)$ is also a basis in $H$ [7].

If $(e_n)$ is an almost normed unconditional basis in $H$, $(e'_n)$ is minimally complete and

$$\sum_{n=1}^{\infty} \|e_n - e'_n\| < \infty, \tag{1.2}$$

then $(e'_n)$ is also an almost normed unconditional basis in $H$ [5]. Bases satifying condition (1.2) were investigated in [2, 8]. Condition (1.2) allows introducing a compact operator $T_0$ defined by the equality

$$T_0\left(\sum_{n=1}^{\infty} c_n e_n\right) = \sum_{n=1}^{\infty} c_n\,(e_n - e'_n)\,; \quad \sum_{n=1}^{\infty} |c_n|^2 < \infty. \tag{1.3}$$

If the operator $I - T_0$ is invertible the system $(e'_n)$ also forms an unconditional basis in $H$ [5].

## 2 Basic notions

If $L(\phi_n)_{n=1}^{\infty} = H$ and $\sum_{n=1}^{\infty} c_n\phi_n = 0$ implies $c_n = 0$ for all $n \in N$ we say that $(\phi_n)$ is a minimally complete system in $H$. Here $L$ is the closure of the linear span of corresponding elements.

If $L(\phi_n)_{n=1}^{\infty} = H$ and every element of system $(\phi_n)$ is outside the closure of all other elements of the system, we say that the system $(\phi_n)$ is minimal and complete in $H$. Note that a minimally complete system is not necessarily minimal and complete [7].

**Remark 1.** Let $L(\psi_n)_{n=1}^{\infty} = F \subset H$. In order to construct a minimally complete subsystem in $F$ one may start with $\psi_1$, then choose the element $\psi_{k_2}$ of the sequences $(\Psi_n)_2^{\infty}$ with the minimal number such that $\psi_1$ and $\psi_{k_2}$ are linearly independent, then the first $(\psi_{k_3})$ such that $(\psi_1)$, $(\psi_{k_2})$ and $(\psi_{k_3})$ are linearly independent and so on.

A bounded linear operator $A$ is a compact operator if it takes any sequence: $x_n \underset{n\to\infty}{\longrightarrow} 0$ to a sequence

$$Ax_n \underset{n\to\infty}{\Longrightarrow} 0 \tag{2.1}$$

Here the symbols $\to$ and $\Rightarrow$ denote the weak and strong convergence respectively.

If relation (2.1) holds for a certain unconditional basis does it follows that $A$ is a compact operator? It will be shown that the answer to this question is positive. In connection with the posed question we introduce the following notion.

**Definition 1.** An almost normed sequence $(\phi_n)$ is called isotropically non-compact if any infinitedimensional orthoprojector $P$ also generates a non-compact sequence $(P\phi_n)$.

We show that the set $M_0$ of all isotropically non-compact sequences contains any almost normed unconditional basis. Recall that a sequence $(\phi_n)$ is called almost normed if

$$0 < \underset{n\in N}{essinf} \, \|\phi_n\|; \; \sup_{n\in N} \|\phi_n\| < \infty. \tag{2.2}$$

## 3 Preliminary results

First we prove the following statement.

**Lemma 3.1.** *Let $(e_n)$ be an almost normed unconditional basis and $Q$ be an orthoprojector in $H$. Then the condition*

$$\lim_{n\to 0} \|Qe_n\| = 0 \tag{3.1}$$

*implies that* $\dim Q < \infty$.

*Proof.* Proof. Any almost normed unconditional basis $(e_n)$ forms an orthonormal basis with respect to a certain scalar product $(x, y)_1$ [5, 14]. Moreover, the norms $\|x\|_1 = \sqrt{(x, x)_1}$ and $\|x\| = \sqrt{(x, x)}$ are equivalent. Hence, without loss of generality we consider an orthonormal basis $(e_n)$. In virtue of (3.1) for any $\epsilon > 0$ there exists a number $n_0 \in N$ such that for all $n > n_0$

$$\|e_n - e_{n,1}\| < \varepsilon, \; e_{n,1} = Pe_n; \; n \geq n_0 + 1, \; P = 1 \ominus Q. \tag{3.2}$$

Let $\tilde{H}_1 = L(e_n)_{n_0+1}^{\infty}$; $H_1 = L(e_{n,1})_{n_0+1}^{\infty}$; $\tilde{H}_2 = H \ominus \tilde{H}_1$; $H_2 = H \ominus H_1$; $\tilde{P}_1(P_1)$ - be the orthoprojector on $\tilde{H}_1(H_1)$; $i = 1, 2$.

Let $L(\phi_n)_{n=1}^{\infty} = F \subset H$.

For all $x \in \tilde{H}_1$ we have $x = \sum_{n_0+1}^{\infty} (x, e_n)e_n$ and

$$P_1 x = \sum_{n=n_0+1}^{\infty} (x, e_n)P_1 e_n = \sum_{n=n_0+1}^{\infty} (x, e_n)e_{n,1}. \tag{3.3}$$

If the system $(e_{n,1})_{n_0+1}^{\infty}$ is minimally complete by (3.3) we have $H_2 \cap \tilde{H}_1 = \{0\}$ and $\dim P_2 = n_0$. Then from $Q \subset P_2$ it follows that $\dim Q \leq n_0$. Assuming the non-minimality of $(e_{n,1})_{n_0+1}^{\infty}$ we select a minimally complete subsystem $(e_{n,1})_1^{\infty}$ in $H_1$. Then we have

$$P_1 x = \sum_{k=1}^{\infty} (x, e_{n_k}) e_{n_k,1}. \tag{3.4}$$

Denote $H_3 = L(e_{n_k,1})_1^{\infty}$; $(\overline{n}_k) = N \backslash (n_k)$. If $\dim H_3 = m$ we obtain $\dim Q \leq n_0 + m$. Since $P_1 y \neq 0$ for all $0 \neq y \in \tilde{H}_1 = L(e_{n_k})_1^{\infty}$ the relation $(\hat{P}_1 P_1 x, y) = (P_1 x, \hat{P}_1 y) = (P_1 x, y) = (x, P_1 y) = 0$; $x \in H_1$ implies that $y = 0$ and we have $\overline{\hat{P}_1 H_1} = \hat{H}_1$. Here $\overline{\hat{P}_1 H_1}$ is the closure of $\hat{P}_1 H_1$. If $(e_{\overline{n}_k,1}) = (e_{n,1}) \backslash (e_{n_k,1})$ is a minimally complete system we also have that $\overline{\hat{P}_2 H_3} = \hat{H}_2 = L(e_{n_k})_1^{\infty}$ and from $\hat{H}_1 = \hat{H}_1 \oplus \hat{H}_2$ the relation

$$\overline{\tilde{P}_1 H_1} = \tilde{H}_1 \tag{3.5}$$

follows. If $(e_{\overline{n}_k,1})$ is a non-minimally complete system we select from it a minimally complete subsystem $(e_{\overline{n}_{k_i},1})$ in $H_3$. The orthoprojector $\hat{P}_{2,1}$ on $\hat{H}_{2,1} = L(e_{\overline{n}_{k_i}})$ maps $H_3$ on $\hat{H}_{2,1}$. If the complement $(e_{\overline{n}_{k_i},1})$ in $(e_{\overline{n}_k,1})$ is a minimally complete subsystems, we obtain $\overline{\hat{P}_2 H_3} = \hat{H}_2$ and again (3.5) takes place. The orthoprojector $\hat{P}_{1,1} = \hat{P} \oplus \hat{P}_{2,1}$ maps $H_1$ on $\hat{H}_{1,1} = \hat{H}_1 \oplus \hat{H}_{2,1}$. As a result of the sequential selection from $(e_{n,1})$ of minimally complete subsystems we have (3.5). Relation (3.5) means the completeness of the system $(\tilde{e}_{n,1})$: $\tilde{e}_{n,1} = \tilde{P}_1 e_{n,1}$; $n \geq n_0 + 1$ in $\tilde{H}_1$.

We consider (3.5) as a natural corollary of the condition $\|e_n - P_1 e_n\| \underset{n\to\infty}{\longrightarrow} 0$ that implies $\|e_n - \tilde{e}_{n,1}\| \underset{n\to\infty}{\longrightarrow} 0$. Now we show that (3.5) implies the equality $\dim H_2 = n_0$. Let $\tilde{H}_{1,0} = \tilde{H}_1 \cap H_2$; $\tilde{H}_{1,1} = \tilde{H}_1 \ominus \tilde{H}_{1,0}$; $\tilde{P}_{1,0}(\tilde{P}_{1,1})$ - the orthoprojector on $\tilde{H}_{1,0}(\tilde{H}_{1,1})$. Because of $\tilde{P}_{1,2} P_1 H_1 = 0$ we obtain $\overline{\tilde{P}_1 H_1} = \tilde{H}_{1,1}$. Then by (3.5) we have $\tilde{H}_{1,0} = \{0\}$ and $\dim H_2 = \dim \tilde{H}_2 = n_0$. Thus $\dim Q \leq n_0$ and the lemma is proved. □

**Remark 2.** By same method we prove the equality $\tilde{H}_{1,0} = \{0\}$ for in almost normed basis $(\phi_n)$. Hence, $\dim H_2 = \dim \tilde{H}_2 = n_0$. Thus $\dim Q \leq n_0$ and the condition $\|\phi_n - P\phi_n\| \underset{n\to\infty}{\longrightarrow} 0$ implies codim $P < \infty$.

**Corollary 3.1.** *An almost normed basis $(\phi_n)$ and $\phi_n \underset{n\to\infty}{\longrightarrow} \{0\}$ is an isotropically non-compact sequence.*

*Proof.* Let $P$ be an infinitedimensional orthoprojector and $(P\phi_n)$ be a compact sequence. Since $P\phi_n \underset{n\to\infty}{\longrightarrow} 0$ and $P\phi_n \underset{n\to\infty}{\Longrightarrow} 0$ and $\|\phi_n - P\phi_n\| \underset{n\to\infty}{\longrightarrow} 0$; $P_1 = 1 \ominus P$. By virtue of Lemma 3.1 we have $\dim P < \infty$ and arrive at a contradiction. □

Now we state a simple criterion of compactness for linear operations in a Hilbert space.

**Theorem 3.1.** *A densely defined closed linear operator $A$ in a Hilbert space is compact if and only if there exists an isotropically non-compact sequence $\phi_n \underset{n\to\infty}{\longrightarrow} 0$ such that*

$$A\phi_n \underset{n\to\infty}{\Longrightarrow} 0. \tag{3.6}$$

*Proof.* The operator $A$ allows a polar representation $A = UB$, where $B = (A^*A)^{1/2}$ and $U$ is an isometric operator [5, 14]. Since $\|A\phi_n\| = \|B\phi_n\|$ condition (3.6) is equivalent to

$$B\phi_n \underset{n\to\infty}{\Longrightarrow} 0. \tag{3.7}$$

We show that the limit spectrum $\sigma_c(B)$ of the selfadjoint operator $B$ consists only of the number $\lambda = 0$.

According to definition for every $\lambda \in \sigma_c(B)$ there exists a bounded non-compact sequence $(x_n)$ such that

$$(B - \lambda I)x_n \underset{n\to\infty}{\Longrightarrow} 0. \tag{3.8}$$

Suppose that $\sigma_c(B)$ contains $\lambda_0 > 0$ and $\lambda_0 \in (\lambda_1, \lambda_2)$; $\lambda_1 > 0$. Let $E_\lambda$ be a spectral function of operator $B$, $P_1 = \Delta E_\lambda$ be the corresponding to $[\lambda_1, \lambda_2)$ orthoprojector, and $B_1$ be a restriction of $B$ on $H_1 = P_1 H$. Since $\lambda_0 \in \sigma_c(B)$ we have $\dim P_1 = \infty$ and by Lemma (3.1) the non-compact sequence $(\phi_{n,1}) = (P_1\phi_n)$. Together with the relation $BP_1 = P_1BP_1$ condition (3.7) implies $B_1\phi_{n,1} \underset{n\to\infty}{\Longrightarrow} 0$ and therefore $0 \in \sigma(B_1)$. Since $\sigma(B_1) \subset [\lambda_1, \lambda_2]$ and $0 \notin [\lambda_1, \lambda_2]$ we arrive at a contraction. Thus the spectum $\sigma(B)$ is discrete.

Let $\{\lambda_n(B)\}$ is the sequence of eigenvalues of the operator $B > 0$. As usual it is assumed that $\lambda_1(B) \geq \lambda_2(B) \geq \ldots \geq \lambda_n(B) \geq \ldots$ where each eigenvalue is repeated as many times as its multiplicity.

Now note that the complement $\Delta_1 = R \backslash \Delta_0$ of any interval $\Delta_0 : 0 \in \Delta_0$ contains only finite number of eigenvalues of $B$. Suppose that there exists the infinitedimensional orthoprojector $P_1$ on the closure of the linear span $L_1$ of all eigenvectors corresponding to eigenvalues belonging to $\sigma_1(B) = \sigma(B) \cap \Delta_1$. Then in virtue of $B\phi_n \underset{n\to\infty}{\Longrightarrow} \{0\}$ and $P_1B = BP_1$ we have $B_1\phi_{n,1} \underset{n\to\infty}{\Longrightarrow} \{0\}$; $\phi_{n,1} = P_n\phi_n$, where $B_1$ is the restriction of $B$ on $L_1$. Since $(\phi_{n,1})$ is a non-compact sequence we obtain $0 \in \sigma(B_1)$ that contradicts to $\sigma(B_1) \subset \Delta_1$.

Hence $\lambda = 0$ is the only point of condensation for $\{\lambda_n(B)\}$ and $\lim\limits_{n\to\infty} \lambda_n(B) = 0$. The operator $B = B^*$ with such spectrum is a compact operator. Then in virtue of $A = UB$ the operator $A$ also is the compact operator. The necessity of condition (3.6) is evident. □

**Remark 3.** If $A$ is a bounded operator then from the known relation [11]

$$B\Delta E \geq \lambda_1 \Delta E;\ \Delta E = E_{\lambda_2} - E_{\lambda_1};\ \lambda_1 < \lambda_2, \tag{3.9}$$

it follows that

$$(BP_1\phi_n, \phi_n) \geq \lambda_1(P_2\phi_n, \phi_n),\ P_1 = \Delta E;\ \lambda_1 > 0, \tag{3.10}$$

$$(P_1\phi_n, B\phi_n) \geq \lambda_1 \|P_1\phi_n\|^2. \tag{3.11}$$

If $B$ is an unbounded operator then estimate (3.11) is deduced by the spectral decomposition [11, 14]:

$$Bx = \int_0^\infty \lambda dE_\lambda x,\ x \in D(B). \tag{3.12}$$

According to (3.12) and the equality $E_\lambda P_1 = 0$; $\lambda < \lambda_1$ we have the relations

$$(BP_1\phi_n, P_1\phi_n) = \int_{\lambda_1}^\infty \lambda d(E_\lambda P_1\phi_n, P_1\phi_n) \tag{3.13}$$

$$(P_1B\phi_n, P_1\phi_n) \geq \lambda_1 \int_{\lambda_1}^\infty d(E_\lambda P_1\phi_n, P_1\phi_n) = \lambda_1\|\phi_{n,1}\|^2. \tag{3.14}$$

Now according to (3.7) and (3.11) we obtain $P_1\phi_n \underset{n\to\infty}{\Longrightarrow} \{0\}$. In virtue of Lemma $\dim P_1 < \infty$ and any interval $(\alpha, \beta) : 0 < \alpha < \beta < \infty$ contains only a finite number of eigenvalues of $B = B^*$. Hence $\lambda_n(B) \underset{n\to\infty}{\longrightarrow} 0$ and $B$ is a compact operator.

**Remark 4.** As the following example shows the condition $\phi_n \to 0$ as $n \to \infty$ is essential for the validity of the proposed criterion. Let $(e_n)$ be an orthonormal basis in $H$. Consider the direct sum $\tilde{H} = H + H$ and the system $(\tilde{\phi}_k)$ in $\tilde{H}$ defined by the equalities: $\tilde{\phi}_{2n-1} = (e_n, 0)$; $\tilde{\phi}_{2n} = (e_n, \alpha_n e_n)$; where $\alpha_n \to 0$ as $n \to \infty$. Let $P_2 : H_1 + H_2 \to H_2$ be an orthoprojector on the second component of $\tilde{H} = H_1 + H_2$; $H_2 = H$. Since $P_2\phi_k \underset{n\to\infty}{\Longrightarrow} 0$ the sequence $(\tilde{\phi}_k)$ is not isotropically non-compact. Now, let $\tilde{A} = B + C$ be an operator in $\tilde{H}$, where $Be_n = \beta_n e_n$; $\beta_n \underset{n\to\infty}{\longrightarrow} 0$ and $C = P_2$. By construction we have $\tilde{A}\tilde{\phi}_k \underset{n\to\infty}{\Longrightarrow} 0$ however $\tilde{A}$ is a non-compact operator in $\tilde{H}$.

## 4 Main result

**Theorem 4.1.** *If $(e_n)$ is an almost normed unconditional basis of $H$ and $(e'_n)$ is a minimal and complete system asymptotically close to $(e_n)$, that is*

$$\lim_{n\to\infty} \|e_n - e'_n\| = 0,$$

*for which its biorthogonal system $(g'_n)$ is also complete, then the system $(e'_n)$ is an almost normed unconditional basis of $H$.*

*Proof.* We consider the linear operator $T$ defined by the equality

$$T\left[\sum_{k=1}^{\infty}(x, g_k)e_k\right] = \sum_{k=1}^{\infty}(x, g_k)(e_k - e'_k) \tag{4.1}$$

on $H$. Here $(g_n)$ is the biorthogonal to $(e_n)$ basis in $H$. According to the relations $(e_n - e'_n, g'_n) = (e_n, g'_n - g_n)$ we have

$$\left(\sum_{k=1}^{\infty}(x, g_k)(e_k - e'_k), \sum_{k=1}^{\infty}(y, e'_k)g'_k\right) = \left(\sum_{k=1}^{\infty}(x, g_k)e_k, \sum_{k=1}^{\infty}(y, e'_k)(g'_k - g_k)\right). \tag{4.2}$$

From (4.2) it follows that the set $D(T^*)$ contains the dense in $H$ linear manifold

$$L_1 = \left\{y \in H : y = \sum_{k=1}^{\infty}(y, e'_k)g'_k\right\} \tag{4.3}$$

and therefore the operator $T$ allows the closure $\overline{T} = (T^*)^*$ [14]. By (4.1) we obtain $\overline{T}e_n \underset{n\to\infty}{\Longrightarrow} 0$ and in virtue of the proposed criterion $\overline{T}$ is a compact operator. The minimality of $(e'_n)$ implies that $\ker(I - \overline{T})P = 0$.

Taking into account that $\overline{T}$ is a compact operator in $H$ we obtain the bounded invertibility of the operator $A = I - \overline{T}$. Thus, the systems $(e'_n)$ and $(g'_n)$ form unconditional bases in $H$. □

**Remark 5.** From the above proof also follows that $\|g_n - g'_n\| \to 0$ as $n \to \infty$. In fact by the relations

$$A^{-1}x = \sum_{n=1}^{\infty}\left(A^{-1}x, g_n\right)e_n;\ x \in H, \tag{4.4}$$

$$x = \sum_{n=1}^{\infty}\left(x, (A^*)^{-1}g_n\right)e'_n;\ x \in H \tag{4.5}$$

we have $g'_n = (A^*)^{-1}g_n$, where $(A^*)^{-1} = 1 - M$ and, moreover, $M$ is a compact operator. Since $g_n - g'_n = Mg_n$ and $Mg_n \underset{n\to\infty}{\Longrightarrow} 0$ we obtain the above noted asymptotic proximity of $(g'_n)$ and $(g_n)$.

Consider the well-known Hilbert and Bessel systems. If $(e_n)$ is an orthonormal basis and $A$ is bounded linear operator a set $(e'_n) = (Ae_n)$ $((e'n) = (A^{-1}e_n))$ is called a Hilbert (Bessel) system.

**Theorem 4.2.** *Let $(\phi_n)$ be a complete in $H$ Hilbert or Bessel system and $(e_n)$ be an almost normed unconditional basis of $H$. Then the condition*

$$\lim_{n\to\infty} \|\phi_n - e_n\| = 0$$

*implies that $(\phi_n)$ forms an almost normed unconditionsl basis of $H$.*

*Proof.* Representing $A(A^{-1})$ in form $A = I - T$ $(A^{-1} = I - L)$ by the proposed criterion we obtain that $T$ and $L$ are compact operators. Moreover, $(I-T)^{-1}$ and $(I-L)^{-1}$ are bounded operartors. This means that the transformations $A : H \to H$ and $A^{-1} : H \to H$ are homeomorphisms. Consequently, the statement of Theorem 4.2 is proved. □

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