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MIKHAIL L'VOVICH GOLDMAN

(to the 75th birthday)

Mikhail L'vovich Goldman was born on April 13, 1945 in Moscow. In 1963 he graduated from school in Moscow and entered the Physical Faculty of the M.V. Lomonosov Moscow State University (MSU) from which he graduated in 1969 and became a PhD student (1969–1972) at the Mathematical Department of this Faculty. In 1972 he has defended the PhD thesis, and in 1988 his DSc thesis "The study of spaces of differentiable functions of many variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor in Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he is a Professor of the S.M. Nikol'skii Mathematical Institute at the Peoples Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman are: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators. Main achievements: optimal embeddings of spaces with generalized smoothness, sharp conditions of the convergence of spectral expansions, descriptions of integral and differential properties of the generalized Bessel and Riesz-type potentials, sharp estimates for operators on cones and optimal envelopes for the cones of functions with properties of monotonicity. Professor M.L. Goldman has over 140 scientific publications in leading mathematical journals.

Under his scientific supervision, 8 candidate theses in Russia and 1 thesis in Kazakhstan were successfully defended. Some of his former students are now professors in Ethiopia, Columbia, Mongolia.

Participation in scientific and organizational activities of M.L. Goldman is well known. He is a member of the DSc Councils at RUDN and MSU, of the PhD Council in the Lulea Technical University (Sweden), a member of the Editorial Board of the Eurasian Mathematical Journal, an invited lector and visiting professor at universities of Russia, Germany, Sweden, UK etc., an invited speaker at many international conferences.

The mathematical community, friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Mikhail L'vovich Goldman on the occasions of his 75th birthday and wish him good health, happiness, and new achievements in mathematics and mathematical education.

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ON SOLVABILITY OF ONE INFINITE SYSTEM OF NONLINEAR FUNCTIONAL EQUATIONS IN THE THEORY OF EPIDEMICS

A.Kh. Khachatryan, Kh.A. Khachatryan

Communicated by E.D. Nursultanov

Key words: nonlinearity, infinite system, monotonicity, bounded solution, iteration, theory of epidemics, p–adic string theory.

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Abstract. In the present paper, an infinite system of nonlinear functional equations arising in the theory of epidemics is investigated. We prove a constructive theorem on the existence of a non-trivial, continuous and bounded solution of the system. In addition, some asymptotic properties of the constructed solution are studied. We conclude the study by applying our theoretical results to two concrete examples arising in spatial-temporal spread of epidemics and in p-adic string theory.

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1 Introduction. Formulation of the main result

In this paper we study the following infinite system of nonlinear functional equations with the Toeplitz matrix:

$$u_i(t) = \sum_{j=-\infty}^{\infty} a_{i-j} \int_0^{\infty} H(\tau) g(u_j(t-\tau)) d\tau, \tag{1.1}$$

where $u_i(t)$, $i \in \mathbb{Z}$ represent the unknown real functions.

For our purposes we assume

$$H(\tau) \geq 0; \quad \tau \in \mathbb{R}^+ := [0, +\infty); \quad \int_0^{\infty} H(\tau) d\tau = 1, \tag{1.2}$$

$$a_{-i} = a_i, \quad \sum_{i=-\infty}^{\infty} a_i = 1; \quad a_i > 0; \quad a_i \downarrow \text{ by } i \text{ on } \mathbb{Z}^+ \equiv \mathbb{N} \cup \{0\}, \quad i \in \mathbb{Z}. \tag{1.3}$$

A solution $\{u_i(t)\}_{i\in\mathbb{Z}}$, to (1.1) has to be defined at least on the interval $(-\infty, T]$.

Let η be the first positive root of the equation

$$g(u) = u. \tag{1.4}$$

We assume that g is a continuous and monotonically increasing function on the interval $[-\eta, \eta]$. In addition we let g be an odd function on $[-\eta, \eta]$ as well as $g(u)$ be convex upward on the interval $[0, \eta]$.

Notice that the monotonicity of the function g implies the existence of the inverse function $Q = g^{-1}$, which increases monotonically on the interval $[-\eta, \eta]$ and is convex downward on the interval $[0,\eta]$.

We additionally assume that for $\varepsilon=\dfrac{1+a_0}{2}$, $\varepsilon\in(0,1)$, there exists a number $\xi\in(0,\eta)$ such that

$$Q(u)\le\varepsilon u,\quad u\in[0,\xi]. \tag{1.5}$$

From the properties of the function g we have

$$Q(-u)=-Q(u),\quad u\in[-\eta,\eta]. \tag{1.6}$$

Observe that, because of $g(0)=0$, system of equations (1.1) has a trivial solution. Further we aim to construct a non-trivial bounded solution of system (1.1).

It is worth to mention that system of equations (1.1) has a direct application in mathematical biology. It describes the geometrical propagation of epidemics (see [1]-[3]). This will be a subject in the last section of the paper, where we consider two boundary value problems arising in spatial-temporal spread of epidemics and in p-adic string theory.

Now let us state the main result of the present work.

Theorem 1.1. *Let conditions* (1.2)-(1.6) *be fulfilled. Then infinite system of nonlinear functional equations* (1.1) *has a non-trivial, continuous and bounded solution* $u_i(t)$, $i\in\mathbb{Z}$. *Moreover,*

a) $u_0(t)=0$ *for each* $t\in(-\infty,T]$,

b) $u_{-i}(t)=-u_i(t)$, $i=0,1,2,...$,

c) $u_i(t)$ *is increasing in* t *on the interval* $(-\infty,T]$, $i\in\mathbb{Z}$,

d) $u_i(t)$ *is increasing in* i *on* $\mathbb{Z}$,

e) $\lim\limits_{i\to\pm\infty}u_i(t)=\pm\eta$ *for any* $t\in(-\infty,T]$.

Before proceeding to the proof of the main result, we prove some auxiliary facts needed hereinafter.

2 Auxiliary facts

We consider the following auxiliary infinite system of nonlinear algebraic equations:

$$Q(x_i)=\sum_{j=-\infty}^{\infty}a_{i-j}x_j,\quad i\in\mathbb{Z} \tag{2.1}$$

with the unknown column vector $x=(\dots x_{-2},x_{-1},x_0,x_1,x_2,\dots)^T$.

Together with equation (2.1) we consider the infinite system of nonlinear algebraic equations

$$Q(\tau_i)=\sum_{j=0}^{\infty}(a_{i-j}-a_{i+j})\tau_j,\quad i\in\mathbb{Z}^+ \tag{2.2}$$

with the unknown column vector

$$\tau=(\tau_0,\tau_1,\tau_2,...)^T. \tag{2.3}$$

Taking into account relation (1.6), it is easy to verify that x_i and τ_i are related by the formula

$$x_i = \begin{cases} \tau_i, & \text{if} \quad i \in \mathbb{Z}^+ \\ -\tau_{-i}, & \text{if} \quad i \in \mathbb{Z} \setminus \mathbb{Z}^+ \end{cases}. \tag{2.4}$$

The following theorem holds.

Theorem 2.1. *Let conditions* (1.3)-(1.5) *be satisfied. Then the infinite system of nonlinear algebraic equations* (2.2) *has a non-negative, monotonically increasing and bounded solution* $\tau = (\tau_0, \tau_1, \tau_2, ...)^T$. *Moreover,*

$$\lim_{i\to\infty} \tau_i = \eta. \tag{2.5}$$

Proof. Let $q > 1$ is a fixed number. First of all we consider the following characteristic equation:

$$\sum_{i=-\infty}^{\infty} a_i q^{-p|i|} = \varepsilon, \tag{2.6}$$

where $\varepsilon = \dfrac{1+a_0}{2}$, $\varepsilon \in (0,1)$.

Observe that equation (2.6) has a unique positive solution. Indeed, denoting

$$f(p) := \sum_{i=-\infty}^{\infty} a_i q^{-p|i|} - \varepsilon, \quad p > 0,$$

we may state that

$$f(p) \downarrow \text{in} \;\; p \;\; \text{on} \;\; \mathbb{R}^+,$$

$$f(0) = \sum_{i=-\infty}^{\infty} a_i - \varepsilon = 1 - \frac{1+a_0}{2} = \frac{1-a_0}{2} > 0,$$

$$f(+\infty) = \frac{a_0 - 1}{2} < 0.$$

Using the Bolzano-Cauchy theorem, we conclude that there exists $p_0 > 0$ such that $f(p_0) = 0$. We fix this number p_0 and bear it in mind for our further discussions.

Now, using equation (2.2), we consider the following iterations:

$$\begin{gathered} Q\left(\tau_i^{(n+1)}\right) = \sum_{j=0}^{\infty} (a_{i-j} - a_{i+j}) \tau_j^{(n)}, \\ \tau_i^{(0)} = \eta; \quad n = 0,1,2,\ldots, \quad i = 0,1,2,\ldots. \end{gathered} \tag{2.7}$$

It is obvious that $\tau_i^{(n)} \geq 0$, $i \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+$. We state that the sequence $\tau_i^{(n)}$ is decreasing in n for every fixed i. Indeed

$$Q\left(\tau_i^{(1)}\right) = \eta \sum_{j=0}^{\infty} (a_{i-j} - a_{i+j}) \leq \eta \sum_{j=0}^{\infty} a_{i-j} \leq \eta = Q(\eta),$$

i. e. $\tau_i^{(1)} \leq \eta = \tau_i^{(0)}$. Assume that the inequality

$$\tau_i^{(n)} \leq \tau_i^{(n-1)}$$

is satisfied for some $n \in \mathbb{N}$. We show that the inequality holds for $n+1$ as well.
Indeed,

$$Q\left(\tau_i^{(n+1)}\right) \le \sum_{j=0}^{\infty}(a_{i-j}-a_{i+j})\tau_j^{(n-1)} = Q\left(\tau_i^{(n)}\right).$$

Since $Q \uparrow$ on $[0,\eta]$ we obtain

$$\tau_i^{(n+1)} \le \tau_i^{(n)}.$$

Next we show that the sequence $\tau_i^{(n)}$ is bounded from below, namely

$$\tau_i^{(n)} \ge \xi(1-q^{-p_0 i}), \quad i \in \mathbb{Z}^+, \tag{2.8}$$

where ξ is the positive root of the equation $Q(u)=\varepsilon u$. Since $\xi \in (0,\eta)$, we get $\eta = \tau_i^{(0)} \ge \xi(1-q^{-p_0 i})$. Assume that (2.8) holds for some natural n. Having in disposal the properties of the function Q, we obtain

$$Q\left(\tau_i^{(n+1)}\right) \ge \xi \sum_{j=0}^{\infty}(a_{i-j}-a_{i+j})(1-q^{-p_0 j}) \ge \xi\varepsilon(1-q^{-p_0 i}) \ge Q(\xi(1-q^{-p_0 i})), \tag{2.9}$$

where we used the inequality

$$\sum_{j=0}^{\infty}(a_{i-j}-a_{i+j})(1-q^{-p_0 j}) \ge \varepsilon(1-q^{-p_0 i}), \quad i \in \mathbb{Z}^+ \tag{2.10}$$

and condition (1.5).

To prove that inequality (2.10) is true, we first show the following inequality:

$$\sum_{i=-\infty}^{m} a_i q^{p_0 i} + q^{2p_0 m}\sum_{i=m+1}^{\infty} a_i q^{-p_0 i} \ge \varepsilon, \quad m \in \mathbb{Z}^+. \tag{2.11}$$

Denote by R_m the following difference:

$$R_m := \sum_{i=-\infty}^{m} a_i q^{p_0 i} + q^{2p_0 m}\sum_{i=m+1}^{\infty} a_i q^{-p_0 i} - \varepsilon.$$

Due to (2.6) we have

$$R_0 = \sum_{i=-\infty}^{0} a_i q^{p_0 i} + \sum_{i=1}^{\infty} a_i q^{-p_0 i} - \varepsilon = \sum_{i=-\infty}^{\infty} a_i q^{-p_0 |i|} - \varepsilon = 0,$$

$$R_{m+1} - R_m = \sum_{j=0}^{\infty} a_{m+j+1} q^{p_0(m+1-j)} \cdot \left(1-q^{-2p_0}\right) \ge 0.$$

Hence $R_m \uparrow$ in m and $R_m \ge R_0 = 0, \quad m \in \mathbb{Z}^+$ which means that inequality (2.11) holds.

Using (2.1), it can be stated by direct calculation that

$$\begin{gathered}\sum_{i=0}^{\infty}(a_{m-i}-a_{m+i})\left(1-q^{-p_0 i}\right) = 1 + a_m - 2\sum_{i=m}^{\infty} a_i - \\ -q^{-p_0 m}\sum_{i=-\infty}^{m} a_i q^{p_0 i} + q^{p_0 m}\sum_{i=m}^{\infty} a_i q^{-p_0 i}.\end{gathered} \tag{2.12}$$

Now employing estimate (2.11), one can prove that

$$1+a_m-2\sum_{i=m}^{\infty}a_i-q^{-p_0\,m}\sum_{i=-\infty}^{m}a_iq^{p_0\,i}+q^{p_0\,m}\sum_{i=m}^{\infty}a_iq^{-p_0\,i}\geq\varepsilon\left(1-q^{-p_0\,m}\right).$$

Denote by L_m the following difference:

$$L_m:=1+a_m-2\sum_{i=m}^{\infty}a_i-q^{-p_0\,m}\sum_{i=-\infty}^{m}a_iq^{p_0\,i}+$$

$$+q^{p_0\,m}\sum_{i=m}^{\infty}a_iq^{-p_0\,i}-\varepsilon\left(1-q^{-p_0\,m}\right),\quad m\in\mathbb{Z}^+.$$

Observe that $L_0=0$. After simple transformations we get

$$L_{m+1}-L_m\geq\left(q^{p_0}-1\right)q^{-p_0(m+1)}\left(\sum_{i=-\infty}^{m}a_iq^{p_0\,i}+q^{2p_0\,m}\sum_{i=m+1}^{\infty}a_iq^{-p_0\,i}-\varepsilon\right).\tag{2.13}$$

Due to (2.11) and (2.13) we obtain

$$L_m\quad\uparrow\text{in}\quad m\quad\text{on}\quad\mathbb{Z}^+.$$

Thus

$$L_m\geq L_0=0,$$

from which the validity of required inequality (2.10) follows immediately.

It should be mentioned that the continuous analogue of inequality (2.10) has been proved by one of the authors of the present paper in [4].

Thus

$$\lim_{n\to\infty}\tau_i^{(n)}=\tau_i,\quad i=0,1,2,\ldots.$$

Now we show that the inequality

$$\tau_{i+1}^{(n)}\geq\tau_i^{(n)},\quad n\in\mathbb{Z}^+,\quad i\in\mathbb{Z}^+.\tag{2.14}$$

holds for every $n\in\mathbb{Z}^+$. This inequality is obviously satisfied for $n=0$. Assume that (2.14) holds for some natural $n\in\mathbb{N}$. We rewrite iteration (2.7) in the form

$$Q\left(\tau_i^{(n+1)}\right)=\sum_{j=-\infty}^{i}a_j\tau_{i-j}^{(n)}-\sum_{j=0}^{\infty}a_{i+j}\tau_j^{(n)}.$$

In view of monotonicity of $\{a_i\}_{i=-\infty}^{\infty}$ with respect to $i\in\mathbb{Z}^+$ we can write

$$\begin{aligned}Q\left(\tau_{i+1}^{(n+1)}\right)&=\sum_{j=-\infty}^{i+1}a_j\tau_{i+1-j}^{(n)}-\sum_{j=0}^{\infty}a_{i+1+j}\tau_j^{(n)}\geq\\&\geq\sum_{j=-\infty}^{i}a_j\tau_{i-j}^{(n)}-\sum_{j=0}^{\infty}a_{i+j}\tau_j^{(n)}=Q\left(\tau_i^{(n+1)}\right).\end{aligned}\tag{2.15}$$

From the last it follows that (2.14) holds. Therefore the sequence $\tau_i=\lim\limits_{n\to\infty}\tau_i^{(n)}$ is also monotonically increasing in i. We pass to the limit as i tends to infinity on both sides of (2.2). Since η is the first positive root of the equation $Q(\eta)=\eta$ and $\xi(1-q^{-p_0i})\leq\tau_i\leq\eta$, we get $\lim\limits_{i\to\infty}\tau_i=\eta$.

Finally, using (2.4), we obtain

$$\lim_{i\to\pm\infty}x_i=\pm\eta.\tag{2.16}$$

$\square$

3 Proof of the basic result

Now we prove Theorem 1.1 formulated in the previous section. To begin with, we consider the following iteration:

$$Q(\psi_i^{(n+1)}(t)) = \int_0^\infty \sum_{j=0}^\infty H(\tau)(a_{i-j} - a_{i+j})\psi_j^{(n)}(t-\tau)d\tau, \qquad (3.1)$$
$$i = 0, 1, 2, \dots, \ n = 0, 1, 2, \dots,$$

where as the first approximation we take

$$\psi_i^{(0)}(t) = \tau_i\,(1 - e^{-(T-t)}), \quad i \in \mathbb{Z}^+ \qquad (3.2)$$

with $\{\tau_i\}_{i\in\mathbb{Z}^+}$ representing solution of (2.2).

First we prove that

$$0 \le \psi_i^{(n)}(t) \le \eta \qquad (3.3)$$

and $\psi_i^{(n)}(t)$ is monotonically increasing in n for each $i \in \mathbb{Z}^+$ and each $t \in (-\infty, T]$. Using the inequality

$$0 \le \psi_i^{(0)}(t) \le \eta, \quad i \in \mathbb{Z}^+$$

for the first approximation, the monotonicity of the function Q and the inequalities

$$a_{i-j} \ge a_{i+j}, \ H(\tau) \ge 0, \ \tau \in [0, +\infty),$$

from (3.1) one can conclude that

$$0 \le \psi_i^{(1)}(t) \le \eta, \quad i \in \mathbb{Z}^+. \qquad (3.4)$$

Next we show that

$$\psi_i^{(1)}(t) \ge \psi_i^{(0)}(t), \ i \in \mathbb{Z}^+, \quad t \in (-\infty, T].$$

Indeed, using (3.1), we can write

$$Q(\psi_i^{(1)}(t)) \le \eta \int_0^\infty H(\tau) \sum_{j=0}^\infty (a_{i-j} - a_{i+j})\, d\tau \le \eta \sum_{j=0}^\infty a_{i-j} \le \eta = Q(\eta),$$

or

$$\psi_i^{(1)}(t) \le \eta, \quad i \in \mathbb{Z}^+.$$

On the other hand, from (3.1) and (2.2) it follows that

$$Q(\psi_i^{(1)}(t)) \ge \int_0^\infty (1 - e^{(t-\tau-T)})H(\tau)d\tau \sum_{j=0}^\infty (a_{i-j} - a_{i+j})\tau_j \ge$$

$$\ge Q(\tau_i) \int_0^\infty (1 - e^{(t-\tau-T)})H(\tau)d\tau = Q(\tau_i)\left(1 - \int_0^\infty e^{t-T}e^{-\tau}H(\tau)d\tau\right) =$$

$$= Q(\tau_i)\left(1 - e^{t-T}\int_0^\infty e^{-\tau}H(\tau)d\tau\right) \ge Q(\tau_i)(1 - e^{t-T}).$$

Since $\tau_i \in [0,\eta]$, $0 \le 1-e^{t-T} \le 1$ and the function Q is convex on $[0,\eta]$, the following inequality follows immediately from the Thales theorem:

$$(1-e^{t-T})Q(\tau_i) \ge Q\left(\tau_i(1-e^{t-T})\right). \tag{3.5}$$

Hence, in view of the monotonicity of the function Q, we have

$$\psi_i^{(1)}(t) \ge \tau_i(1-e^{t-T}) = \psi_i^{(0)}(t), \quad i \in \mathbb{Z}^+.$$

Now we assume that for some $n \in \mathbb{N}$ the following inequalities hold:

$$\psi_i^{(n)}(t) \ge \psi_i^{(n-1)}(t) \text{ and } 0 \le \psi_i^{(n)}(t) \le \eta,\ i \in \mathbb{Z}^+,\ t \in (-\infty, T].$$

Taking into account (3.1), the inequalities $a_{i-j} \ge a_{i+j}, \quad i, j \in \mathbb{Z}^+, \quad H(t) \ge 0, \quad t \in (-\infty, T]$, it can be easily shown by induction on n that

$$0 \le \psi_i^{(n)}(t) \le \psi_i^{(n+1)}(t) \le \eta. \tag{3.6}$$

Inequalities (3.3) ensure that the limit

$$\lim_{n\to\infty} \psi_i^{(n)}(t) = \psi_i(t)$$

exists, where $\psi_i(t)$ satisfies the equation

$$Q(\psi_i(t)) = \sum_{j=0}^{\infty} \int_0^{\infty} H(\tau)(a_{i-j} - a_{i+j})\psi_j(t-\tau)d\tau, \quad i \in \mathbb{Z}^+. \tag{3.7}$$

It remains to show that

$$\psi_i^{(n)}(t) \text{ in} \uparrow\ i, \quad i \in \mathbb{Z}^+.$$

Thus can be done in a similar way as for the sequence τ_i (see also below the proof of statement **d**)). Passing to the limit in both sides of equation (3.7) as i tends to infinity, we get

$$Q(\psi_\infty(t)) = \int_{-\infty}^{t} \psi_\infty(\tau)H(t-\tau)d\tau, \tag{3.8}$$

where $\psi_\infty(t) := \lim\limits_{i\to\infty} \psi_i(t)$.

Here we used the well known convolution property (see[5])

$$\lim_{i\to\infty} \sum_{j=0}^{\infty}(a_{i-j} - a_{i+j})\psi_j(t-\tau) = \lim_{i\to\infty} \sum_{j=-\infty}^{\infty} a_{i-j}\psi_j(t-\tau) = \psi_\infty(t-\tau) \sum_{j=-\infty}^{\infty} a_j = \psi_\infty(t-\tau).$$

It has been proved in [3] that equation (3.8) has only the trivial solution $\psi_\infty(t) \equiv \eta$, $t \in (-\infty, T]$, in the class of positive, continuous and bounded by η functions.

We construct the function $W_i(t)$ as follows:

$$W_i(t) := \begin{cases} \psi_i(t), & \text{if } \ i \in \mathbb{Z}^+, \quad t \in (-\infty, T], \\ -\psi_{-i}(t), & \text{if } \ i \in \mathbb{Z}\backslash\mathbb{Z}^+,\ t \in (-\infty, T]. \end{cases} \tag{3.9}$$

By direct checking one can see that the sequence of functions $W_i(t)$ is increasing in t for $t \in (-\infty, T], \quad i \in \mathbb{Z}$.

Now we show that the sequence $Q(W_i(t))$, $i \in \mathbb{Z}$, satisfies the initial infinite system of nonlinear functional equations (1.1).

Indeed, first we verify the case, where $i \in \mathbb{Z}^+$. We have

$$Q(W_i(t)) = Q(\psi_i(t)) = \int_0^\infty H(\tau) \sum_{j=0}^\infty (a_{i-j} - a_{i+j})\, \psi_j(t-\tau)\, d\tau =$$

$$= \int_0^\infty H(\tau) \sum_{k=0}^\infty a_{i-k}\psi_k(t-\tau)\, d\tau - \int_0^\infty H(\tau) \sum_{j=1}^\infty a_{i+j}\psi_j(t-\tau)\, d\tau =$$

$$= \int_0^\infty H(\tau) \sum_{k=0}^\infty a_{i-k}\psi_k(t-\tau)\, d\tau - \int_0^\infty H(\tau) \sum_{k=-\infty}^{-1} a_{i-k}\psi_{-k}(t-\tau)\, d\tau =$$

$$= \int_0^\infty H(\tau) \sum_{k=0}^\infty a_{i-k}W_k(t-\tau) d\tau + \int_0^\infty H(\tau) \sum_{k=-\infty}^{-1} a_{i-k}W_k(t-\tau)\, d\tau =$$

$$= \int_0^\infty H(\tau) \sum_{k=-\infty}^{\infty} a_{i-k}W_k(t-\tau)\, d\tau.$$

Here we bear in mind that $\psi_0(t) = 0$.

Let $i \in \mathbb{Z}\backslash\mathbb{Z}^+$, $t \in (-\infty, T]$, then

$$Q(W_i(t)) = Q(-\psi_{-i}(t)) = -Q(\psi_{-i}(t)) =$$

$$= -\int_0^\infty H(\tau) \sum_{j=0}^\infty (a_{-i-j} - a_{-i+j})\, \psi_j(t-\tau)\, d\tau =$$

$$= \int_0^\infty H(\tau) \sum_{j=0}^\infty (a_{i-j} - a_{i+j})\psi_j(t-\tau)\, d\tau =$$

$$= \int_0^\infty H(\tau) \sum_{j=0}^\infty a_{i-j}W_j(t-\tau)\, d\tau - \int_0^\infty H(\tau) \sum_{j=1}^\infty a_{i+j}\psi_j(t-\tau)\, d\tau =$$

$$= \int_0^\infty H(\tau) \sum_{j=0}^\infty a_{i-j}W_j(t-\tau) d\tau - \int_0^\infty H(\tau) \sum_{k=-\infty}^{-1} a_{i-k}\psi_{-k}(t-\tau) d\tau =$$

$$= \int_0^\infty H(\tau) \sum_{j=0}^\infty a_{i-j}W_j(t-\tau) d\tau + \int_0^\infty H(\tau) \sum_{j=-\infty}^{-1} a_{i-j}W_j(t-\tau) d\tau =$$

$$= \int_0^\infty H(\tau) \sum_{j=-\infty}^{\infty} a_{i-j}W_j(t-\tau) d\tau.$$

Since Q is the inverse function of g on $[-\eta, \eta]$ and

$$-\eta \le W_i(t) \le \eta, \quad t \in (-\infty, T],$$

from (1.1) it follows that

$$u_i(t) := Q(W_i(t)), \quad i \in \mathbb{Z} \tag{3.10}$$

satisfy initial system (1.1).

It remains to prove statements **a**)–**e**) of Theorem 1.1.

Statement a). In virtue of $a_{-i} = a_i$ we have

$$u_0(t) = Q(W_0(t)) = \int_0^\infty H(\tau) \sum_{j=0}^\infty (a_{-j} - a_j)\psi_0(t-\tau)d\tau = 0. \tag{3.11}$$

Statement b). Recalling that Q is an odd function, we obtain

$$u_{-i}(t) = Q(W_{-i}(t)) = Q(-W_i(t)) = -Q(W_i(t)) = -u_i(t). \tag{3.12}$$

Statement c). Let $t_1 > t_2$. Due to the monotonicity of $W_i(t)$ in t we have

$$\begin{aligned} u_i(t_1) = Q(W_i(t_1)) = \int_0^\infty H(\tau) \sum_{j=-\infty}^\infty a_{i-j} W_j(t_1 - \tau)d\tau \geq \\ \geq \int_0^\infty H(\tau) \sum_{j=-\infty}^\infty a_{i-j} W_j(t_2-\tau)d\tau = Q(W_i(t_2)) = u_i(t_2). \end{aligned} \tag{3.13}$$

Statement d). First of all by induction we prove that

$$\psi_i^{(n)}(t) \uparrow \text{ in } i, \quad i \in \mathbb{Z}^+.$$

Since $\tau_{i+1} \geq \tau_i$, then the inequality is obviously satisfied for $n = 0$,

$$\psi_i^{(0)}(t) \leq \psi_{i+1}^{(0)}(t).$$

We assume that it is true for some natural n. Let us verify the statement for $n+1$:

$$Q(\psi_{i+1}^{(n+1)}(t)) = \int_0^\infty \sum_{j=0}^\infty H(\tau)(a_{i+1-j} - a_{i+1+j})\psi_j^{(n)}(t-\tau)d\tau$$

$$= \int_0^\infty H(\tau) \sum_{k=-\infty}^{i+1} a_k \psi_{i+1-k}^{(n)}(t-\tau)d\tau - \int_0^\infty H(\tau) \sum_{j=0}^\infty a_{i+1+j}\psi_j^{(n)}(t-\tau)d\tau$$

$$\geq \int_0^\infty H(\tau) \sum_{k=-\infty}^{i} a_k \psi_{i-k}^{(n)}(t-\tau)d\tau - \int_0^\infty H(\tau) \sum_{j=0}^\infty a_{i+j}\psi_j^{(n)}(t-\tau)d\tau = Q(\psi_i^{(n+1)}(t)).$$

In view of monotonicity of the function Q it follows that

$$\psi_{i+1}^{(n+1)}(t) \geq \psi_i^{(n+1)}(t).$$

Hence the limit function as $n \to \infty$ will be increasing in i. Since $W_i(t)$ are the odd extensions of $\psi_i(t)$ on $\mathbb{Z} \backslash \mathbb{Z}^+$, then $W_i(t) \uparrow$ in i. Due to the monotonicity of Q we can assert that the functions $u_i(t) := Q(W_i(t))$ are also increasing in i.

Statement e). Using the continuity of Q and taking into account the fact

$$\lim_{i \to \pm\infty} W_i(t) = \pm\eta,$$

we obtain

$$\lim_{i \to +\infty} u_i(t) = \lim_{i \to +\infty} Q(\psi_i(t)) = Q(\lim_{i \to +\infty} \psi_i(t)) = Q(\psi_\infty(t)) = Q(\eta) = \eta.$$

Analogously, $\lim\limits_{i \to -\infty} u_i(t) = -\eta$.

4 Application to epidemic and p-adic string theories

I) Theory of epidemics. There exist different types of infectious diseases in the world. Each year over a million of people die from these diseases and few millions are infected. Such infectious diseases circulate among animals and plants as well. There are various mathematical models describing the spread of infectious diseases. Among these models the one describing the spatial-temporal propagation of epidemics is of particular importance. In 1978, O. Diekmann [1] derived a model driven by a nonlinear integral equation. To describe the spread of epidemic several authors distingwish between at least two groups of population: susceptible individuals S and infective individuals I (see [1]-[3]). In the framework of continuous time many ecological systems are better modelled when one considers discrete space coordinates in contrast to continuous ones. It is also assumed that the population size is large and constant, i.e. $S + I = const$.

Let now $S_i(t) := S(t, x_i)$ be the density of susceptible individuals at the time t and at the position x_i, $i \in \mathbb{Z}$. Let further $A_{ij}(\tau) := A(\tau, x_i, x_j)$ be a matrix function which describes the infection of a susceptible person at the position x_i from a person being infected τ time ago and located at the point x_j. Let $B_i(t) = B(t, x_i)$ be defined as the rate at which susceptible individuals become infective,

$$B_i(t) = \sum_{j=-\infty}^{\infty} \int_0^{\infty} A_{ij}(\tau) \frac{\partial S_j(t-\tau)}{\partial t} d\tau, \quad i, j \in \mathbb{Z}. \tag{4.1}$$

Using all above notations, the basic equation describing spatial-temporal spread of epidemic can be written as follows:

$$\frac{\partial S_i(t)}{\partial t} = -S_i(t) B_i(t), \quad i \in \mathbb{Z}. \tag{4.2}$$

We complete the last differential equation with the appropriate boundary conditions, namely

$$S(-\infty, x_i) = S_0 = const, \quad i \in \mathbb{Z}. \tag{4.3}$$

The solution $S_i(t)$ of system (4.1)-(4.2) is defined for $t \in (-\infty, T]$. We assume that the duration of illness τ and space variables x_i are independent,

$$A_{ij}(\tau) = \gamma H(\tau) b_{ij}, \quad i, j \in \mathbb{Z}. \tag{4.4}$$

We assume also that the spread of epidemics depends only on the distance between two individuals, which in turn means that

$$b_{ij} = a_{i-j}, \quad i, j \in \mathbb{Z}. \tag{4.5}$$

The function $H(\tau)$ in (4.4) describes the evolution of the spread of epidemics and satisfies conditions (1.2). The constant γ is the so-called epidemic parameter. Taking into account (4.1), (4.4), (4.5) and boundary condition (4.3) and integrating equation (4.2) with respect to t, we obtain the following infinite system of nonlinear functional equations:

$$u_i(t) = \sum_{j=-\infty}^{\infty} \int_0^{\infty} a_{i-j} H(\tau) S_0 \gamma (1 - e^{-u_j(t-\tau)}) d\tau, \quad i \in \mathbb{Z} \tag{4.6}$$

with the concrete nonlinearity

$$g(u) = S_0 \gamma (1 - e^{-u}). \tag{4.7}$$

Here

$$u_i(t) = -ln \frac{S_i(t)}{S_0}, \quad i \in \mathbb{Z}. \tag{4.8}$$

It is obvious that the inverse function Q can be written as

$$Q(u)=\begin{cases} ln\dfrac{\gamma S_0}{\gamma S_0-u}, & \text{if} \quad u\in[0,S_0\gamma],\\ -ln\dfrac{\gamma S_0}{\gamma S_0+u}, & \text{if} \quad u\in[-S_0\gamma,0]. \end{cases} \tag{4.9}$$

As for the number η, we have $\eta\in(S_0\,\gamma-1,S_0\,\gamma)$.

It can be easily verified that all conditions of Theorem 1.1 are satisfied.

Remark 1. In 1978, O. Diekmann first derived the mathematical model of epidemics driven by the following integral equation:

$$u(t,x)=\int\limits_0^\infty\int\limits_{\mathbb{R}^n} V(x-y)H(\tau)S_0\gamma(1-e^{-u(t-\tau,y)})d\tau dy, \quad t\in(-\infty,T], \quad x\in\mathbb{R}^n \tag{4.10}$$

(see [1]). In [3] the existence theorems for equation (4.10) in the case of general nonlinearity have been proved. Notice that equation (4.6) is a discrete analogue of integral equation (4.10), where $n=1$.

Remark 2. It is clear that the one-dimensional case for the spread of epidemics (i.e. spread along the line) is not realistic. More realistic two-dimensional medium must be therefore considered, but it is the scope of the present work.

II) p-adic string theory. The string theory is one of the most developing branches of mathematical physics. p-adic string theory has special and important place in the general string theory. It is based on nonlinear pseudo-differential equations. Among them an important role plays Vladimirov's equation [6]

$$u(x)=\int\limits_{-\infty}^{\infty} K(x-t)\sqrt[p]{u(t)}\,dt, \quad x\in\mathbb{R}, \tag{4.11}$$

where $K(x)=\dfrac{1}{\sqrt{\pi}}e^{-x^2}$ and $p>2$ is an odd number.

It has been shown in [6] that equation (4.11) has a bounded, monotonically increasing solution. This result of V.S. Vladimirov and Ya.I. Volovich has been generalised for a arbitrary kernel K satisfying $K(x)>0$; $\int\limits_{-\infty}^{\infty}K(x)dx=1$; $K(-x)=K(x)$ by one of the authors of the present work (see [4], [7]).

The discrete analogue of equation (4.11) takes the form

$$u_i=\sum_{j=-\infty}^{\infty} a_{i-j}\sqrt[p]{u_j}, \tag{4.12}$$

where $a_i=K(x_i)$, $i\in\mathbb{Z}$, $p>2$ is an odd number.

Notice that in the special case when $H(\tau)=\delta(\tau)$, where δ is the Dirac delta-function, equation (1.1) becomes system (4.12) with the nonlinearity $g(u)=\sqrt[p]{u}$.

Observe that the inverse function Q of the function g is given by $Q(u)=u^p$, $u\in[-\eta,\eta]$, where η is the first positive root of the equation $u^p=u$, i. e. $\eta=1$. It is easy to check that all conditions imposed on g and Q are fulfilled.

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