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MIKHAIL L'VOVICH GOLDMAN

(to the 75th birthday)

Mikhail L'vovich Goldman was born on April 13, 1945 in Moscow. In 1963 he graduated from school in Moscow and entered the Physical Faculty of the M.V. Lomonosov Moscow State University (MSU) from which he graduated in 1969 and became a PhD student (1969–1972) at the Mathematical Department of this Faculty. In 1972 he has defended the PhD thesis, and in 1988 his DSc thesis "The study of spaces of differentiable functions of many variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor in Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he is a Professor of the S.M. Nikol'skii Mathematical Institute at the Peoples Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman are: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators. Main achievements: optimal embeddings of spaces with generalized smoothness, sharp conditions of the convergence of spectral expansions, descriptions of integral and differential properties of the generalized Bessel and Riesz-type potentials, sharp estimates for operators on cones and optimal envelopes for the cones of functions with properties of monotonicity. Professor M.L. Goldman has over 140 scientific publications in leading mathematical journals.

Under his scientific supervision, 8 candidate theses in Russia and 1 thesis in Kazakhstan were successfully defended. Some of his former students are now professors in Ethiopia, Columbia, Mongolia.

Participation in scientific and organizational activities of M.L. Goldman is well known. He is a member of the DSc Councils at RUDN and MSU, of the PhD Council in the Lulea Technical University (Sweden), a member of the Editorial Board of the Eurasian Mathematical Journal, an invited lector and visiting professor at universities of Russia, Germany, Sweden, UK etc., an invited speaker at many international conferences.

The mathematical community, friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Mikhail L'vovich Goldman on the occasions of his 75th birthday and wish him good health, happiness, and new achievements in mathematics and mathematical education.

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HYPERBOLICITY WITH WEIGHT OF POLYNOMIALS IN TERMS OF COMPARING THEIR POWER

H.G. Ghazaryan, V.N. Margaryan

Communicated by V.I. Burenkov

Key words: hyperbolic by Gårding polynomial, weak hyperbolic polynomial, hyperbolic with the weight polynomial, completely regular Newtons polyhedron.

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Abstract. For a given completely regular Newton polyhedron $\Re$, and a given vector $N \in \mathbb{R}^n$, we give conditions under which a weakly hyperbolic polynomial (with respect to the vector N) $P(\xi) = P(\xi_1, ..., \xi_n)$ is $\Re-$hyperbolic (with respect to the vector N). For polynomials of two variables, the largest number $s > 0$ is determined for which an $\Re-$hyperbolic (with respect to the vector N) polynomial is $s-$hyperbolic.

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1 Introduction

We use the following standard notation: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_0^n = \mathbb{N}_0 \times ... \times \mathbb{N}_0$ is the set of all n- dimensional multi-indices, i.e. points $\alpha = (\alpha_1, ..., \alpha_n) : \alpha_j \in \mathbb{N}_0$ $j = 1, ..., n$, $\mathbb{R}^n$ and $\mathbb{E}^n$ are the n-dimensional Euclidean spaces of points (vectors) $\xi = (\xi_1, ..., \xi_n)$ and $x = (x_1, ..., x_n)$ respectively, $\mathbb{R}^{n,+} := \{\xi \in \mathbb{R}^n, \xi_j \geq 0, \ j = 1, ..., n\}$ $\mathbb{C}^n = \mathbb{R}^n \times i\mathbb{R}^n$.

For $\xi, \eta \in \mathbb{R}^n$, $\zeta \in \mathbb{C}^n$ $\alpha \in \mathbb{N}_0^n$ and $\nu \in \mathbb{R}^{n,+}$ we put $(\xi, \eta) := \xi_1 \eta_1 + ... + \xi_n \eta_n$, $|\xi| := \sqrt{\xi_1^2 + ... + \xi_n^2}$, $|\zeta| := \sqrt{|\zeta_1|^2 + ... + |\zeta_n|^2}$, $\xi^\alpha = \xi_1^{\alpha_1} ... \xi_n^{\alpha_n}$, $|\zeta^\nu| := |\zeta_1|^{\nu_1} ... |\zeta_n|^{\nu_n}$ and $|\nu| := \nu_1 + ... + \nu_n$.

Let $\mathfrak{A} := \alpha^1, ..., \alpha^M$ be a finite set of points in $\mathbb{R}^{n,+}$. By the **Newton polyhedron** of the set $\mathfrak{A}$ we mean the minimal convex polyhedron $\Re = \Re(\mathfrak{A}) \subset \mathbb{R}^{n,+}$ containing all points of $\mathfrak{A} \cup \{0\}$ (see, for instance, [18], [12], [5]). A polyhedron $\Re \subset \mathbb{R}^{n,+}$ with vertices in $\mathbb{R}^{n,+}$ is said to be **complete** (see [18]) if $\Re$ has a vertex at the origin and (distinct from the origin) on each coordinate axis of $\mathbb{R}^n$. A complete polyhedron $\Re$ is called regular (completely regular), if all coordinates of the outward (relative to $\Re$) normals (futher $\Re-$normals) to its noncoordinate $(n-1)-$ dimensional faces are non - negative (postive) ([5] and [11]).

For a completely regular polyhedron $\Re \subset \mathbb{R}^{n,+}$ we denote

1) $\Re^0-$ the set of its vertices,

2) $\Lambda(\Re)-$ the set of its $\Re-$ normals of $(n-1)-$dimensional noncoordinate faces $\{\lambda = (\lambda_1, ..., \lambda_n)\}$, normalized so that $\min\limits_{1 \leq j \leq n} \lambda_j = 1$,

3) $\rho(\Re) := \max\limits_{\nu \in \Re} |\nu|$,

4)$d(\Re) := \max\limits_{\nu \in \Re, \, \lambda \in \Lambda(\Re)} (\lambda, \nu)$,

5) $h_\Re(\zeta) := \sum\limits_{\nu \in \Re^0} |\zeta^\nu|, \quad \zeta \in \mathbb{C}^n$.

By $\mathfrak{B}_n$ ($\mathfrak{B}'_n$ respectively) we denote the set of completely regular polyhedrons $\Re \subset \mathbb{R}^{n,+}$, for which $\rho(\Re) < 1$ ($d(\Re) < 1$ respectively). It is obvious that $\mathfrak{B}'_n \subset \mathfrak{B}_n$.

First we give examples of polyhedrons both belonging and not belonging to the sets $\mathfrak{B}_n$ or $\mathfrak{B}'_n$.

Example 1. Let $n = 2$, and let $\Re$ be the Newton polyhedron with vertices $\{(0,0),(\frac{2}{5},0)(\frac{3}{10},\frac{1}{5}),(0,\frac{2}{5})\}$. Then $\Lambda(\Re) = \{(2,1),(1,\frac{3}{2})\}$, $\rho(\Re) = \frac{1}{2}$, $d(\Re) = \frac{4}{5}$, therefore $\Re \in \mathfrak{B}'_2 \subset \mathfrak{B}_2$.

Example 2. Let $n = 2$, and let $\Re$ be the Newton polyhedron with vertices $\{(0,0),(\frac{3}{5},0),(\frac{1}{2},\frac{3}{10}),(0,\frac{3}{5}))\}$. Then $\Lambda(\Re) = \{(3,1),(1,\frac{5}{3})\}$, $\rho(\Re) = \frac{4}{5}$, $d(\Re) = \frac{9}{5}$, therefore $\Re \in \mathfrak{B}_2$ and $\Re \notin \mathfrak{B}'_2$.

Let $n \geq 2$ and let $\mathfrak{M}$ be a completely regular polyhedron. Then for any $\vartheta \in (0, \frac{1}{d(\mathfrak{M})})$ the polyhedron $\Re := \vartheta \mathfrak{M} \in \mathfrak{B}'_n$. Moreover if $\rho(\mathfrak{M}) < d(\mathfrak{M})$, then for any $\vartheta \in (\frac{1}{d(\mathfrak{M}}, \frac{1}{\rho(\mathfrak{M})})$ we have that $\rho(\Re) < 1$, and $d(\Re) > 1$ therefore $\Re := \vartheta.\mathfrak{M} \notin \mathfrak{B}'_n$, but $\Re \in \mathfrak{B}_n$.

For $\Re \in \mathfrak{B}'_n$ by $\Re^*$ we denote the Newton polyhedron of the set

$$\{(0,...,0, \max_{\nu\in\Re,\, \lambda\in\Lambda(\Re)} \frac{(\lambda,\nu)}{\lambda_j}, 0,...,0)\}_{j=1}^n.$$

It is obvious that $\Re \subset \Re^* \in \mathfrak{B}'_n$, and $\rho(\Re^*) = d(\Re^*) = d(\Re)$ for any $\Re \in \mathfrak{B}'_n$ Then (see [15]) there exists a number $c = c(\Re) > 0$ such that

$$h_{\Re}(\zeta+\eta) \leq c\,[h_{\Re}(\zeta) + h_{\Re^*}(\eta)] \quad \forall \zeta,\eta \in \mathbb{C}^n. \tag{1.1}$$

Since $\Re^* \in \mathfrak{B}'_n$ for $\Re \in \mathfrak{B}'_n$, then $\rho(\Re^*) < 1$. Therefore, from (1.1) it follows that for any $\varepsilon > 0$ there is a number $c_\varepsilon > 0$ such that

$$h_{\Re}(\zeta+\eta) \leq c_\varepsilon\, h_{\Re}(\zeta) + \varepsilon\,|\eta| \quad \forall \zeta,\eta \in \mathbb{C}^n. \tag{1.1$'$}$$

Let $P(\xi) = \sum_\alpha \gamma_\alpha \xi^\alpha$ be a polynomial, where the sum is taken over a finite set of multi-indices $(P) = \{\alpha \in \mathbb{N}_0^n, \gamma_\alpha \neq 0\}$ and $m := \max_{\alpha\in(P)} |\alpha|$. We represent the polynomial P as sum of homogeneous polynomials

$$P(\xi) = \sum_{j=0}^m P_j(\xi) := \sum_{j=0}^m [\sum_{|\alpha|=j} \gamma_\alpha \xi^\alpha],\ \xi \in \mathbb{R}^n. \tag{1.2}$$

Definition 1. (see [6] or [7] Definition 12.3.3 and Theorem 12.4.1) An operator $P(D)$ (a polynomial $P(\xi)$) is called **hyperbolic (by Gårding),** with respect to the vector $N \in \mathbb{R}^n$, if $P_m(N) \neq 0$ (see representation (1.2)) and there exists a number $\tau_0 > 0$ such that $P(\xi + i\,\tau\, N) \neq 0$ for all $(\xi,\tau) \in \mathbb{R}^{n+1}$ and $|\tau| > \tau_0$.

It is easy to verify that the polynomial P is hyperbolic with respect to vector $N \in \mathbb{R}^n$ if and only if $P_m(N) \neq 0$, and there exists a number $\tau_0 > 0$ such that $P(\xi + i\tau\, N) \neq 0$ for all $\xi \in \mathbb{R}^n$, $\tau \in \mathbb{C}$ and $|Re\tau| > \tau_0$.

A polynomial P is called **weakly hyperbolic** with respect to the vector $N \in \mathbb{R}^n$ (see, for instance, [13], [19] or [9], [10]) if $P_m(N) \neq 0$ and zeros of the polynomial $P_m(\xi + \tau\, N)$ with respect to τ are real for any point $\xi \in \mathbb{R}^n$, while there are multiple roots among them.

A polynomial P is called $s-$**hyperbolic** $(1 < s < \infty)$ (see. [13] or [19]) with respect to vector $N \in \mathbb{R}^n$ if $P_m(N) \neq 0$ and there exists a number $c > 0$ such that $P(\xi + i\,\tau\, N) \neq 0$ for all $(\xi,\tau) \in \mathbb{R}^{n+1}$, $|\tau| > c\,(1 + |\xi|^{1/s})$.

It was proved in [13] that the polynomial P is s- hyperbolic with respect to vector N if and only if $P_m(N) \neq 0$ and there exists a number $c > 0$ such that $P(\xi + i\,\tau\, N) \neq 0$ for all $\xi \in \mathbb{R}^n$, $\tau \in \mathbb{C}$ and $|Re\tau| \geq c\,(1 + |\xi|^{1/s})$.

A function g, defined on $\mathbb{R}^n$, is called a weight of hyperbolicity (see [14] or [16]), if
a) $g(\xi) \geq \kappa_0 > 0 \ \forall \xi \in \mathbb{R}^n$,
b) $g(\xi)/|\xi| \to 0$ as $|\xi| \to \infty$.

Definition 2. A polynomial P is called **hyperbolic with weight** g, (further $g-$**hyperbolic**) with respect to the vector $0 \neq N \in \mathbb{R}^n$ if $P_m(N) \neq 0$ and there exists a number $c > 0$ such that

$$P(\xi + i\,\tau\,N) \neq 0 \quad \forall \xi \in \mathbb{R}^n, \tau \in \mathbb{C}, |Re\,\tau| \geq c\,g(\xi).$$

In [6] (see also [7], Theorem 12.5.6) it was proved that for hyperbolic by Gȧrding (with respect to any vector N) operators, the non - characteristic Cauchy problem has one and only one solution from a certain class of smooth functions. Similar results (when $N = (1, 0, ..., 0)$) for s-hyperbolic operators were obtained in [13] by E. Larsson, and for $\Re-$hyperbolic operators in [1] by D. Calvo. In [14] - [16] (see also [17]), similar results were obtained for $\Re-$hyperbolic operators for more general Newton polyhedrons $\Re$ and for arbitrary vectors N.

It follows from the definition of a function $h_\Re$ that for any polyhedron $\Re \in \mathfrak{B}_n'$ the function $h_\Re$ is a weight of hyperbolisity. Therefore, further, if $\Re \in \mathfrak{B}_n'$ and the polynomial P is hyperbolic with weight $h_\Re$, then, for brevity we call the polynomial P **$\Re$-hyperbolic.**

Remark 1. It is easy to see that any $\Re$- hyberbolic ($\Re \in \mathfrak{B}_n'$) polynomial is $s := 1/\rho(\Re)-$hyberbolic and any $s := 1/\min_{0 \neq \nu \in \Re^0} |\nu|-$hyperbolic polynomial is $\Re-$hyberbolic.

It is known that

1) if the function g is bounded on $\mathbb{R}^n$, then the $g-$hyperbolic with respect to the vector $0 \neq N \in \mathbb{R}^n$ polynomial P is hyperbolic (by Gärding) with respect to the same vector. In addition, for an arbitrary function $f \in C_0^\infty$ with $supp\, f \subset \Omega_N := \{x, (x, N) \geq 0\}$ equation $P(D)u = f$ has a solution $u \in C^\infty$ with $supp\, u \subset \Omega_N$ (see [7], Theorems 12.4.5 and 12.5.4);

2) any weakly hyperbolic polynomial is $s := \frac{r}{r-1}-$ hyperbolic, where r is the maximal multiplicity of the zeros of the polynomial $P_m(\xi + \tau\,N)$ (with respect to τ) (see.[19] or [1]);

3) each $s-$ or $\Re-$hyperbolic polynomial (if $\Re \in \mathfrak{B}_n'$) is weakly hyperbolic. In addition, the equation $P(D)u = f$ for $f \in G^{0,s}$ with $supp\, f \subset \Omega_N$ has a solution $u \in G^s$ such that $supp\, u \subset \Omega_N$ (see [13]), where $G^s := G^s(\mathbb{R}^n)$ is an isotropic Gevrey space (see [4] or [8], paragraph 8.4), and $G^{0,s} := G^s \cap C_0^\infty$ with the appropriate topology (see [13], [19], [3] or [21]);

4) for the $\Re-$hyperbolic operator P (if $\Re \in \mathfrak{B}_n'$) and for any function $f \in G^{0,\Re}(\Omega_N)$ the equation $P(D)u = f$ has a solution $u \in G^\Re$ with $supp\, u \subset \Omega_N$, where $G^\Re$ is an anisotropic Gevrey space (see [14]).

Searches for the widest possible classes of linear partial differential equations for which the Cauchy problem can be posed correctly (necessary and sufficient conditions) led to the concept of the hyperbolicity of the equation (operator, polynomial). These searches continue to this day. Thus, hyperbolic equations are distinguished from the general class of equations by the fact that the Cauchy problem for such equations can be posed correctly.

However, the correctness of the Cauchy problem depends not only on the type of equation, but also on the functional space where this problem is studied.

The process of finding suitable function spaces where the Cauchy problem for a weakly hyperbolic operator is posed correctly led to the Gevrey classes (see [4]). These classes are intermediate between classes of infinitely differentiable and real analytic functions. The condition of strict hyperbolicity is sufficient for the correctness of the Cauchy problem in C^∞, generally speaking, this problem can be posed incorrectly for weakly hyperbolic equations. This can be easily verified even with the example of the heat conduction operator, for which the Cauchy problem can be posed incorrectly in the Gevrey class G^s for $s > 2$ as well in C^∞ (see, for instance, [2] or [19], Section 4.2]).

Searches for the widest possible classes of linear partial differential equations for which the Cauchy problem can be posed correctly (nacessary and sufficient conditions) led to the concept of the hyperbolicity of the equation (operator, polynomial). These searches continue to this day. Thus, hyperbolic equations are distinguished from the general class of equations by the fact that the Cauchy problem for such equations can be posed correctly.

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The present work is devoted to finding conditions under which a given polinomial is hyperbolic and to a comparison of hyperbolic polynomials of different types. In partcular,

1) necessary and sufficient conditions are obtained for $\Re-$hyperbolicity of polynomials in $n \geq 2$ variables (Theorem 2.2),

2) conditions are obtained under which a $\Re-$hyperbolic, with respect to a nonzero vector N^0 polynomial is $\Re-$hyperbolic with respect to any vector N in a neighborhood of the vector N^0 (Theorem 2.3),

3) for polynomials in two variables conditions are obtained under which an $\Re-$hyperbolic polynomial is $s-$hyperbolic, where the number s is uniquely determined by the polyhedron $\Re$ (Theorem 3.1).

2 Polynomials in $n \geq 2$ variables

Let $\zeta = (\zeta_1, \zeta') = (\zeta_1, \zeta_2, ..., \zeta_n)$, $\xi = (\xi_1, \xi') = (\xi_1, \xi_2, ..., \xi_n)$, $N^0 = (1, 0')$ and $P(\xi) = P(\xi_1, \xi')$ is a polynomial, represented in form (1.2), for which $P_m(N^0) \neq 0$. For any $\zeta' \in \mathbb{C}^{n-1}$, we denote $\mathfrak{D}(P, \zeta') := \{\zeta_1 = \zeta_1(\zeta') \in \mathbb{C}, P(\zeta) = 0\}$ and by $\lambda(\zeta')$ we denote an element in the set $\mathfrak{D}(P, \zeta')$, for which $|\lambda(\zeta')| = \max_{\zeta_1(\zeta') \in \mathfrak{D}(P, \zeta')} |\zeta_1|$ (if there are several of them, then we will take any one of them.). Let polynomial (1.2) be represented as follows

$$P(\xi) = \sum_{j=0}^{m} \xi_1^j Q_j(\xi') := \sum_{j=0}^{m} \xi_1^j \sum_{(j,\alpha') \in (P)} \gamma_{(j,\alpha')} (\xi')^{\alpha'}$$

$$= \gamma_{(m,0')} \xi_1^m + \sum_{j=0}^{m-1} \xi_1^j Q_j(\xi'). \quad (1.2')$$

Theorem 2.1. *Let $\Re \in \mathfrak{B}_{n-1}$ and P be a polynomial of degree m of form (1.2′) such that $P_m(N^0) \neq 0$. Then the following conditions are equivalent:*

1) there exists a number $\kappa_0 = \kappa_0(\Re, P) > 0$ such that for all $j = 0, 1, ..., m-1$

$$|Q_j(\xi')| := | \sum_{(j,\alpha') \in (P)} \gamma_{(j,\alpha')} (\xi')^{\alpha'} | \leq \kappa_0 \, h_{\Re}^{m-j}(\xi'), \;\; \xi' \in \mathbb{R}^{n-1}, \quad (2.1)$$

2) there exists a number $\kappa_1 > 0$ such that

$$|\lambda(\zeta')| \leq \kappa_1 \, h_{\Re}(\zeta'), \;\; \zeta' \in \mathbb{C}^{n-1}. \quad (2.2)$$

Proof. We show that (2.1) implies (2.2). Since $\{\alpha' \in \mathbb{N}_0^{n-1}, \; (j, \alpha') \in (P)\} \subset (m-j)\,\Re$, when (2.1) is fulfilled, hence, by the definition of the function $h_{\Re}$, relation (2.1) is equivalent to the fact that there exists a constant $\kappa_0^{'} > 0$ such that

$$|Q_j(\zeta')| \leq \kappa_0^{'} h_{\Re}^{m-j}(\zeta'), \ \zeta' \in \mathbb{C}^{n-1}, \ j = 0, 1, ..., m-1. \qquad (2.1').$$

Therefore, it suffices to prove that, under the conditions of the theorem, $(2.1^{'})$ implies (2.2). Assume the converse that if $(2.1')$ holds, there exists a sequence $\{(\zeta')^k\}_{k=1}^{\infty} \subset \mathbb{C}^{n-1}$ such that for $k \to \infty$

$$|\lambda((\zeta^{'})^k)|/h_{\Re}((\zeta')^k) \to \infty. \qquad (2.3)$$

Since $h_{\Re}(\zeta') \geq 1 \ \forall \zeta^{'} \in \mathbb{C}^{n-1}$, this implies that for $k \to \infty$ we get

$$|\lambda((\zeta^{'})^k)| \to \infty. \qquad (2.4)$$

On the other hand, due to the fact that $\lambda((\zeta^{'})^k) \in \mathfrak{D}(P, (\zeta^{'})^k)$ $(k = 1.2....)$ and $P_m(N^0) \neq 0$, by virtue of inequality $(2.1^{'})$ and relations (2.3) - (2.4) we have

$$0 = |P(\lambda((\zeta^{'})^k), (\zeta^{'})^k)| = |\gamma_{(m,0')}(\lambda((\zeta^{'})^k)^m + \sum_{j=0}^{m-1}[\lambda((\zeta^{'})^k)]^j \ Q_j(\zeta^{'})^k)|$$

$$\geq |P_m(N^0)| \, |\lambda(\zeta^{'})^k)|^m - \sum_{j=0}^{m-1} |\lambda((\zeta^{'})^k)|^j \ |Q_j(\zeta^{'})^k)| \geq |P_m(N^0)| \, |\lambda(\zeta^{'})^k)|^m$$

$$-\kappa_0^{'} \sum_{j=0}^{m-1} |\lambda((\zeta^{'})^k)|^j \ h_{\Re}^{m-j}\left((\zeta^{'})^k\right) = |P_m(N^0)| \, | \, |\lambda(\zeta^{'})^k)|^m [1 + o(1)] \to \infty$$

as $k \to \infty$.

The obtained contradiction proves, that under the conditions of the theorem, estimate (2.2) follows from estimate (2.1).

Now, let us prove the converse, namely, that estimate (2.1) follows from estimate (2.2). Since $P_m(N^0) \neq 0$, for any fixed point $\zeta' \in \mathbb{C}^{n-1}$ the polynomial (with respect to τ) $P(\tau, \zeta')$ has m roots : $\tau_1(\zeta'), ..., \tau_m(\zeta')$. Then for all $\zeta' \in \mathbb{C}^{n-1}$ we have

$$P(\tau, \zeta') = \sum_{j=0}^{m-1} \tau^j \, Q_j(\zeta') + P_m(N^0) \, \tau^m = P_m(N^0) \prod_{j=1}^{m} (\tau - \tau_j(\zeta')).$$

Since, by virtue of the Vieta theorem

$$Q_j(\zeta') = (-1)^{m-j} \sum_{1 \leq r_l \leq m,} \tau_{r_1}(\zeta')...\tau_{r_{m-j}}(\zeta'), \ \zeta' \in \mathbb{C}^{n-1} \ j = 0, 1, ..., m-1,$$

by virtue of the definition of $\lambda(\zeta^{'})$ from estimate (2.2) we have

$$|Q_j(\zeta')| \leq \kappa_1 \, C_m^{m-j} \, h_{\Re}^{m-j}(\zeta') \ \ \forall \zeta' \in \mathbb{C}^{n-1}, j = 0, 1, ..., m-1,$$

where $\{C_m^{m-j}\}$ are the binomial coefficients. Denoting by κ_0' the maximum of $\{\kappa_1 \, C_m^{m-j}\}_{j=1}^{m-1}$, we obtain (2.1'). □

Theorem 2.2. *Let $0 \neq N \in \mathbb{R}^n$, a polynomial P of degree m, represented in form (1.2) such that $P_m(N) \neq 0$ and $\Re \in \mathfrak{B}_n^{'}$. Then the polynomial P is $\Re-$hiperbolic with respect to the vector N if and only if there exists a constand $c > 0$ such that*

$$P(\xi + i \, \tau \, N) \neq 0 \ \ \xi \in \mathbb{R}^n, \, \tau \in \mathbb{C}, \ |Re \, \tau| \geq c \, g_{\Re, N}(\xi), \qquad (2.5)$$

where $g_{\Re, N}(\xi) := \inf_{t \in \mathbb{R}^1} h_{\Re}(\xi - t \, N)$.

Remark 2. It is easy to verify that for $\Re \in \mathfrak{B}_n'$ the function $g_{\Re, N}$ is a weight of hyperbolicity.

Proof. *Neccessity* Let a polynomial P be $\Re-$hyperbolic with respect to a vector N. Let us prove that for some constant $c > 0$ relation (2.5) holds. By the definition of $\Re-$hyperbolicity of the polynomial P (with respect to N) there exists a number $c_1 > 0$ such that for all pairs $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{C}$ for which $P(\xi + i\,\tau\, N) = 0$ the following inequaliy holds

$$|Re\, \tau| \le c_1\, h_{\Re}(\xi). \tag{2.6}$$

Since for any $t \in \mathbb{R}^1$ for the mentioned pairs $\{\xi, \tau\}$ the relation $P(\xi - t\,N + i\,(\tau - i\,t)\,N) = P(\xi + i\,\tau\,N)$ holds, then, by estimate (2.6), for such pairs $\{\xi, \tau\}$ we have that for such pairs $P(\xi + i\,\tau\,N) = 0$ and $|Re\,\tau| = |Re\,(\tau - i\,t)| \le c_1\, h_{\Re}(\xi - t\,N)$ for any $t \in \mathbb{R}$. Since t is arbitrary, it follows that for all $\xi \in \mathbb{R}^n$ and $\tau \in \mathbb{C}$ such that $P(\xi + i\,\tau\,N) = 0$ the inequality $|Re\,\tau| \le c_1\, g_{\Re, N}(\xi)$ holds.

From here, in turn, it follows that $P(\xi + i\,\tau\,N) \ne 0$ for all pairs $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{C}$ such that $|Re\,\tau| > c_1\, g_{\Re, N}(\xi)$. Relation (2.5) is proved.

Sufficiency. Let relation (2.5) be satisfied. Let us prove that the polynomial P is $\Re-$hyperbolic. Since $h_{\Re}(\xi) \ge g_{\Re, N}(\xi)$ for all $\xi \in \mathbb{R}^n$, it follows from (2.5) that $P(\xi + i\,\tau\,N) \ne 0$ for all $\xi \in \mathbb{R}^n$ and $\tau \in \mathbb{C}$ such that $|Re\,\tau| > c\, h_{\Re}(\xi)$. From this and the condition $P_m(N) \ne 0$ of the theorem it follows that the polynomial P is $\Re-$hyperbolic with respect to N. □

We will use the following proposition in the proof of Theorem 2.3

Proposition 2.1 Let $\mathfrak{M} \subset \mathfrak{B}_{n-1}'$, and $\Re \in \mathbb{R}^{n,+}$ be the Newton polyhedron of the set $\{(0, \nu') : \nu' \in \mathfrak{M}^0\} \cup \{(d(\mathfrak{M}), 0')\}$. Then $\Re \in \mathfrak{B}_n'$, and $\rho(\Re) = d(\Re) = d(\mathfrak{M})$ $(= \rho(\mathfrak{M}^*)$.

Proof. For any $\lambda' \in \Lambda(\mathfrak{M})$ we denote $d_{\lambda'}(\mathfrak{M}) := \max_{\nu' \in \mathfrak{M}}(\lambda', \nu')$, then $d(\mathfrak{M}) = max_{\lambda' \in \Lambda(\mathfrak{M})} d_{\lambda'}(\mathfrak{M})$. Since $\Re = \{\nu \in \mathbb{R}^{n,+}, (\nu, (1, \lambda')) \le d_{\lambda'}(\mathfrak{M}) \quad \forall \lambda' \in \Lambda(\mathfrak{M})\}$, then $\Lambda(\Re) = \{(1, \lambda'), \lambda' \in \Lambda(\mathfrak{M})\}$. Recall that by the definition of the set $\Lambda(\mathfrak{M})$, $min_{2 \le j \le n} \lambda_j' = 1$, $\forall \lambda' \in \Lambda(\mathfrak{M})$, therefore $\Re = \{\nu \in \mathbb{R}^{n,+}, (\lambda, \nu) \le d_{\lambda'}(\mathfrak{M})\}$ $\forall \lambda = (1, \lambda') \in \Lambda(\Re)$, where $\lambda' \in \Lambda(\mathfrak{M})\}$. It immediately follows that $d(\Re) = \max_{\lambda' \in \Lambda(\mathfrak{M})} d_{\lambda'}(\mathfrak{M}) = d(\mathfrak{M})$.

Since $\mathfrak{M} \in \mathfrak{B}_{n-1}'$, then $d(\mathfrak{M}) < 1$, therefore $d(\Re) < 1$.

On the other hand, by the definition of the polyhedron $\Re$, the point $(d(\mathfrak{M}), 0') \in \Re^0$ and $min_{1 \le j \le n} \lambda_j = 1$, $\forall \lambda \in \Lambda(\Re)$, therefore $\rho(\Re) = d(\Re)$. □

Theorem 2.3. *Let $\mathfrak{M} \subset \mathfrak{B}_{n-1}'$, $\Re \in \mathbb{R}^{n,+}$ be the Newton polyhedron of the collection $\{(0, \nu') : \nu' \in \mathfrak{M}^0\} \cup (d(\mathfrak{M}), 0')$, U be a neighborhood of the vector N^0, such that $U \subset \{N; |N - N^0| < 1/2\}$ and $P_m(N) \ne 0$ for any $N \in U$. Let P be a polynomial of degree m, represented in form (1.2′), for which, with some constant $c > 0$, the following relation holds*

$$|Q_j(\xi')| = |\sum_{(j,\alpha') \in (P)} \gamma_{(j,\alpha')}(\xi')^{\alpha'}| \le c\, h_{\mathfrak{M}}^{m-j}(\xi') \;\; \xi' \in \mathbb{R}^{n-1}, \;\; j = 1, ..., m-1.$$

Then for any $N \in U$ the polynomial P is $\Re-$hyperbolic with respect to the vector N.

Proof. Let $N \in U$. Since $P_m(N) \ne 0$, for any point $\xi \in \mathbb{R}^n$ the number of zeros of the polynomial $P(\xi + i\,\tau\,N)$ (with respect to τ) is m. Let $\{\tau_j(\xi)\}_{j=1}^m$ be the zeros of the polynomial $P(\xi + i\,\tau\,N)$ and $\zeta = (\zeta_1, \zeta') := (\xi_1 + i\,\tau_j(\xi)\,N_1, \xi' + i\,\tau_j(\xi)\,N')$. Then from the second statement of Theorem 2.1 (see inequality (2.2)) it follows that $|\zeta_1(\zeta')| \le \kappa_1\, h_{\mathfrak{M}}(\zeta')$, or, which is the same

$$|\,\xi_1 + i\,\tau_j(\xi)\,N_1| \le \kappa_1\, h_{\mathfrak{M}}(\xi' + i\,\tau_j(\xi)\,N') \;\; \xi \in \mathbb{R}^n, \;\; j = 1, ..., m.$$

Hence, by virtue of inequality (1.1), for some constant $\kappa_2 > 0$ we have

$$|\,\xi_1 + i\,\tau_j(\xi)\,N_1| \le \kappa_2\,[h_{\mathfrak{M}}(\xi') + h_{\mathfrak{M}^*}(\tau_j(\xi)\,N')]\ \ \xi \in \mathbb{R}^n,\ j = 1, ..., m.$$

Since $\rho(\mathfrak{M}^*) = d(\mathfrak{M})$ by virtue of $\mathfrak{M}^*$ and condition $\mathfrak{M} \in \mathfrak{B}'_{n-1}$, for some constant $\kappa_3 > 0$ we obtain

$$|\,\xi_1 + i\,\tau_j(\xi)\,N_1| \le \kappa_3\,[h_{\mathfrak{M}}(\xi') + |\tau_j(\xi)|^{\rho(\mathfrak{M}^*)}]$$
$$= \kappa_3\,[h_{\mathfrak{M}}(\xi') + |\tau_j(\xi)|^{d(\mathfrak{M})}]\ \ \forall \xi \in \mathbb{R}^n,\ j = 1, ..., m.$$

Since $N_1 > 1/2$, for $N \in U$, for some positive constants κ_4, κ_5 we have

$$|\,\xi_1 + i\,\tau_j(\xi)\,N_1| \le \kappa_4\,[h_{\mathfrak{M}}(\xi') + |i\,\tau_j(\xi)\,N_1|^{d(\mathfrak{M})}]$$

$$= \kappa_4\,[h_{\mathfrak{M}}(\xi') + |\xi_1 + i\,\tau_j(\xi)\,N_1 - \xi_1|^{d(\mathfrak{M})}]$$

$$\le \kappa_5\,[h_{\mathfrak{M}}(\xi') + |\,\xi_1 + i\,\tau_j(\xi)\,N_1|^{d(\mathfrak{M})} + |\xi_1|^{d(\mathfrak{M})}]\ \ \forall \xi \in \mathbb{R}^n,\ j = 1, ..., m.$$

Since $d(\mathfrak{M}) \in (0,1)$ for $\mathfrak{M} \in \mathfrak{B}'_{n-1}$, it follows that for any $\varepsilon > 0$ there exists a number $c_\varepsilon > 0$ such that for all $j = 1, ..., m$ we have

$$|\,\xi_1 + i\,\tau_j(\xi)\,N_1| \le \kappa_5\,[h_{\mathfrak{M}}(\xi') + \varepsilon\,|\,\xi_1 + i\,\tau_j(\xi)\,N_1| + c_\varepsilon + |\xi_1|^{d(\mathfrak{M})}]\ \ \forall \xi \in \mathbb{R}^n.$$

Since $h_{\mathfrak{M}}(\xi') \ge 1\ \ \forall \xi' \in \mathbb{R}^{n-1}$, it follows that for $\kappa_5\,\varepsilon = 1/2$ and some constant $\kappa_6 > 0$ we obtain

$$|\,\xi_1 + i\,\tau_j(\xi)\,N_1| \le \kappa_6\,[h_{\mathfrak{M}}(\xi') + |\xi_1|^{d(\mathfrak{M})}]\ \ \forall \xi \in \mathbb{R}^n,\ j = 1, ..., m.$$

By the definition of the polyhedrons $\Re$ and $\mathfrak{M}$ we have $h_{\mathfrak{M}}(\xi') + |\xi_1|^{d(\mathfrak{M})} = h_{\Re}(\xi)$ for $\xi \in \mathbb{R}^n$. Therefore,

$$|\,\xi_1 + i\,\tau_j(\xi)\,N_1| \le \kappa_6\,h_{\Re}(\xi)\ \ \forall \xi \in \mathbb{R}^n,\ j = 1, ..., m.$$

Bearing in mind that $N_1 \ge 1/2$, we obtain

$$|Re\,\tau_j(\xi)| \le 2\,\kappa_6\,h_{\Re}(\xi)\ \ \forall \xi \in \mathbb{R}^n,\ j = 1, ..., m. \tag{2.7}$$

So, we have proved that for any root $\tau(\xi)$ of the polynomial $P(\xi + i\,\tau\,N)$ (in variable τ) inequality (2.7) holds, i.e. $P(\xi + i\,\tau\,N) \ne 0$ for $\xi \in \mathbb{R}^n$, $|Re\tau| > 2\,\kappa_6\,h_{\Re}(\xi)$. This means that the polynomial P is $\Re-$hyperbolic with respect to vector N. □

We give an example of a polynomial that is not hyperbolic by Gårding, but is $\Re-$hyperbolic for a certain completely regular polyhedron $\Re$.

Example 3. Let $n = 3$, $P(\xi) = \xi_1^{15} + \xi_1^{10}\,\xi_2^2\,\xi_3^2 + \xi_1^{10}\,(\xi_2^3 + \xi_3^3) + \xi_2^6\,\xi_3^6 + \xi_2^9 + \xi_3^9$.

Here $m = 15$, $P(\xi) = P_{15}(\xi) + P_{14}(\xi) + P_{13}(\xi) + P_{12}(\xi) + P_9(\xi)$, $\mathfrak{M}$ is the Newton polyhedron of the set $\{(0,3/5,0),\ (0,0,3/5),\ (0,2/5,2/5)\}$, $d(\mathfrak{M}) = 4/5$, $\Re \in \mathbb{R}^{n,+}-$Newton polyhedron of the set $\{(0,\nu') : \nu' \in \mathfrak{M}^0\} \ \cup \{(d(\mathfrak{M}), 0')\} = \{(0,3/5,0),\ (0,0,3/5),\ (0,2/5,2/5), (4/5,0,0)\}$, $\xi = (\xi_1, \xi')$, $\xi' = (\xi_2, \xi_3)$. Simple calculations show that $h_{\mathfrak{M}}(\xi') = h_{\mathfrak{M}}(\xi_2, \xi_3) = 1 + |\xi_2|^{3/5} + |\xi_3|^{3/5} + |\xi_2|^{2/5}\,|\xi_3|^{2/5}$ and $Q_0(\xi') = \xi_2^6\,\xi_3^6 + \xi_2^9 + \xi_3^9$, $Q_j(\xi') \equiv 0$ $(j = 1, ..., 9)$, $Q_{10}(\xi') = \xi_2^2\,\xi_3^2 + \xi_2^3 + \xi_3^3$, $Q_j(\xi') \equiv 0$ $(j = 11, ..., 14)$.

Since the polynomial P_{15} is not stronger (according to L. Hörmander) (see [7], Definition 10.3.4) than (for example) polynomial P_{14}, it follows from the Svensson theorem on the necessary conditions for hyperbolicity by Gårding (see [20]) that the polynomial P is not hyperbolic by Gårding with respect to the vector $N^0 = (1,0,0)$.

Now we show that the polynomial P is $\mathfrak{R}-$hyperbolic with respect to any vector $N \in \mathbb{R}^3$: $|N - N^0| < 1/2$. Obviously $P_{15}(N) \neq 0$ for the indicated N. Verification of the existence of $c > 0$ for which the inequality

$$|Q_j(\xi')| \leq c\, h^{m-j}_{\mathfrak{M}}(\xi') \;\; \forall \xi' \in \mathbb{R}^2, \;\; j = 1, ..., 14$$

is satisfied, reduces to proving the following easily verified estimates for all $(\xi_2, \xi_3) \in \mathbb{R}^2$

$$|\xi_2^3 + \xi_3^3 + \xi_2^2\, \xi_3^2| \leq \; (1 + |\xi_2|^{3/5} + |\xi_3|^{3/5} + |\xi_2|^{2/5}\, |\xi_3|^{2/5})^5,$$

$$|\xi_2^9 + \xi_3^9 + \xi_2^6\, \xi_3^6| \;\leq\; (1 + |\xi_2|^{3/5} + |\xi_3|^{3/5} + |\xi_2|^{2/5}\, |\xi_3|^{2/5})^{15},$$

Thus, by Theorem 2.3, the considered polynomial is $\mathfrak{R}-$hyperbolic.

3 Polynomials in $n = 2$ variables

In this section, we find conditions under which an $\mathfrak{R}-$hyperbolic polynomial of two variables is $s-$hyperbolic. We preliminarly prove several propositions that we will need to prove the main theorem of this section (Theorem 3.1). In this case, we use the notation $h_{\mathfrak{R},N}(\xi) := \inf_{t \in \mathbb{R}^1} h_{\mathfrak{R}}(\xi - t\,N)$.

Lemma 3.1. *Let* $n = 2$, $\mathfrak{R} \in \mathfrak{B}'_2$, $(\sigma_1, 0)$ *and* $(0, \sigma_2)$ *are nonzero vertices of the polyhedron* $\mathfrak{R}$, *lying on the coordinate axes* $\mathbb{R}^{2,+}$, $s := 1/min\{\sigma_1, \sigma_2\}$, $N = (N_1, N_2) \in \mathbb{R}^2$, $N_1 N_2 \neq 0$. *Then there exists a number* $c > 0$ *such that for all* $\xi \in \mathbb{R}^2$ *we have*

$$c^{-1}\,(1 + |N_2\, \xi_1 - N_1\, \xi_2|^{1/s}) \leq h_{\mathfrak{R},N}(\xi) \leq c\,(1 + |N_2\, \xi_1 - N_1\, \xi_2|^{1/s}). \tag{3.1}$$

Proof. It is obvious that for all $\xi \in \mathbb{R}^2$

$$h_{\mathfrak{R},N}(\xi) \leq h_{\mathfrak{R}}(\xi - \frac{\xi_2}{N_2}\, N) = h_{\mathfrak{R}}(\xi_1 - \frac{N_1}{N_2}\, \xi_2, 0) = 1 + |\, \xi_1 - \frac{N_1}{N_2}\, \xi_2|^{\sigma_1})$$

and

$$h_{\mathfrak{R},N}(\xi) \leq h_{\mathfrak{R}}(\xi - \frac{\xi_1}{N_1}\, N) = h_{\mathfrak{R}}(0,\, \xi_2 - \frac{N_2}{N_1}\, \xi_1) = 1 + |\, \xi_2 - \frac{N_2}{N_1}\, \xi_1|^{\sigma_2},$$

where the right-hand side of inequality (3.1) with any constant $c > max\{1, |N_1|, |N_2|\}$ follows immediately.

Let us prove the left-hand side of estimate (3.1). By the definition of the function $h_{\mathfrak{R}}$ and the property $h_{\mathfrak{R}}(\eta) \geq 1\; \forall \eta \in \mathbb{R}^2$, for arbitrary $t \in \mathbb{R}^1$ and $\xi \in \mathbb{R}^2$, we have

$$1 + |\, \xi_1 - \frac{\xi_2}{N_2}\, N_1|^{1/s} = 1 + |\, \xi_1 - N_1\, t + \frac{N_1}{N_2}\, [(N_2\, t - \xi_2)]\, |^{1/s} \leq 1 + |\xi_1 - N_1\, t|^{1/s}$$

$$+ |\, \frac{N_1}{N_2}\, |^{1/s}\, |\xi_2 - N_2\, t\, |^{1/s} \leq 2 + |\xi_1 - N_1\, t|^{\sigma_1} + |\, \frac{N_1}{N_2}\, |^{1/s}\, [1 + |\, \xi_2 - N_2\, t\, |^{\sigma_2})$$

$$\leq 1 + h_{\mathfrak{R}}(\xi_1 - N_1\, t, 0) + |\, \frac{N_1}{N_2}\, |^{1/s}\; h_{\mathfrak{R}}(0, \xi_2 - N_2\, t) \leq 1 + (1 + |\frac{N_1}{N_2}|^{1/s})\, h_{\mathfrak{R}}(\xi - Nt)$$

$$\leq (2 + |\frac{N_1}{N_2}|^{1/s})\, h_{\mathfrak{R}}(\xi - N\, t).$$

Hence, due to the arbitrariness of the number $t \in \mathbb{R}^1$, we obtain the left hand side of estimate (3.1). □

For a polynomial $R(\xi) = R(\xi_1, ..., \xi_n)$ and a number $\tau \in \mathbb{R}^1$ we denote (L. Hörmander's function): $\tilde{R}(\xi, \tau) := \sum_\alpha |R^{(\alpha)}(\xi)|\, |\tau|^{|\alpha|}$ and introduce the following comparison relation (with weight) of two polynomials

Definition 3. Let g be a weight of hyperbolicity. We say that polynomial P is $g-$stronger than a polynomial Q (a polynomial Q is $g-$weaker than a polynomial P) and write : $Q \prec^g P$ if, for some constant $c > 0$

$$\tilde{Q}(\xi, g(\xi)\,) \leq c\, \tilde{P}(\xi, g(\xi)\,) \quad \forall \xi \in \mathbb{R}^n.$$

This definition directly implies

Lemma 3.2. *Let g_1 and g_2 be weights of hyperbolicity such that $g_1(\xi) \geq c\, g_2(\xi)$ $\forall \xi \in \mathbb{R}^n$ for some constant $c > 0$. If $Q \prec^{g_2} P$, then $Q \prec^{g_1} P$.*

Lemma 3.3. *Let g_1 and g_2 be weights of hyperbolicity and $g := g_1 + g_2$. Then $Q \prec^g P$, if and only if $Q \prec^{g_j} P$ $(j = 1, 2)$.*

Proof. The proof immediately follows from Lemma 3.2 and the following easily verified inequality for any $k \in \mathbb{N}$

$$g_1^k(\xi) + g_2^k(\xi) \leq g^k(\xi) \leq 2^k\, \left(g_1^k(\xi) + g_2^k(\xi)\right) \quad \forall \xi \in \mathbb{R}^n.$$

□

Lemma 3.4. *Let $N^0 = (1, 0)$, $\Re \in \mathfrak{B}_2'$, $(0, \sigma_2)$ be a nonzero vertex of the polyhedron $\Re$, lying on the axis $0\xi_2$, $\Delta \in (0, 1)$, $\mathfrak{M}_\Delta$ be the Newton polyhedron of points $\{(\Delta, 0), (0, \sigma_2)\}$ and P be a $\Re-$hyperbolic polynomial with respect to a vector N^0. Then P is 1)$\mathfrak{M}_\Delta-$hyperbolic and 2) $s_1 := 1/\sigma_2-$hyperbolic polynomial with respect to the vector N^0.*

Proof. To prove part one, it suffices to note that under the conditions of the lemma $h_{\mathfrak{M}_\Delta, N}(\xi) = 1 + |\xi_2|^{\sigma_2} = \inf_{t \in \mathbb{R}^1} h_\Re(\xi - t\, N)$ for any $\Delta \in (0, 1)$. Part 2) follows directly from Part 1). □

Lemma 3.5. *Let $\Re \in \mathfrak{B}_2'$, $(\sigma_1, 0)$ and $(0, \sigma_2)$ be vertices of the polyhedron $\Re$, lying on the coordinate axes $\mathbb{R}^2$, $s := 1/min\{\sigma_1, \sigma_2\}$, $N = (N_1, N_2) \in \mathbb{R}^2$, $N_1\, N_2 \neq 0$, and P be an $\Re-$hyperbolic polynomial with respect to the vector N. Then P is $s-$hyperbolic with respect to the same vector.*

Proof. We need to prove that there exists a number $\kappa_0 > 0$ such that

$$P(\xi + i\, \tau\, N) \neq 0 \;\; \forall (\xi,\, \tau) \in \mathbb{R}^2 \times \mathbb{C} : |Re \tau| \geq \kappa_0\, (1 + |\xi|^{\frac{1}{s}}). \tag{3.2}$$

It is obvious that for any $\varepsilon > 0$ there exists a constant $\kappa_1 = \kappa_1(\varepsilon) > 0$ such that

$$1 + |\xi|^\varepsilon \geq \kappa_1\, (1 + |N_2\, \xi_1 - N_1\, \xi_2|^\varepsilon), \;\; \xi \in \mathbb{R}^2.$$

On the other hand, by virtue of Lemma 3.1 there exists a number $\kappa_2 > 0$ such that

$$h_{\Re,\, N}(\xi) \leq \kappa_2\, (1 + |N_2\, \xi_1 - N_1\, \xi_2|^{1/s}) \;\; \forall \xi \in \mathbb{R}^2.$$

From the last two relations we obtain with some constant $\kappa_3 > 0$

$$1 + |\xi|^{\frac{1}{s}} \geq \kappa_3\, h_{\Re,\, N}(\xi) \;\; \forall \xi \in \mathbb{R}^2. \tag{3.3}$$

Since the polynomial P is $\Re-$hyperbolic with respect to the vector N, i.e. for some constant $\kappa_4 > 0$ we have the relation

$$P(\xi + i\, \tau\, N) \neq 0 \;\; \forall \xi,\, \tau \in \mathbb{R}^2 \times \mathbb{C} : |Re\, \tau| \geq \kappa_4\;\, h_{\Re,\, N}(\xi),$$

then this and (3.3) imply relation (3,2) with the constant $\kappa_0 = \kappa_4/\kappa_3$. This proves that the polynomial P is $s-$hyperbolic with respect to the vector N. □

Theorem 3.1. *Let* $N^0 = (1,0)$, *and let* $U = U(N^0)$ *be a neighbourhood of the vector* N^0, *such that* $U \subset \{N; |N - N^0| < 1/2\}$.

Let $\Re \in \mathfrak{B}_2^{'}$, $(\sigma_1, 0)$ *and* $(0, \sigma_2)$ *be nonzero vertices of the polyhedron* $\Re$, *lying on the coordinat axes of* $\mathbb{R}^{2,+}$, $s := 1/min\{\sigma_1, \sigma_2\}$, P_m *be a homogeneous polynomial of degree* m, *hyperbolic (by Gȧrding) with respect to the vector* N^0, $P_m(N) \neq 0$ *for* $N \in U(N^0)$. *and* Q *be a polynomial of degree less than* m.

1) If there exists a vector $N^1 \in U = U(N^0)$ *that is not collinear to the vector* N^0 *such that* $Q \prec^{h_{\Re,N^j}} P_m$ $(j = 0, 1)$, *then the polynomial* $P_m + Q$ *is* $s_1 := 1/\sigma_2-$*hyperbolic with respect to any vector* $N \in U(N^0)$.

2) If there are noncollinear vectors N^1, $N^2 \in U(N^0)$, *all of which coordinates are nonzero and such that* $Q \prec^{h_{\Re,N^j}} P_m$ $(j = 1, 2)$, *then the polynomial* $P_m + Q$ *is* $s-$*hyperbolic with respect to any vector* $N \in U(N^0)$.

Proof. **Statement 1).** From the assumption of part one it follows that $Q \prec^{h_{\Re,N^0} + h_{\Re,N^1}} P_m$. Since the vectors N^0 and N^1 are not collinear, then (see Lemma 3.1) there exists a number $c_1 > 0$ such that

$$c_1^{-1}\,(1 + |\xi_1|^{1/s} + |\xi_2|^{\sigma_2}) \leq h_{\Re,N^0}(\xi) + h_{\Re,N^1}(\xi) \leq c_1^{-1}\,(1 + |\xi_1|^{1/s} + |\xi_2|^{\sigma_2}), \;\; \xi \in \mathbb{R}^2,$$

consequently

$$g(\xi) := 1 + |\xi|^{\sigma_2} \geq \frac{1}{2}\, c_1^{-1}\,[h_{\Re,N^0}(\xi) + h_{\Re,N^1}(\xi)], \;\; \xi \in \mathbb{R}^2.$$

Therefore, by Lemma 3.2, we obtain $Q \prec^{g} P_m$. Since the polynomial P_m is hyperbolic (by Gȧrding) with respect to any vector $N \in U(N^0)$, (see [6] or [7], theorems 12.4.4 and 12.4.2), by virtue of Theorem 3.3 of [16] for any $N \in U(N^0)$, there exists a number $c_2 > 0$ such that

$$(P_m + Q)\,(\xi + i\,\tau\,N) \neq 0, \;\; (\xi, \tau) \in \mathbb{R}^{n+1}, |\tau| \geq c_2\, g(\xi),$$

i.e. the polynomial $P_m + Q$ is $1/\sigma_2-$hyperbolic with respect to any vector $N \in U(N^0)$ (see the definition of s - hyperbolicity).

Statement 2). From the conditions on the vectors N^1, N^2 and from Lemma 3.1, we have for some constant $c_3 > 0$

$$c_3^{-1}\,(1 + |\xi|)^{1/s} \leq h_{\Re,N^1}(\xi) + h_{\Re,N^2}(\xi) \leq c_3\,(1 + |\xi|)^{1/s}, \;\; \xi \in \mathbb{R}^2.$$

Carrying out calculations similar to those performed in the proof of part 1, we obtain that the polynomial $P_m + Q$ is $s-$hyperbolic with respect to any vector $N \in U(N^0)$. □

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