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MIKHAIL L'VOVICH GOLDMAN

(to the 75th birthday)

Mikhail L'vovich Goldman was born on April 13, 1945 in Moscow. In 1963 he graduated from school in Moscow and entered the Physical Faculty of the M.V. Lomonosov Moscow State University (MSU) from which he graduated in 1969 and became a PhD student (1969–1972) at the Mathematical Department of this Faculty. In 1972 he has defended the PhD thesis, and in 1988 his DSc thesis "The study of spaces of differentiable functions of many variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor in Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he is a Professor of the S.M. Nikol'skii Mathematical Institute at the Peoples Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman are: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators. Main achievements: optimal embeddings of spaces with generalized smoothness, sharp conditions of the convergence of spectral expansions, descriptions of integral and differential properties of the generalized Bessel and Riesz-type potentials, sharp estimates for operators on cones and optimal envelopes for the cones of functions with properties of monotonicity. Professor M.L. Goldman has over 140 scientific publications in leading mathematical journals.

Under his scientific supervision, 8 candidate theses in Russia and 1 thesis in Kazakhstan were successfully defended. Some of his former students are now professors in Ethiopia, Columbia, Mongolia.

Participation in scientific and organizational activities of M.L. Goldman is well known. He is a member of the DSc Councils at RUDN and MSU, of the PhD Council in the Lulea Technical University (Sweden), a member of the Editorial Board of the Eurasian Mathematical Journal, an invited lector and visiting professor at universities of Russia, Germany, Sweden, UK etc., an invited speaker at many international conferences.

The mathematical community, friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Mikhail L'vovich Goldman on the occasions of his 75th birthday and wish him good health, happiness, and new achievements in mathematics and mathematical education.

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ON THE SPECTRAL ANALYSIS OF A DIFFERENTIAL OPERATOR WITH AN INVOLUTION AND GENERAL BOUNDARY CONDITIONS

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Abstract. We study first-order differential operators with an involution and non-periodic boundary conditions. We exhibit their spectral properties such as the asymptotic estimates of their eigenvalues, eigenvectors and spectral projections. We also use these properties to estimate the groups generated by the differential operators we study. The results were obtained by using the method of similar operators.

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1 Introduction

Let $H = L_2[0,\omega]$ be the complex Hilbert space of all (equivalence classes of) square integrable functions on the closed interval $[0,\omega]$. The inner product in the space $L_2[0,\omega]$ is defined, as usually, by

$$\langle x, y\rangle = \frac{1}{\omega}\int_0^\omega x(s)\overline{y(s)}\,ds$$

and the $L_2[0,\omega]$ norm is induced by this inner product. We denote by $W_2^1[0,\omega]$ the Sobolev space of all absolutely continuous functions from $[0,\omega]$ whose derivatives (existing almost everywhere on $[0,\omega]$) belong to also in $L_2[0,\omega]$.

We consider a differential operator $L : D(L) \subset L_2[0,\omega] \to L_2[0,\omega]$ generated by the differential expression with an involution:

$$(lx)(s) = \frac{dx}{ds}(s) - q_0(s)x(s) - q_1(s)x(\omega - s), \quad s \in [0,\omega], \tag{1.1}$$

where $x \in L_2[0,\omega]$ and q_0, $q_1 \in L_2[0,\omega]$.

The domain $D(L)$ of the operator L is defined by the general boundary conditions

$$D(L) = \{x \in W_2^1[0,\omega] : x(0) = \gamma x(\omega)\}, \quad \gamma \in \mathbb{C}, \quad \gamma \neq 0. \tag{1.2}$$

In this paper, we explore the spectral properties of the differential operator L. We employ the method of similar operators (see [1, 3, 4, 5, 8, 9] and references therein).

The operators with an involution arise in various applications, such as the filtering and prediction theory [22] and the study of subharmonic oscillations [26, 27]. In addition, operators with an involution are interesting because of their relation to the Dirac operators [5, 13, 15, 20, 28].

First-order differential operators with periodic boundary conditions, an involution, and a smooth potential were studied in a series of papers by A.P. Khromov and M.Sh. Burlutskaya (see [13, 14, 15, 16, 17]). Second-order differential operators with an involution were investigated by A.M. Sarsenbi and L.V. Kritskov (see [24] and references therein). In this paper, we shall study the spectral properties of differential operators with an involution defined by differential expression (1.1) and general boundary conditions (1.2). Other authors considered similar problems in [18, 19, 23, 25, 29, 30].

The authors of this paper studied first-order differential operators with an involution and periodic boundary conditions in a series of papers [6, 7, 10, 11, 12]. The method of similar operators was also used there. In [6] and [10], operators with a matrix potential were studied. In [6], estimates for the weighted averages of eigenvalues and the equiconvergence of spectral decompositions were proved. In [10], a differential operator was transformed into an orthogonal direct sum of finite rank operators (see Definition 2). An operator group was then constructed to describe mild solutions of a mixed problem. In this paper, we introduce a similarity transform which allows us to reduce the operator L defined by (1.1) and (1.2) to the operator L_1 with periodic boundary conditions and $q_0(s) \equiv \mathrm{const}$, $s \in [0, \omega]$. For the operator L_1, the results from [6, 7, 10, 11, 12] can be applied.

The main result of this paper is Theorem 5.1, where we establish the similarity of the operator L and an operator given by a direct sum of a single finite-rank operator and countably many rank-one operators. This theorem allows us to obtain structural results about L and the group generated by it. In particular, asymptotic estimates of the eigenvalues and eigenvectors of L appear in Theorems 5.2, 5.3, and the group generated by the operator L is exhibited in Theorem 5.4.

2 Notation and the first similarity transform

In this section, we introduce the necessary notation following [7, 10, 11, 12].

For a function $v \in L_2[0, \omega]$, its Fourier series is given by

$$v(s) \sim \sum_{n \in \mathbb{Z}} \widehat{v}(n) e^{i \frac{2\pi n}{\omega} s}, \quad s \in [0, \omega],$$

where the Fourier coefficients are

$$\widehat{v}(n) = \frac{1}{\omega} \int_0^\omega v(s) e^{-i \frac{2\pi n}{\omega} s} \, ds = \langle v, e_n \rangle, \quad n \in \mathbb{Z}.$$

Throughout this section, by $\mathcal{H}$ we denote an abstract Hilbert space and by $\operatorname{End} \mathcal{H}$ the Banach algebra of all bounded linear operators in $\mathcal{H}$. We shall also make use of the ideal of Hilbert-Schmidt operators in $\mathcal{H}$ denoted by $\mathfrak{S}_2(\mathcal{H})$. The norm in $\mathfrak{S}_2(\mathcal{H})$ is $\|X\|_2 = (\operatorname{tr} XX^*)^{1/2} = \left(\sum_{n=1}^{\infty} |s_n|^2\right)^{1/2}$, where (s_n) is the sequence of singular values of the operator X. We refer to [21] for the standard properties of $\mathfrak{S}_2(\mathcal{H})$ used in this paper.

Let $\ell_p(\mathbb{Z})$, $p \geqslant 1$, be the complex Banach space of all p-summable sequences. If $p = 2$, then $\ell_2(\mathbb{Z})$ is the Hilbert space with the inner product and the norm defined by $\langle x, y \rangle = \sum_{n \in \mathbb{Z}} x(n) \overline{y(n)}$ and $\|x\|_2 = \left(\sum_{n \in \mathbb{Z}} |x(n)|^2\right)^{1/2}$, x, $y \in \ell_2(\mathbb{Z})$, respectively.

We begin with the following definitions.

Definition 1 ([3])**.** Two linear operators $A_i : D(A_i) \subset \mathcal{H} \to \mathcal{H}$, $i = 1, 2$, are called *similar*, if there exists a continuously invertible operator $U \in \operatorname{End} \mathcal{H}$ such that

$$A_1 U x = U A_2 x, \quad x \in D(A_2), \quad U D(A_2) = D(A_1).$$

The operator U is called the *similarity transform* of A_1 into A_2.

Directly from Definition 1, we have the following result on spectral properties of similar operators.

Lemma 2.1 ([9]). *Let $A_i : D(A_i) \subset \mathcal{H} \to \mathcal{H}$, $i = 1, 2$, be two similar operators with the similarity transform U. Then the following properties hold.*

(1) *We have $\sigma(A_1) = \sigma(A_2), \sigma_p(A_1) = \sigma_p(A_2)$, and $\sigma_c(A_1) = \sigma_c(A_2)$, where σ_p denotes the point spectrum and σ_c denotes the continuous spectrum.*

(2) *Assume that the operator A_2 admits a decomposition $A_2 = A_{21} \oplus A_{22}$ with respect to a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $A_{21} = A_2|\mathcal{H}_1$ and $A_{22} = A_2|\mathcal{H}_2$ are the restrictions of A_2 to the respective subspaces. Then the operator A_1 admits a decomposition $A_1 = A_{11} + A_{12}$ with respect to the direct sum $\mathcal{H} = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$, where $A_{11} = A_1|\widetilde{\mathcal{H}}_1$ and $A_{12} = A_1|\widetilde{\mathcal{H}}_2$ are the restrictions of A_1 to the respective invariant subspaces. Moreover, if P is the projection onto $\mathcal{H}_1$ parallel to $\mathcal{H}_2$, then $\widetilde{P} = UPU^{-1}$ is the projection onto $\widetilde{\mathcal{H}}_1$ parallel to $\widetilde{\mathcal{H}}_2$.*

(3) *If λ_0 is an eigenvalue of the operator A_2 and x is a corresponding eigenvector, then $y = Ux$ is an eigenvector of the operator A_1 corresponding to the same eigenvalue λ_0.*

(4) *If A_2 is a generator of a C_0-semigroup (group) $T_2 : \mathbb{J} \to \operatorname{End} \mathcal{H}$, $\mathbb{J} = \{\mathbb{R}, \mathbb{R}_+\}$, then the operator A_1 generates the C_0-semigroup (group) $T_1(t) = UT_2(t)U^{-1}$, $t \in \mathbb{J}$, $T_1 : \mathbb{J} \to \operatorname{End} \mathcal{H}$, $\mathbb{J} = \{\mathbb{R}, \mathbb{R}_+\}$.*

We begin the study of the operator L defined by (1.1) and (1.2) with a similarity transform of L into the operator $\widetilde{L}$ with

$$D(\widetilde{L}) = \{x \in W_2^1[0, \omega] : x(0) = x(\omega)\}. \tag{2.1}$$

By $\ln \gamma$, $\gamma \neq 0$, we denote the complex number z_0 satisfying $e^{z_0} = \gamma$ and $0 \leqslant \arg z_0 \leqslant \arg z$ for all $z \in \mathbb{C}$ with $e^z = \gamma$. We then let

$$g_0(s) = q_0(s) + \frac{\ln \gamma}{\omega}, \quad g_1(s) = q_1(s)\gamma^{\frac{2s}{\omega}-1}, \quad s \in [0, \omega], \tag{2.2}$$

and define $W \in \operatorname{End} L_2[0, \omega]$ by

$$(Wz)(s) = \gamma^{-\frac{s}{\omega}} z(s), \quad s \in [0, \omega]. \tag{2.3}$$

The main result of this section is

Theorem 2.1. *The operator L is similar to the operator $\widetilde{L} : D(\widetilde{L}) \subset L_2[0, \omega] \to L_2[0, \omega]$ defined by*

$$(\widetilde{L}y)(s) = y'(s) - g_0(s)y(s) - g_1(s)y(\omega - s), \tag{2.4}$$

and domain (2.1). The functions g_0, $g_1 \in L_2[0, \omega]$ in (2.4) are defined by (2.2). The similarity transform W of L into $\widetilde{L}$ is given by (2.3).

Proof. We have

$$(LWz)(s) = \gamma^{-\frac{s}{\omega}} z'(s) - \frac{\ln \gamma}{\omega}\gamma^{-\frac{s}{\omega}} z(s) - q_0(s)\gamma^{-\frac{s}{\omega}} z(s) - q_1(s)\gamma^{\frac{s-\omega}{\omega}} z(\omega - s)$$

and

$$\widetilde{L} = (W^{-1}LWz)(s) = z'(s) - \left(\frac{\ln \gamma}{\omega} + q_0(s)\right) z(s) - q_1(s)\gamma^{\frac{2s}{\omega}-1} z(\omega - s).$$

The assertion about the domains is straightforward. $\square$

3 Direct sums

We need to extend Property (2) in Lemma 2.1 to the case of countable direct sums [7, 10, 11, 12]. To do this, we assume that the abstract Hilbert space $\mathcal{H}$ can be written as

$$\mathcal{H} = \bigoplus_{l\in\mathbb{Z}} \mathcal{H}_l,$$

where each $\mathcal{H}_l$, $l \in \mathbb{Z}$, is a closed nonzero subspace of $\mathcal{H}$, $\mathcal{H}_j$ is orthogonal to $\mathcal{H}_l$ for $l \neq j \in \mathbb{Z}$, and each $x \in \mathcal{H}$ satisfies $x = \sum_{l\in\mathbb{Z}} x_l$, where $x_l \in \mathcal{H}_l$ and $\|x\|^2 = \sum_{l\in\mathbb{Z}} \|x_l\|^2$. In other words, we have a disjunctive resolution of the identity

$$\mathcal{P} = \{P_l, l \in \mathbb{Z}\},$$

that is a system of idempotents with the following properties:

1) $P_l^* = P_l$, $l \in \mathbb{Z}$;
2) $P_j P_l = \delta_{jl} P_l$, j, $l \in \mathbb{Z}$, where δ_{jl} is the Kronecker delta;
3) the series $\sum_{l\in\mathbb{Z}} P_l x$ converges unconditionally to $x \in \mathcal{H}$ and $\|x\|^2 = \sum_{l\in\mathbb{Z}} \|P_l x\|^2$;
4) equalities $P_l x = 0$, $l \in \mathbb{Z}$, imply $x = 0 \in \mathcal{H}$;
5) $\mathcal{H}_l = \operatorname{Im} P_l$, $x_l = P_l x$, $l \in \mathbb{Z}$.

Definition 2 ([7, 11])**.** We say that a closed linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is represented as an *orthogonal direct sum* of bounded operators $A_l \in \operatorname{End} \mathcal{H}_l$, $l \in \mathbb{Z}$, that is

$$A = \bigoplus_{l\in\mathbb{Z}} A_l,$$

if the following three properties hold.

1. $D(A) = \{x \in \mathcal{H} : \sum_{l\in\mathbb{Z}} \|A_l x_l\|^2 < \infty, x_l = P_l x, l \in \mathbb{Z}\}$ and $\mathcal{H}_l \subset D(A)$ for all $l \in \mathbb{Z}$.
2. For each $l \in \mathbb{Z}$, the subspace $\mathcal{H}_l$ is an invariant subspace of the operator A and A_l is the restriction of A to $\mathcal{H}_l$. The operators A_l, $l \in \mathbb{Z}$, are called the *parts* of the operator A.
3. $Ax = \sum_{l\in\mathbb{Z}} A_l x_l$, $x \in D(A)$, where $x_l = P_l x$, $l \in \mathbb{Z}$, and the series converges unconditionally in $\mathcal{H}$.

Definition 3 ([7, 11])**.** Given a continuously invertible operator $U \in \operatorname{End} \mathcal{H}$ and an orthogonal decomposition of $\mathcal{H}$, a *U-orthogonal decomposition* of $\mathcal{H}$ is the orthogonal direct sum

$$\mathcal{H} = \bigoplus_{l\in\mathbb{Z}} U\mathcal{H}_l.$$

Definition 4 ([7, 11])**.** Given a continuously invertible operator $U \in \operatorname{End} \mathcal{H}$, we say that a closed linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is a U-orthogonal direct sum of bounded linear operators $\widetilde{A}_l$, $l \in \mathbb{Z}$, if $\widetilde{A}_l = UA_lU^{-1}$, $l \in \mathbb{Z}$, and

$$U^{-1}AU = \bigoplus_{l\in\mathbb{Z}} A_l.$$

We remark that U-orthogonal decompositions and direct sums can be viewed as orthogonal with respect to the inner product

$$\langle x, y\rangle_U = \langle Ux, Uy\rangle, \quad x, y \in \mathcal{H}.$$

We write $\widetilde{L} = L_0 - V$, where $L_0 : D(L_0) = D(\widetilde{L}) \subset L_2[0, \omega] \to L_2[0, \omega]$ is the differential operator $L_0 = d/ds$ with periodic boundary conditions (2.1) and

$$(Vy)(s) = g_0(s)y(s) + g_1(s)y(\omega - s), \quad s \in [0, \omega].$$

The operator V is well defined because $D(L_0) \subset D(V)$. We shall treat the operator $\widetilde{L}$ as a perturbation of L_0 by V.

We shall illustrate the notion of direct sums of operators with the help of an operator L_0. We shall call this operator *unperturbed* or *free*. The operator L_0 possesses simple eigenvalues $\lambda_n = i2\pi n/\omega$, $n \in \mathbb{Z}$, and its spectrum satisfies $\sigma(L_0) = \cup_{n\in\mathbb{Z}}\sigma_n$, where $\sigma_n = \{\lambda_n\}$, $n \in \mathbb{Z}$. An eigenvector that corresponds to an eigenvalue λ_n, $n \in \mathbb{Z}$, is given by $e_n(s) = e^{i\frac{2\pi n}{\omega}s}$, $s \in [0,\omega]$. The spectral projections $P_n = P(\sigma_n, L_0)$, $n \in \mathbb{Z}$, are defined by

$$(P_n y)(s) = \widehat{y}(n)e^{i\frac{2\pi n}{\omega}s}, \quad y \in L_2[0,\omega], \quad n \in \mathbb{Z}, \quad s \in [0,\omega].$$

We let $\mathcal{H}_n = \operatorname{Im} P_n$, $n \in \mathbb{Z}$. Then L_0 is an orthogonal direct sum of the operators $L_{0l} = L_0|\mathcal{H}_l = \lambda_l I_l$, where I_l is the identity operator on $\mathcal{H}_l = \operatorname{Im} P_l$, $l \in \mathbb{Z}$. In other words, $L_0 = \bigoplus_{l\in Z} \lambda_l I_l$. This representation is with respect to the orthogonal decomposition of $L_2[0,\omega]$ given by $L_2[0,\omega] = \bigoplus_{l\in\mathbb{Z}} \mathcal{H}_l$.

Given $m \in \mathbb{Z}_+$, consider a new resolution of the identity

$$\mathcal{P}^{(m)} = \{P_{(m)}\} \cup \{P_l, |l| > m, l \in \mathbb{Z}\},$$

where $P_{(m)} = \sum_{|l|\leq m} P_l$. Then the operator L_0 may also be represented as an orthogonal direct sum

$$L_0 = L_{0(m)} \oplus \Big(\bigoplus_{|l|>m, l\in\mathbb{Z}} L_{0l}\Big) = L_{0(m)} \oplus \Big(\bigoplus_{|l|>m, l\in\mathbb{Z}} i\frac{2\pi l}{\omega} I_l\Big), \tag{3.1}$$

where $L_{0(m)}$ is the restriction of L_0 to $\mathcal{H}_{(m)} = \operatorname{Im} P_{(m)}$. Representation (2.1) is with respect to the orthogonal decomposition $L_2[0,\omega] = \mathcal{H}_{(m)} \oplus \big(\bigoplus_{|l|>m, l\in\mathbb{Z}} \mathcal{H}_l\big)$. Observe that $L_{0(m)} = \bigoplus_{|l|\leqslant m, l\in\mathbb{Z}} i\frac{2\pi l}{\omega} I_l$ with respect to the decomposition $\mathcal{H}_{(m)} = \bigoplus_{|l|\leqslant m, l\in\mathbb{Z}} \mathcal{H}_j$.

4 The second similarity transform

Theorem 4.1. *We have* $L_1 = \widetilde{\widetilde{W}}^{-1}\widetilde{L}\widetilde{\widetilde{W}}$, *where*

$$(\widetilde{\widetilde{W}} y)(s) = (\exp \int_0^s (g_0(\tau) - \widehat{g}_0(0))\, d\tau) y(s), \quad s \in [0,\omega], \tag{4.1}$$

$$(L_1 y)(s) = \frac{dy}{ds} - \big(\frac{\ln\gamma}{\omega} - \widehat{q}_0(0)\big) y(s) - v(s) y(\omega - s),$$

$$v(s) = g_1(s)\exp\Big(\int_s^{\omega - s} (g_0(\tau) - \widehat{g}_0(0))\, d\tau\Big) = \sum_{l\in\mathbb{Z}} \widehat{v}(l) e^{i\frac{2\pi l}{\omega}s}, \tag{4.2}$$

and $D(L_1) = D(L)$.

Proof. Firstly, $(\widetilde{W}^{-1}y)(s) = \exp\big(-\int_0^s (g_0(\tau)-\widehat{g}_0(0))\,d\tau\big)y(s)$ and

$$
\begin{aligned}
&(\widetilde{L}\widetilde{W}x)(s) = \widetilde{L}\Big(\exp\big(\int_0^s (g_0(\tau)-\widehat{g}_0(0))\,d\tau\big)x(s)\Big) =\\
&= (g_0(s)-\widehat{g}_0(0))\exp\big(\int_0^s (g_0(\tau)-\widehat{g}_0(0))\,d\tau\big)x(s)+\\
&+\exp\big(\int_0^s (g_0(\tau)-\widehat{g}_0(0))\,d\tau\big)\frac{dx}{ds} - g_1(s)\exp\big(\int_0^{\omega-s} (g_0(\tau)-\\
&-\widehat{g}_0(0))\,d\tau\big)x(\omega-s) - g_0(s)\exp\big(\int_0^s (g_0(\tau)-\widehat{g}_0(0))\,d\tau\big)x(s) =\\
&= \exp\big(\int_0^s (g_0(\tau)-\widehat{g}_0(0))\,d\tau\big)\frac{dx}{ds} - g_1(s)\exp\big(\int_0^{\omega-s} (g_0(\tau)-\\
&-\widehat{g}_0(0))\,d\tau\big)x(\omega-s) - \Big(\frac{\ln\gamma}{\omega}+\widehat{q}_0(0)\Big)\exp\big(\int_0^s (g_0(\tau)-\widehat{g}_0(0))\,d\tau\big)x(s).
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
L_1 = (\widetilde{W}^{-1}\widetilde{L}\widetilde{W}x)(s) &= \frac{dx}{ds} - g_1(s)\exp\big(-\int_0^s (g_0(\tau)-\\
&-\widehat{g}_0(0))\,d\tau\big)\exp\big(\int_0^{\omega-s} (g_0(\tau)-\widehat{g}_0(0))\,d\tau\big)x(\omega-s)-\\
&-\Big(\frac{\ln\gamma}{\omega}+\widehat{q}_0(0)\Big)x(s) = \frac{dx}{ds} - g_1(s)\exp\big(\int_s^{\omega-s} (g_0(\tau)-\\
&-\widehat{g}_0(0))\,d\tau\big)x(\omega-s) - \Big(\frac{\ln\gamma}{\omega}-\widehat{q}_0(0)\Big)x(s) =\\
&= \frac{dx}{ds} - v(s)x(\omega-s) - \Big(\frac{\ln\gamma}{\omega}+\widehat{q}_0(0)\Big)x(s),
\end{aligned}
$$

where v was defined by (4.2).

Let $z(s) = (\widetilde{W}y)(s)$, where $y(0) = y(\omega)$ and $\widetilde{W}$ is defined by (4.1). Then

$$
\begin{aligned}
&z(0) = \exp\big(\int_0^0 (g_0(\tau)-\widehat{g}_0(0))\,d\tau\big)y(0) = y(0),\\
&z(\omega) = \exp\big(\int_0^\omega (g_0(\tau)-\widehat{g}_0(0))\,d\tau\big)y(\omega) =\\
&\quad = \exp\big(\omega(\widehat{g}_0(0)-\widehat{g}_0(0))\big)y(\omega) = y(\omega),
\end{aligned}
$$

and the result is proved. □

5 Main results

Our main Theorem 5.1 exhibits several spectral properties of the operator L. It follows by Lemma 2.1 and an application of [7, Theorem 2.6], [10, Theorem 1], [11, Theorem 3], and [12, Theorem 1] to the operator L_1. In the formulation of the theorem, we use the notation introduced in the previous sections.

Theorem 5.1. *There exists a number $n \in \mathbb{Z}_+$, such that the operator L is similar to the operator $L_0 - \big(\frac{\ln\gamma}{\omega}+\widehat{q}_0(0)\big)I - B$, where the operator B belongs to the space $\mathfrak{S}_2(L_2[0,\omega])$, and the subspaces*

$\mathcal{H}_{(m)} = \operatorname{Im} P_{(m)}$, $\mathcal{H}_l = \operatorname{Im} P_l$, $|l| > m$, $l \in \mathbb{Z}_+$, *are invariant subspaces of the operator* B. *The operator* $L_0 - B$ *is the orthogonal direct sum:*

$$L_0 - B = \Big((L_{0(m)} - B_{(m)}) \oplus \Big(\bigoplus_{|l|>m} (i\frac{2\pi l}{\omega} I_l - B_l)\Big)\Big)$$

with respect to the orthogonal decomposition $L_2[0,\omega] = \mathcal{H}_{(m)} \oplus \big(\bigoplus_{|l|>m} \mathcal{H}_l\big)$; *the dimension of* $\mathcal{H}_{(m)}$ *is* $2m+1$ *and the dimension of each* $\mathcal{H}_l$, $|l| > m$, *is one. Moreover, there exists* $U \in \mathfrak{S}_2(L_2[0,\omega])$, *such that with* $\widetilde{U} = (I+U)\widetilde{W}W$, *the operator* L *is the* $\widetilde{U}$*-orthogonal direct sum:*

$$L = \widetilde{U}\Big(\Big((L_{0(m)} - B_{(m)}) \oplus \Big(\bigoplus_{|l|>m} \Big(i\frac{2\pi l}{\omega} I_l - B_l\Big)\Big)\Big) - \\ - \Big(\frac{\ln\gamma}{\omega} + \widehat{q}_0(0)\Big)I\Big)\widetilde{U}^{-1}$$

with respect to the $\widetilde{U}$*-orthogonal decomposition* $L_2[0,\omega] = \widetilde{U}\mathcal{H}_{(m)} \oplus \Big(\bigoplus_{l>m} \widetilde{U}\mathcal{H}_l\Big)$.

The operator U in Theorem 5.1 can be effectively calculated as the limit of a sequence of operators that emerge when applying the method of simple iterations to a Riccati-type equation (see [1, 3, 4, 5, 6, 7, 8, 9, 11] for details).

A result similar to Theorem 5.1 for the operator $L_0 - V$, where $(Vy)(s) = v(s)y(\omega - s)$, was proved in [6] using the approach from [5]. Theorem 1 from [10] and Theorem 1 from [12] are a reformulation of Theorems 3.3 and 3.4 from [6] that takes into account the terminology introduced above. Theorem 5.1 follows from [11, Theorem 3] or [7, Theorem 2.6], and Theorems 1 and 2 in this paper.

It follows from Theorem 5.1, Lemma 2.1, and an argument in [5, Remark 2] that the spectrum $\sigma(L)$ of the operator L coincides with the union of the spectra of its parts. Therefore, we have the following assertion.

Theorem 5.2. *The spectrum* $\sigma(L)$ *of the operator* L *can be represented in the form*

$$\sigma(L) = \sigma_{(m)} \cup \Big(\bigcup_{|l|>m} \sigma_l\Big),$$

where $\sigma_{(m)}$ *consists of no more than* $2m+1$ *eigenvalues. The sets* σ_l, $|l| > m$, *are singletons:* $\sigma_l = \{\widetilde{\lambda}_l\}$ *with*

$$\widetilde{\lambda}_l = i\frac{2\pi l}{\omega} - \widehat{q}_0(0) - \frac{\ln\gamma}{\omega} - \mu_l,$$

where the sequence $\{\mu_l, |l| > m\}$ *belongs to* ℓ_2. *More precisely,*

$$\mu_l = \widehat{v}(2l) + \frac{\omega}{2\pi i}\sum_{n\neq -j} \frac{\widehat{v}(n+j)^2}{n+j} + \eta_l,$$

where the sequence $\{\eta_l, |l| > m\}$ *belongs to* ℓ_1.

The normalized eigenvectors u_l, $|l| > m$, *of the operator* L *form a Riesz basis in the space* $L_2[0,\omega]$. *They satisfy the asymptotic estimate* $\|u_l - \widetilde{u}_l\|_2 \leqslant \alpha_l$, $|l| > m$,

$$\widetilde{u}_l(s) = \gamma^{-\frac{s}{\omega}s} e^{\int\limits_0^s (g_0(\tau) - \widehat{g}_0(0))\,d\tau} e^{i\frac{2\pi l}{\omega}s}, \quad s \in [0,\omega],$$

where the sequence $\{\alpha_l, |l| > m\}$ *belongs to* ℓ_2.

Let $\widetilde{P}_m = P(\sigma_{(m)}, L)$ and $\widetilde{P}_l = P(\{\lambda_l\}, L)$, $|l| > m$, be the spectral projections corresponding to the sets $\sigma_{(m)}$ and $\sigma_l = \{i\frac{2\pi l}{\omega} - \widehat{q}_0(0) - \frac{\ln\gamma}{\omega} - \mu_l\}$, $|l| > m$, respectively.

Theorem 5.3. *We have*

$$\lim_{n\to\infty} \|\widetilde{P}_{(m)} + \sum_{|l|>m} \widetilde{P}_l - \sum_{l=-n}^{n} \widetilde{U} P_l \widetilde{U}^{-1}\|_2 = 0.$$

Definition 5. Two families of idempotents $\{P_n, n \in \mathbb{Z}\}$ and $\{\widetilde{P}_n, n \in \mathbb{Z}\}$ are called equiconvergent with respect to the U-orthogonal decomposition of $\mathcal{H}$ or U-equiconvergent, if

$$\lim_{N\to\infty} \left\| \sum_{|n|\leqslant N} (\widetilde{P}_n - \widetilde{U} P_n \widetilde{U}^{-1}) \right\| = 0.$$

Thus, Theorem 5.3 can be stated as a result on U-equiconvergence.

Theorem 5.4. *The differential operator L is a generator of a C_0-group $T : \mathbb{R} \to \operatorname{End} L_2[0,\omega]$. This group is similar to the group $\widetilde{T} : \mathbb{R} \to \operatorname{End} L_2[0,\omega]$ that admits the orthogonal decomposition*

$$\widetilde{T}(t) = e^{t(L_{0(m)} - B_{(m)})} \oplus \Big(\bigoplus_{|l|>m} e^{(i\frac{2\pi l}{\omega} - \frac{\ln\gamma}{\omega} - \widehat{q}_0(0) + \mu_l)t} I_l \Big), \quad t \in \mathbb{R},$$

with respect to the decomposition $L_2[0,\omega] = \mathcal{H}_{(m)} \oplus \big(\bigoplus_{|l|>m} \mathcal{H}_l\big)$ of the space $L_2[0,\omega]$ with $T(t) = \widetilde{U}\widetilde{T}(t)\widetilde{U}^{-1}$, $t \in \mathbb{R}$. The operator group $\widetilde{T}$ can also be written as

$$\widetilde{T}(t) = e^{(L_{0(m)} - B_{(m)})t} P_{(m)} + \Big(\sum_{|l|>m} e^{(i\frac{2\pi l}{\omega} - \frac{\ln\gamma}{\omega} - \widehat{q}_0(0) + \mu_l)t} P_l \Big), \quad t \in \mathbb{R},$$

The above result motivates the terminology in the following definition.

Definition 6. A strongly continuous operator (semi)group $T_0 : \mathbb{J} \to \operatorname{End}\mathcal{H}$, $\mathbb{J} = \{\mathbb{R}, \mathbb{R}_+\}$, is called a *basic (semi)group* for a strongly continuous operator (semi)group $T : \mathbb{J} \to \operatorname{End}\mathcal{H}$ if there exists a strongly continuous operator-valued function $V : \mathbb{J} \to \operatorname{End}\mathcal{H}$ such that $T(t) = T_0(t)V(t)$ and $\lim\limits_{t\to\infty} \|V(t)\| = 0$. If $\|V(t)\| \leqslant e^{-\beta t}$, $t \geqslant 0$, for some $\beta > 0$, then we call T an *exponentially basic (semi)group*.

If $\operatorname{Re}\frac{\ln\gamma}{\omega} > 0$ and $\operatorname{Re}\widehat{q}_0(0) > 0$, we have $\widetilde{T}(t) = T_0(t)V(t)$, where

$$\widetilde{T}_0(t) = e^{(L_{0(m)} - B_{(m)})t} \oplus \Big(\bigoplus_{|l|>m} e^{-i\frac{2\pi l}{\omega}t} I_l \Big), \quad t \in \mathbb{R}, \tag{5.1}$$

$$V(t) = \bigoplus_{|l|>m} e^{\left(-\frac{\ln\gamma}{\omega} - \widehat{q}_0(0) + \mu_l\right)t} I_l, \quad t \in \mathbb{R}.$$

It follows that the group $T_0 : \mathbb{R} \to \operatorname{End}\mathcal{H}$ is exponentially basic for the group $\widetilde{T} : \mathbb{R} \to \operatorname{End}\mathcal{H}$.

The existence of the operator group T, guaranteed by Theorem 5.4, is important because it allows one to use the results, for example, from [2] on exponential dichotomy and the estimates for Green's function.

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