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MIKHAIL L'VOVICH GOLDMAN

(to the 75th birthday)

Mikhail L'vovich Goldman was born on April 13, 1945 in Moscow. In 1963 he graduated from school in Moscow and entered the Physical Faculty of the M.V. Lomonosov Moscow State University (MSU) from which he graduated in 1969 and became a PhD student (1969–1972) at the Mathematical Department of this Faculty. In 1972 he has defended the PhD thesis, and in 1988 his DSc thesis "The study of spaces of differentiable functions of many variables with generalized smoothness" at the S.L. Sobolev Institute of Mathematics in Novosibirsk. Scientific degree "Professor in Mathematics" was awarded to him in 1991.

From 1974 to 2000 M.L. Goldman was successively an assistant Professor, Full Professor, Head of the Mathematical Department at the Moscow Institute of Radio Engineering, Electronics and Automation (technical university). Since 2000 he is a Professor of the S.M. Nikol'skii Mathematical Institute at the Peoples Friendship University of Russia (RUDN University).

Research interests of M.L. Goldman are: the theory of function spaces, optimal embeddings, integral inequalities, spectral theory of differential operators. Main achievements: optimal embeddings of spaces with generalized smoothness, sharp conditions of the convergence of spectral expansions, descriptions of integral and differential properties of the generalized Bessel and Riesz-type potentials, sharp estimates for operators on cones and optimal envelopes for the cones of functions with properties of monotonicity. Professor M.L. Goldman has over 140 scientific publications in leading mathematical journals.

Under his scientific supervision, 8 candidate theses in Russia and 1 thesis in Kazakhstan were successfully defended. Some of his former students are now professors in Ethiopia, Columbia, Mongolia.

Participation in scientific and organizational activities of M.L. Goldman is well known. He is a member of the DSc Councils at RUDN and MSU, of the PhD Council in the Lulea Technical University (Sweden), a member of the Editorial Board of the Eurasian Mathematical Journal, an invited lector and visiting professor at universities of Russia, Germany, Sweden, UK etc., an invited speaker at many international conferences.

The mathematical community, friends and colleagues and the Editorial Board of the Eurasian Mathematical Journal cordially congratulate Mikhail L'vovich Goldman on the occasions of his 75th birthday and wish him good health, happiness, and new achievements in mathematics and mathematical education.

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SURJECTIVE QUADRATIC JORDAN ALGEBRAS

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Abstract. We introduce the concepts of surjectivity and linear minimality for quadratic Jordan algebras, then we present a partial classification of such algebras of characteristic 2. As a corollary, we obtain that in substance non-trivial minimal quadratic Jordan algebras are fields.

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1 Introduction

This article is not on algebra, as someone might think, rather on the model theory, which is located at the junction of logic and algebra. Recall that an infinite model $\mathfrak{M}$ is called *minimal* if any first-order formula over $\mathfrak{M}$ defines either a finite or cofinite subset in it. This concept of minimality played a major role in the development of the model theory, starting a model-theoretic generalization of the important algebraic concepts such as dependence, exchange principle, basis and dimension.

The study of minimal rings was undertaken in [2]. Later in [3], the concept of linear minimality was introduced for rings, as a weakened version of the minimality, and it was shown that it can be naturally extended to algebras. The motivation of this research was the following hypothesis.

Hypothesis. An infinite field is interpreted in any non-trivial minimal ring.

In [2] this hypothesis was confirmed for Jordan rings of characteristic different from 2, furthermore it has been proved that the minimality can be replaced by the linear minimality [1]. With regard to the characteristic 2, in this case, Jordan rings do not allow classical linear axiomatization in which all previous studies were fulfilled. There are many different approaches, and in this paper we choose the quadratic axiomatization.

Quadratic Jordan algebras are defined as vector spaces with additional operators U and σ, where U is a binary operator that is quadratic in the first variable and linear in the second one; σ is a quadratic unary operator. Here, as it is customary to the theory of quadratic Jordan algebras, we write $U_x y$ and x^2 instead of $U(x, y)$ and $\sigma(x)$ respectively. We also denote by U_x a linear operator $U(x, \cdot)$. The following multilinear operators are defined in the standard way:

$$\{x, y, z\} := V_{x,y}z := U_{x,z}y := U_{x+z}y - U_x y - U_z y;$$

$$V_x y := (x + y)^2 - x^2 - y^2.$$

In this signature quadratic Jordan algebras distinguished by the following axioms:

(QJ1) $\quad V_{x,x} = V_{x^2}$

(QJ2) $\quad U_x V_x = V_x U_x$

(QJ3) $U_x x^2 = (x^2)^2$

(QJ4) $U_{x^2} = U_x^2$

(QJ5) $U_x U_y x^2 = (U_x y)^2$

(QJ6) $U_{U_x y} = U_x U_y U_x$.

In addition, it is required that these axioms hold not only in the algebra, but that they also continue to hold in all scalar extensions of it. This turns out to be equivalent to the condition that all formal linearizations of these identities (QJ1)-(QJ6) remain valid on the algebra itself.

2 Surjectivity of quadratic Jordan algebras

Initially, the linear minimality and the surjectivity properties were introduced for rings and algebras in a general linear axiomatization context [3], [4]. Here we adapt them for quadratic Jordan algebras.

First of all it is necessary to clarify the definition of the multiplication algebra.

Definition 1. Let $\mathfrak{Q} = \langle Q; +, U, \sigma, 0\rangle$ be a quadratic Jordan algebra over a field F. The *multiplication algebra* $\mathfrak{T}(\mathfrak{Q})$ of the algebra $\mathfrak{Q}$ is generated by the identity mapping $\mathrm{id}_Q : x \mapsto x$ together with maps of the form U_a, $U_{a,b}$, V_a, $V_{a,b}$, $a, b \in Q$, in the algebra $\mathfrak{E}(\mathfrak{Q})$ of all linear transformations of the vector space $\langle Q; +, 0\rangle$. So, $\mathfrak{T}(\mathfrak{Q})$ is a unital associative algebra over F with the unit id_Q.

Note that when $char F \neq 2$ this definition of the multiplication algebra coincides with the classical one [5].

Definition 2. An infinite quadratic Jordan algebra $\mathfrak{Q}$ is *surjective* provided that every nonzero linear mapping from the multiplication algebra $\mathfrak{T}(\mathfrak{Q})$ is a surjective mapping; it is *linearly minimal* (briefly, *l-minimal*) if it is surjective and every nonzero linear mapping from $\mathfrak{T}(\mathfrak{Q})$ has a finite kernel.

Now let us recall some definitions from [7].

Definition 3. An element a of a quadratic Jordan algebra is *weakly trivial* if $U_a \equiv 0$. An element b is *trivial* if it is weakly trivial and $b^2 = 0$. An algebra is (*weakly*) *trivial* if each element is (weakly) trivial.

In a weakly trivial algebra $\mathfrak{Q}$ we have $U_a \equiv U_{a,b} \equiv V_{a,b} \equiv 0$ for all $a, b \in Q$. In such an algebra (QJ2), (QJ4)-(QJ6) hold trivially; (QJ1) becomes

(QJ1)$'$ $V_{x^2} \equiv 0$

and (QJ3) takes the form

(QJ3)$'$ $(x^2)^2 \equiv 0$.

The following is Surjective Unit Lemma 18.1.4 from [7]. We present it here with a proof for an arbitrary characteristic (to provide a completeness).

Proposition 2.1. *If a surjective quadratic Jordan algebra contains an element which is not weakly trivial, then the algebra is unital.*

Proof. So, assume that $\mathfrak{Q} = \langle Q; +, U, \sigma, 0\rangle$ is a surjective quadratic Jordan algebra and there exists $a \in Q$ with nonzero U_a. Then the operator U_a is surjective. Let the element $b \in Q$ be such that $U_a b = a$. Then by (QJ6) we have

$$U_a = U_a U_b U_a. \tag{2.1}$$

Hence, the operators U_b, U_bU_a are also nonzero, therefore they are both surjective too. Now from (2.1) we get Ker U_a =Ker $U_b = \{0\}$: if Ker $U_a \setminus \{0\}$ is non-empty, then for some $c \in Q$ we have $U_bU_ac \in$ Ker $U_a \setminus \{0\}$, so

$$U_ac \neq 0 = U_aU_bU_ac$$

which contradicts (2.1); similarly, if Ker $U_b \setminus \{0\}$ is non-empty, then an element $c \in Q$ such that

$$U_ac \in \mathrm{Ker} U_b \setminus \{0\}$$

leads to a contradiction. Thus, both operators U_a, U_b are invertible. From (2.1) we get also $U_aU_b = \mathrm{id}_Q$ or $U_b = U_a^{-1}$.

Let us denote $e = U_ab^2$, then by (QJ6) and (QJ4) we obtain

$$U_e = U_aU_{b^2}U_a = U_aU_b^2U_a = \mathrm{id}_Q.$$

Let $E := \{x \in Q : U_x = \mathrm{id}_Q\}$. This set is non-empty, since $e \in E$. Moreover, from (QJ4) it follows that $e^2 \in E$. Furthermore, e^2 is an idempotent, since, by virtue of (QJ3), we have

$$(e^2)^2 = U_ee^2 = e^2.$$

Hence, by Proposition 5.2.4 from [7], e^2 is the unit element of algebra $\mathfrak{Q}$. □

Remark 1. (a) The proof of Proposition 2.1 shows that in a surjective quadratic Jordan algebra each element is either weakly trivial or invertible. If $U_a \not\equiv 0$ then the unique b such that $U_ab = a$ is called the *inverse* of a and can be denoted by $b = a^{-1}$; for this b we have $U_b = U_a^{-1}$.
(b) In a unital algebra the operator σ can be defined via U: $\sigma x = U_x1$, where 1 stands for the unit element. Therefore there is no need to consider it separately (it is sufficient to work only with U). In particular, in a unital Jordan algebra the notions of triviality and weakly triviality coincide, since $U_x \equiv 0$ implies $x^2 = U_x1 = 0$.

Lemma 2.1. *The identity*

$$U_{x+y}(x+y) = U_x(x+y) + U_y(x+y) + V_{x^2}y + V_{y^2}x$$

holds in any quadratic Jordan algebra.

Proof. By the definition of multilinear functions and (QJ1) we have

$$\begin{aligned} U_{x+y}(x+y) &= U_x(x+y) + U_y(x+y) + \{x, x+y, y\} = U_x(x+y) + U_y(x+y) + \{x,x,y\} + \{x,y,y\} \\ &= U_x(x+y) + U_y(x+y) + V_{x,x}y + V_{y,y}x = U_x(x+y) + U_y(x+y) + V_{x^2}y + V_{y^2}x. \end{aligned}$$

□

Lemma 2.2. *In a surjective quadratic Jordan algebra weakly trivial elements form a subspace.*

Proof. There is no problem if the algebra itself is weakly trivial, otherwise by Proposition 2.1 the algebra is unital and each weakly trivial element is trivial (see Remark 1.(b) above). Let a, b be any trivial elements and $\lambda \in F$. We have to show that λa and $a+b$ are trivial. For λa we have

$$U_{\lambda a} = \lambda^2 U_a = \lambda^2 0 \equiv 0.$$

There is no problem if $a+b=0$. Let $a+b \neq 0$. By Lemma 2.1 we have

$$U_{a+b}(a+b) = U_a(a+b) + U_b(a+b) + V_{a^2}b + V_{b^2}a = 0+0+0+0 = 0.$$

So, $0 \neq a+b \in$ Ker U_{a+b} and the operator U_{a+b} is not invertible. Then by Remark 1.(a) above $a+b$ is (weakly) trivial. □

Notation. From now on we fix a surjective (non-trivial) quadratic Jordan algebra $\mathfrak{Q} = \langle Q; +, U, \sigma, 0\rangle$ over F and denote by A its subspace of weakly trivial elements. If $\mathfrak{Q}$ is unital we denote its unit by 1 and, identifying each $\lambda \in F$ by $\lambda 1 \in Q$, we can assume that $F \subseteq Q$. Since $U_1 = \mathrm{id}_Q$, obviously $1 \notin A$.

We notice that the subspace A is U-invariant, that is if $a \in A$ then $U_x a \in A$ for any $x \in Q$. Indeed, $U_a \equiv 0$ implies $U_{U_x a} = U_x U_a U_x \equiv 0$ by (QJ6).

Lemma 2.3. *For any $a \in A$ and for all $x, y \in Q$ we have*

$$\{x, a, y\} = U_{x+y}a - U_x a - U_y a \equiv 0.$$

Proof. If $\{b, a, c\} \neq 0$ for some $a \in A$ and $b, c \in Q$ then $A \neq Q$ and by the surjectivity property the map $x \overset{\varphi}{\mapsto} \{b, a, x\}$ will be surjective, giving $Q = \mathrm{Im}\ \varphi \subseteq A+A+A = A$, which is a contradiction. □

Lemma 2.4. *$A = \{0\}$ or A is an infinite subspace.*

Proof. Let $0 \neq a \in A$. By Lemma 2.3, the mapping $x \overset{\varphi_a}{\mapsto} U_x a$ is an endomorphism with Ker $\varphi_a = A$ and Im $\varphi_a \subseteq A$. Thus, $Q^+/A \cong \mathrm{Im}\ \varphi_a$ implies the infinity of A. □

The following proposition is true in a more general context. We use the surjectivity just to get a shorter proof (see 1.8.3 in [7] for the definition of ideals).

Proposition 2.2. *A is an (inner) ideal in $\mathfrak{Q}$, so the quotient algebra $\overline{\mathfrak{Q}} = \mathfrak{Q}/A$ is well-defined.*

Proof. If $A = \{0\}$ or $A = Q$ then the statement of the proposition is obvious. Let $A \neq \{0\}$ and $A \neq Q$. By Proposition 2.1, $\mathfrak{Q}$ is unital. We have already seen that if $a \in A$ then $U_a \equiv 0$, $a^2 = U_a 1 = 0$, $U_x a \in A$ and $\{x, a, y\} = 0$ for any $x, y \in Q$. So, it remains to show that $U_{x,a}y \in A$ for any $x, y \in Q$ (no need to check $V_x a \in A$ because of $V_x a = U_{x,a}1$).

Let $a \in A$. The identity (see the identity (QJ7) on the page 22 of [6])

$$U_x U_y U_a + U_a U_y U_x + U_{x,a} U_y U_{x,a} = U_{U_a y, U_x y} + U_{U_{x,a}y}$$

gives $U_{x,a}U_y U_{x,a} = U_{U_{x,a}y}$. Since $A \subseteq \mathrm{Ker}(U_{x,a}U_y U_{x,a})$, the map $U_{U_{x,a}y}$ is not invertible. So, $U_{x,a}y \in A$ (in fact, we have $U_{x,a} \equiv 0$: from $U_{x,a}1 \in A$ it follows $(U_{x,a})^2 = U_{x,a}U_1 U_{x,a} = U_{U_{x,a}1} \equiv 0$ and then the surjectivity implies $U_{x,a} \equiv 0$). □

Definition 4. We say that the algebra $\mathfrak{Q}$ has *a trivial linear part* if $\{x, y, z\} \equiv 0$ and $V_x y \equiv 0$ for all $x, y, z \in Q$

Remark 2. If $V_x y \equiv 0$, in particular, if the algebra Q has a trivial linear part, then σ is an endomorphism of the abelian group $\langle Q; +, 0\rangle$. Moreover, if *char*$F \neq 2$ then $\sigma \equiv 0$. Indeed, on one hand, σ is an endomorphism and $\sigma(2x) = \sigma(x + x) = \sigma x + \sigma x = 2(\sigma x)$ for each $x \in Q$. On the other hand, since σ is quadratic, $\sigma(2x) = 4(\sigma x)$. Then the equality $2(\sigma x) = 4(\sigma x)$ together with *char*$F \neq 2$ implies $\sigma x = 0$.

Lemma 2.5. *If the algebra $\mathfrak{Q}$ is weakly trivial then $V_a \equiv 0$ for all $a \in Q$; that is $\mathfrak{Q}$ has a trivial linear part.*

Proof. Suppose to the contrary that $V_a \not\equiv 0$ for some $a \in Q$. By the surjectivity property, V_a is surjective and Ker V_a is a proper subspace of $\mathfrak{Q}$. For any $b \in Q$, $\sigma b = b^2 \in \mathrm{Im}\ \sigma$ we have $V_a b^2 = V_{b^2} a = 0$ by (QJ1)$'$. So Im $\sigma \subseteq$ Ker V_a. Then from the identity $V_a x = (x + a)^2 - a^2 - x^2$ we get that

$$\mathrm{Im}\ V_a \subseteq \mathrm{Im}\ \sigma - \mathrm{Im}\ \sigma - \mathrm{Im}\ \sigma \subseteq \mathrm{Ker}\ V_a,$$

contradicting the surjectivity of V_a. □

Proposition 2.3. *If $char F \neq 2$ then $\mathfrak{Q}$ is a division algebra.*

Proof. Assume the contrary, let $A \neq \{0\}$. If $A = Q$ then, by Lemma 2.5, $\mathfrak{Q}$ has trivial linear part; and by Remark 2 the algebra $\mathfrak{Q}$ is trivial, a contradiction.

So, let $A \neq Q$. Then $\mathfrak{Q}$ is unital. Since $char F \neq 2$, for any $0 \neq a \in A$ we have $\{1, a, 1\} = U_{1+1}a - U_1 a - U_1 a = 4a - a - a = 2a \neq 0$. This contradicts Lemma 2.3. $\square$

Proposition 2.3 improves the result of [1]; there the division property was proved under the stronger assumption of the linear minimality.

3 Triviality of linear part

From here point to the end of the article, we will assume that $char F = 2$.

Definition 5. In a unital quadratic Jordan algebra the powers of elements can be defined as usual: $x^0 = 1$, $x^1 = x$ and $x^{2n} = U_{x^n}1$, $x^{2n+1} = U_{x^n}x$ for positive integer n (see page 26 of [6]). If $x^n = 1$ for some positive integer n, then the smallest n with this property is called the *order* of x and we write $|x| = n$; otherwise, we say that x is of *infinite order* and write $|x| = \infty$.

First we recall some basic properties of orders.

Claim 1. (1) If there is an element of even order then there is an element of order 2; in particular, σ is not injective.
(2) If $|x| = 2n - 1$ then $x^{2n} = x$, so $x \in \operatorname{Im} \sigma$.

Proof. (1) If $|x| = 2n$ then by definition $x^{2n} = U_{x^n}1 = 1$ and $x^n \neq 1$. So for $y = x^n$ we have $y \neq 1$ and $y^2 = U_y = 1$. This means that $|y| = 2$.
(2) Let $|x| = 2n-1$ then $U_x^{2n-1} = U_{x^{2n-1}} = U_1 = \mathrm{id}_Q$, so there is the inverse mapping $U_x^{-1} = U_x^{2n-2} = U_{x^{2n-2}} = U_y$ for $y = x^{2n-2}$. We have

$$U_y x = U_{x^{2n-2}}x = U_{x^{n-1}}(U_{x^{n-1}}x) = U_{x^{n-1}}x^{2n-1} = U_{x^{n-1}}1 = x^{2n-2} = y,$$

so $x = U_y^{-1}y = U_x y = U_x(U_{x^{n-1}}1) = U_{x^n}1 = x^{2n}$. $\square$

Proposition 3.1. *If $\mathfrak{Q}$ is l-minimal, then $V_x \equiv 0$ for any $x \in Q$.*

Proof. Assume the contrary: $V_x \not\equiv 0$ for some $x \in Q$. Then, by the minimality, V_x is a surjective map with a finite kernel $\operatorname{Ker} V_x$. Since $char(F) = 2$, we notice that

$$V_x x = (x + x)^2 + x^2 + x^2 = 0.$$

So, $\{0, x\} \subseteq \operatorname{Ker} V_x$, whence $|\operatorname{Ker} V_x| > 1$.

Now, by Lemma 2.5 the algebra $\mathfrak{Q}$ is not weakly trivial, hence by Proposition 2.1 it is unital. Since $char(F) = 2$, as a consequence of the following identity (see the identity (QJ20) on the page 24 of [6])

$$V_{a^2} = V_a^2 - 2U_a\,,$$

we get $V_{a^2} = V_a^2$ for each $a \in Q$. We can proceed further by induction and show that the identity

$$V_{a^k} = V_a^k$$

holds for each $k = 2^n$, where n is positive integer. Hence for such k we have

$$V_x x^k = V_{x^k}x = V_x^k x = V_x^{k-1}(V_x x) = V_x^{k-1}0 = 0\,.$$

This means that $\{x, x^2, x^4, \ldots, x^{2^n}, \ldots\} \subseteq \operatorname{Ker} V_x$. But $\operatorname{Ker} V_x$ is finite, so for some $m \neq n$ we have $x^{2^m} = x^{2^n}$. This is a contradiction, as on one hand, we have $V_{x^{2^m}} = V_{x^{2^n}}$, and on the other hand

$$|\operatorname{Ker} V_{x^{2^m}}| = |\operatorname{Ker} V_x^{2^m}| = |\operatorname{Ker} V_x|^{2^m} \neq |\operatorname{Ker} V_x|^{2^n} = |\operatorname{Ker} V_x^{2^n}| = |\operatorname{Ker} V_{x^{2^n}}|.$$ $\square$

Now let us give a conclusion on the structure of the algebra $\mathfrak{Q}$ when it is weakly trivial. By Lemma 2.5 (see also Remark 2), σ is an endomorphism of the abelian group $\langle Q; +, 0\rangle$ and (QJ3)′ provides that Im $\sigma \subseteq$ Ker σ. Vice versa, it is easy to check that any vector space together with a quadratic endomorphism σ of $\langle Q; +, 0\rangle$ such that Im $\sigma \subseteq$ Ker σ, provides a weakly trivial, linearly minimal quadratic Jordan algebra, since its multiplication algebra consists of only scalar multiplications.

Example. Let F be an infinite field of characteristic 2 and $Q = F \times F = F^2$. If we define $U \equiv (0, 0)$ and a quadratic function $\sigma : Q \to Q$ so that $\sigma(x, y) = (0, x^2)$, then Im $\sigma \subset$ Ker σ, and we get a weakly trivial, linearly minimal quadratic Jordan algebra.

We will not classify linearly minimal weakly trivial quadratic Jordan algebras: firstly, they are not so interesting, since the presence of U-operators is fictitious; secondly, the linear minimality condition does not have a structural impact on them. They exist, as shown above, and we leave them at that.

In what follows we assume that our algebra contains a non weakly trivial element. By Proposition 1, it is equivalent to the following assumption.

Assumption 1. $A \neq Q$ and $\mathfrak{Q}$ is unital.

So, from now on A is the subspace of trivial elements (see Remark 1.(b)).

Lemma 3.1. *If $\mathfrak{Q}$ is l-minimal then $U_xU_y1 = U_yU_x1$ for any $x, y \in Q$.*

Proof. We use the identity (QJ30) from [6] : $(V_xy)^2 = U_xy^2 + U_yx^2 + V_yU_xy$. Since by Proposition 3.1 $(V_xy)^2 = V_yU_xy = 0$, we have $U_xU_y1 = U_xy^2 = U_yx^2 = U_yU_x1$. □

Proposition 3.2. *If $\mathfrak{Q}$ is l-minimal then it has a trivial linear part.*

Proof. Assume the contrary: $\{a, b, c\} \neq 0$ for $a, b, c \in Q$. The l-minimality, Lemmas 2.3 and 2.4 imply $A = \{0\}$. Also, by Proposition 3.1 and Remark 2, the map σ is an endomorphism. Moreover, it is a monomorphism, since Ker $\sigma = A = \{0\}$ (we note that by Remark 1.(a) we have $U_a1 = a^2 = 0 \Leftrightarrow U_a \equiv 0 \Leftrightarrow a \in A \Leftrightarrow a = 0$). Let $x, y \in Q$ be arbitrary elements and $z \in$ Im σ. Let $z = w^2$ for $w \in Q$. Then by Lemma 3.1, we get $U_{x+y}z = U_{x+y}U_w1 = U_wU_{x+y}1 = U_w(x + y)^2 = U_wx^2 + U_wy^2 = U_xw^2 + U_yw^2 = U_xz + U_yz$, which exactly means that

$$\{x, z, y\} = U_{x,y}z = U_{x+y}z - U_xz - U_yz = 0.$$

Since Im σ is infinite, the linear minimality implies $\{x, z, y\} \equiv 0$ for all $x, y, z \in Q$. This contradicts $\{a, b, c\} \neq 0$. □

Let us note that if $\mathfrak{Q}$ has a trivial linear part, then the multiplication algebra $\mathfrak{T}(\mathfrak{Q})$ is generated by the operators of the form U_x.

4 Linearly minimal division Jordan algebras

Let $\mathfrak{Q}$ be a unital quadratic Jordan algebra with a trivial linear part. We will use the following corollaries of the triviality: $U_{x+y} = U_x + U_y$ and $(x + y)^2 = x^2 + y^2$ for all $x, y \in Q$.

Remark 3. Until Assumption 2, neither an infinity nor the l-minimality of algebra $\mathfrak{Q}$ will be used.

Lemma 4.1. *The algebra $\mathfrak{T}(\mathfrak{Q})$ is commutative.*

Proof. We use the identity $U_xU_yU_z + U_zU_yU_x + U_{x,z}U_yU_{x,z} = U_{U_zy,U_xy} + U_{U_{x,z}y}$ again (see the proof of Propostion 2.2). Since the algebra has a trivial linear part, this identity gives $U_xU_yU_z = U_zU_yU_x$. Putting $z = 1$ we get $U_xU_y = U_yU_x$ for arbitrary $x, y \in Q$. So, since the generators commute, the algebra $\mathfrak{T}(\mathfrak{Q})$ is commutative. □

Definition 6. We say that an element $c \in Q$ is *normal* if $U_c b^2 \in \mathrm{Im}\,\sigma$ for all $b \in Q$. Let N be a set of normal elements in $\mathfrak{Q}$.

Proposition 4.1. *N is a subspace of $\mathfrak{Q}$ and* $\mathrm{Im}\,\sigma \subseteq N$.

Proof. Let $c, d \in N$, $b \in Q$ and $\lambda \in F$. Assume that $U_c b^2 = x^2$, $U_d b^2 = y^2 \in \mathrm{Im}\ \sigma$. Then $U_{\lambda c} b^2 = \lambda^2 U_c b^2 = (\lambda x)^2 \in \mathrm{Im}\ \sigma$. Moreover, by the triviality of the linear part

$$U_{c+d} b^2 = U_c b^2 + U_d b^2 = x^2 + y^2 = (x+y)^2 \in \mathrm{Im}\sigma\,.$$

Hence, N is a subspace. Also, by (QJ4), Lemma 4.1 and (QJ6), we have for any $a \in Q$

$$U_{a^2} b^2 = U_a U_a U_b 1 = U_a U_b U_a 1 = (U_a b)^2 \in \mathrm{Im}\sigma\,.$$

Thus $\mathrm{Im}\ \sigma \subseteq N$. □

Let $\Phi = \langle \{a^2 : a \in F\}; +, \cdot, 0, 1\rangle$. Then Φ is a subfield of F and it is isomorphic to F via σ. Now $\mathfrak{Q}$ can be considered as an algebra $\mathfrak{Q}_\Phi$ over Φ. If so, $\Sigma_\Phi = \langle \mathrm{Im}\,\sigma; +, U, 0, 1\rangle$ becomes its subalgebra, since $\mathrm{Im}\,\sigma$ is obviously closed under multiplications by scalars from Φ.

Lemma 4.2. *The mapping $\sigma : \mathfrak{Q}_F \to \Sigma_\Phi$ is a homomorphism, hence $\mathfrak{Q}_F/A \cong \Sigma_\Phi$.*

Proof. Let $x, y \in Q$ and $\lambda \in F$. Then since σ is quadratic, by definition we have

$$\sigma(\lambda x) = \lambda^2 \sigma x = \sigma\lambda \cdot \sigma x\,.$$

Since the algebra has a trivial linear part

$$\sigma(x+y) = \sigma x + \sigma y\,.$$

By (QJ5), Lemma 4.1 and (QJ4), we have

$$\sigma(U_x y) = (U_x y)^2 = U_x U_y x^2 = U_x U_y U_x 1 = U_x U_x U_y 1 = U_x U_x y^2 = U_{x^2} y^2 = U_{\sigma x} \sigma y\,.$$

We know that $\mathrm{Ker}\ \sigma = A$. Thus σ is a homomorphism indicating that $\mathfrak{Q}_F/A$ and Σ_Φ are isomorphic algebras. □

The remaining of the section is devoted to complete characterization of linearly minimal division Jordan algebras. So from now we continue using the following assumption.

Assumption 2. $\mathfrak{Q}$ is a linearly minimal division Jordan algebra, that is $A = \{0\}$.

First let us discuss some immediate corollaries of this assumption.

Observations 1. The map $\sigma : \mathfrak{Q}_F \to \Sigma_\Phi$ is an isomorphism. So the inverse mapping $\rho := \sigma^{-1} : \Sigma_\Phi \to \mathfrak{Q}_F$ is well-defined.

2. If all elements have finite order, then σ is an automorphism of Q^+, since each element must have odd order (by Claim 1 in Section 3). In this case $N = \mathrm{Im}\ \sigma = Q$.

Lemma 4.3. *If $c \in N$ then*

(1) $U_a c \in N$ *for any* $a \in Q$;

(2) $\rho(U_c d^2) \in N$ *for any* $d \in N$;

(3) $c^{-1} \in N$.

Proof. The commutativity of the algebra $\mathfrak{T}(\mathfrak{Q})$ is used in the proof of each item. Let $b \in Q$ be an arbitrary element.

(1) If $U_c b^2 = x^2$, then by (QJ6) we have $U_{U_a c} b^2 = U_a U_c U_a b^2 = U_a U_a U_c b^2 = U_a U_a x^2 = U_a U_a U_x 1 = U_a U_x U_a 1 = U_{U_a x} 1 = (U_a x)^2$. This means that $U_a c \in N$.

(2) $U_c d^2 \in \mathrm{Im}\sigma \subseteq N$, so $U_{\rho(U_c d^2)} b^2 = U_{\rho(U_c d^2)} U_b 1 = U_b U_{\rho(U_c d^2)} 1 = U_b (U_c d^2) \in \mathrm{Im}\sigma$ by (1). Thus, $\rho(U_c d^2) \in N$.

(3) By (QJ6) $U_{c^{-1}} b^2 = U_{c^{-1}} U_b 1 = U_c U_{c^{-1}} U_b U_{c^{-1}} 1 = U_c U_{U_{c^{-1}} b} 1 = U_c (U_{c^{-1}} b)^2 \in \mathrm{Im}\sigma$. So, $c^{-1} \in N$. $\square$

Definition 7. We define the symmetric bilinear form on N as follows: for $x, y \in N$

$$x \cdot y := \rho(U_x U_y 1) = \rho(U_x y^2)\,.$$

Proposition 4.2. *$\mathcal{N} = \langle N; +, \cdot, 0, 1\rangle$ is a field and the subalgebra $\mathfrak{N} = \langle N; +, U, 0, 1\rangle$ is a special algebra $\mathcal{N}^{(+)}$.*

Proof. In order to prove that $\mathcal{N}$ is a field, the only identity we need to check is the associativity law for $\cdot$. We know that $\cdot$ is a commutative operation and $\mathfrak{T}(\mathfrak{Q})$ is commutative too. So, for any $a, b, c \in N$ we have

$$U_a U_b c^2 = U_b U_a c^2,$$

which is equivalent to

$$a \cdot (b \cdot c) = b \cdot (a \cdot c)\,.$$

So, for any $a, b, c \in N$

$$a \cdot (b \cdot c) = a \cdot (c \cdot b) = c \cdot (a \cdot b) = (a \cdot b) \cdot c\,.$$

Thus, $\cdot$ is associative.

It is easy to verify the second statement of the proposition: $x \cdot y \cdot x = x^2 \cdot y = \rho(U_{x^2} y^2) = U_x y$. $\square$

Definition 8. Let $\mathcal{K} = \langle K; +, \cdot, 0, 1\rangle$ be a field with $char\mathcal{K} = 2$ and let $\mathfrak{P}$ be a quadratic Jordan algebra over a subfield of $\mathcal{K}$. We say that the special algebra $\mathfrak{P}$ *lives in the field* $\mathcal{K}$ if $\mathfrak{P}$ is a subalgebra of $\mathcal{K}^{(+)}$, where $\mathcal{K}^{(+)} = \langle K; +, U, 0, 1\rangle$ with $U_x y = x^2 \cdot y$.

By Proposition 4.1 Σ_Φ lives in the field $\mathfrak{N}$. This leads to an important conclusion.

Corollary 4.1. *The algebra $\mathfrak{Q}$ lives in a field.*

This corollary and the next lemma give a complete classification of l-minimal quadratic division Jordan algebras.

Lemma 4.4. *Any infinite quadratic Jordan division algebra living in a field is linearly minimal.*

Proof. Let $\mathfrak{P} = \langle P; +, U, 0, 1\rangle$ be an infinite quadratic Jordan division algebra living in a field Ψ. Let $f \in \mathfrak{T}(\mathfrak{P})$ be a non-zero linear function. Then $f(x) = ax$ for some $a \in \Psi^*$ and obviously $\mathrm{Ker}\, f = \{0\}$. First, we notice that $a = f(1) \in P$, hence $a^{-1} \in P$, since $\mathfrak{P}$ is a division algebra. Now let $b \in P$ be any element. Then $U_{a^{-1}} b = a^{-2} b \in P$. Next, $f(a^{-2}b) = a^{-1}b \in P$, then $f(a^{-1}b) = b \in P$. So, f is surjective and $\mathfrak{P}$ is linearly minimal. $\square$

Example. Let $\mathcal{N} = \mathbb{Z}_2(x, y)$, i.e. $\mathcal{N}$ is the field of all rational functions over $\mathbb{Z}_2$ in two variables x, y. Let $\Phi = \mathbb{Z}_2(x^2, y^2)$ be the subfield of squares in $\mathcal{N}$. Of course, any subfield of $\mathcal{N}$ containing Φ would be a subalgebra of $\mathcal{N}_\Phi^{(+)}$. Let P denote the subspace of $\mathcal{N}$ generated (over Φ) by the set of vectors $\{1, x, y\}$. Then $\mathfrak{P} = \langle P; +, U, 0, 1\rangle$ is a subalgebra of $\mathcal{N}_\Phi^{(+)}$. So, $\mathfrak{P}$ lives in the field $\mathcal{N}$ and it is not a subfield, since $xy \notin P$.

5 Main theorems

Now we are ready to prove the main results of the paper.

Theorem 5.1. *Let $\mathfrak{Q}$ be a l-minimal, unital quadratic Jordan algebra and $char F = 2$. Then*
(1) *if $A = \{0\}$ then the algebra $\mathfrak{Q}$ lives in a field;*
(2) *if $A \neq \{0\}$ then the quotient algebra $\overline{\mathfrak{Q}} = \mathfrak{Q}/A$ lives in a field.*

Proof. (1) just repeats Corollary 4.1.

(2) The quotient algebra $\overline{\mathfrak{Q}}$ inherits all properties of the algebra $\mathfrak{Q}$ such as unitality, triviality of the linear part and commutativity of the multiplication algebra. Moreover, it is a division algebra. So, if $\overline{\mathfrak{Q}}$ is infinite, then all the results of the previous section, where we nowhere relied on the l-minimality of algebra, are applicable to $\overline{\mathfrak{Q}}$. Therefore, repeating the same reasoning that we conducted in Section 4, we prove that $\overline{\mathfrak{Q}}$ lives in a field.

Now let $\overline{\mathfrak{Q}}$ be a finite division algebra. All the results of the previous section, where we nowhere relied on the infinity of algebra, remain valid for $\overline{\mathfrak{Q}}$, since $\overline{\mathfrak{Q}}$ inherits from $\mathfrak{Q}$ all the properties necessary for them. According to Observation 2, all elements of $\overline{\mathfrak{Q}}$ are normal. Finally, Proposition 4.2 implies that $\overline{\mathfrak{Q}} = \Psi^{(+)}$ for some field Ψ. □

Example. Let F be a field of characteristic 2 (it may be finite) and let A be an infinite vector space over F. We define operations over $F \times A$ as follows: for $(f, a), (g, b) \in F \times A$ we put

$$(f, a) + (g, b) := (f + g, a + b)$$

and

$$(f, a)^2 := (f^2, 0); \quad U_{(f,a)}(g, b) := (f^2 g, f^2 b)\,.$$

Then we obtain the quadratic Jordan algebra $\mathfrak{Q}_{F,A}$ on $F \times A$, which is indeed the algebra $F \cdot 1 + A$ of trivial bilinear form [7]. It is easy to see that $\mathfrak{Q}_{F,A}$ is a linearly minimal, unital algebra with a trivial linear part; $\mathcal{A} = \{0\} \times A$ is the inner ideal of trivial elements and the quotient algebra $\mathfrak{Q}_{F,A}/\mathcal{A}$ is isomorphic to $F^{(+)}$.

Question. Let $\mathfrak{Q}$ be a linearly minimal, unital quadratic Jordan algebra with the infinite ideal of trivial elements, $char F = 2$. Is $\mathfrak{Q}$ always special?

Theorem 5.2. *Any minimal non-trivial quadratic Jordan algebra lives in a field with the same set of elements.*

Proof. Let $\mathfrak{Q} = \langle Q; +, U, \sigma, 0\rangle$ be a minimal quadratic Jordan algebra. Since each element of $\mathfrak{T}(\mathfrak{Q})$ is definable by a first-order formula, $\mathfrak{Q}$ is linearly minimal. By [2] (see also [1]) we can assume $char F = 2$.

If $\mathfrak{Q}$ is weakly minimal, then σ is an endomorphism of Q^+ with $\mathrm{Im}\ \sigma \subseteq \mathrm{Ker}\ \sigma$ (see the discussion after Proposition 3.1). Since $Q^+/\,\mathrm{Ker}\ \sigma \cong \mathrm{Im}\ \sigma$, the subgroup $\mathrm{Ker}\ \sigma$ is infinite. But $\mathrm{Ker}\ \sigma$ is definable by the formula $\sigma x = 0$, so by minimality $\mathrm{Ker}\ \sigma = Q$, i.e. $\mathfrak{Q}$ is trivial.

Now, assume that $\mathfrak{Q}$ is not weakly minimal, then it is unital by Proposition 2.1. The subspace $A \subset Q$ is definable by the formula $\forall y(U_x y = 0)$, so it must be finite. Further, we get $A = \{0\}$ by Lemma 2.4.

Finally, let us note that the subspace N of normal elements is definable by the formula $\forall y \exists z(U_x \sigma y = \sigma z)$ and, by Proposition 4.1, it is an infinite subspace containing $\mathrm{Im}\ \sigma$. So, $N = Q$, again by the minimality of $\mathfrak{Q}$. Thus, by Proposition 4.2, $\mathcal{Q} = \langle Q; +, \cdot, 0, 1\rangle$ is a field and $\mathfrak{Q} = \mathcal{Q}^{(+)}$. □

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